Chapter 4

Entire Functions

4.1 Infinite Products

Definition 4.1.1. Let $\{z_n\}$ be a sequence of complex numbers, and we write $p_n = \prod_1^n z_i$ to denote the nth-partial product of $\{z_n\}$. If $p_n \to p$ as $n \to \infty$. We say the infinite product exists and denote the limit by $p = \lim p_n = \prod_1^\infty z_i$. If p_n does not tend to a finite number or $p_n \to 0$, then we say $\prod_1^\infty z_i$ diverges.

Example 4.1.2. Determine the convergence of $(1+1)(1-\dfrac{1}{2})(1+\dfrac{1}{3})\cdots$.

Solution. Define

$$\begin{aligned}
p_n &= \begin{cases} (1+1)(1-\frac{1}{2})\,(1+\frac{1}{3})\,\cdots(1-\frac{1}{n}), & n \text{ even;} \\ (1+1)(1-\frac{1}{2})\,(1+\frac{1}{3})\,\cdots(1+\frac{1}{n}), & n \text{ odd.} \end{cases} \\
&= \begin{cases} (1+1)\dfrac{1}{2}\dfrac{4}{3}\dfrac{3}{4}\cdots\dfrac{n}{n-1}\dfrac{n-1}{n} = 1, & n \text{ even;} \\ (1+1)\dfrac{1}{2}\dfrac{4}{3}\dfrac{3}{4},\cdots\dfrac{n-1}{n-2}\dfrac{n-2}{n-1}\dfrac{n+1}{n} = 1+\dfrac{1}{n}, & n \text{ odd.} \end{cases}
\end{aligned}$$

Hence $p_n \to 1$ as $n \to \infty$, and we conclude that

$$(1+1)(1-\frac{1}{2})(1+\frac{1}{3})\cdots = 1.$$

Note that we also have

$$(1+1)(1-\frac{1}{2})(1+\frac{1}{3})\cdots = \prod_{n=1}^{\infty}\left(1-\frac{(-1)^n}{n}\right).$$

□

From Example 4.1.2, we see that the last number $1-\frac{(-1)^n}{n}$ in the partial product p_n tends to one as $n\to\infty$. This is true in general. For suppose $p_n\to p$, then $z_N=\frac{\prod_1^N z_i}{\prod_1^{N-1} z_i}\to\frac{p}{p}=1$ as $N\to\infty$. In view of this observation, it will be more convenient for us to consider infinite product of the form $\prod_1^\infty(1+a_n)$ where $a_n\to 0$ as $n\to\infty$ if the infinite product converges. We now prove a fundamental convergence criterion.

Theorem 4.1.3. *The infinite product $\prod_1^\infty(1+a_n)$ is convergent if and only if given $\epsilon>0$, there exists an $N>0$ such that*

$$|(1+a_{n+1})\cdots(1+a_m)-1|<\epsilon$$

for all $m>n\geq N$.

Proof. Suppose $\prod_1^\infty(1+a_i)=p$. Let p_n be the nth-partial product of $\prod_1^\infty(1+a_i)$, then $\{p_n\}$ is a Cauchy sequence in $\mathbb{C}$. That is, given $\epsilon>0$, there exists an N such that $|p_n|>\frac{|p|}{2}$ and

$$|p_n-p_m|<\epsilon\frac{|p|}{2}$$

for all $m>n\geq N$. Thus

$$\begin{aligned}|(1+a_{n+1})\cdots(1+a_m)-1| &= |p_n|\left|\frac{p_m}{p_n}-1\right|\frac{1}{|p_n|}\\ &= |p_m-p_n|\frac{1}{|p_n|}\\ &< \epsilon\frac{|p|}{2}\frac{2}{|p|}=\epsilon\end{aligned}$$

for all $m > n \geq N$, as required.

Conversely, suppose given $1 > \epsilon > 0$, there exists an N such that for $m > n \geq N$ we have

$$\left|\frac{p_m}{p_n} - 1\right| < \epsilon.$$

Let $p'_k = \dfrac{p_k}{p_N}$ for all $k \geq N$ and N fixed, then

$$1 - \epsilon < |p'_k| < 1 + \epsilon < 2.$$

Notice that the assumption is equivalent to

$$\left|\frac{p'_m}{p'_n} - 1\right| < \epsilon.$$

That is

$$|p'_m - p'_n| < \epsilon|p'_n| < 2\epsilon$$

for all $m > n \geq N$. Hence $\{p'_m\}$ is a Cauchy sequence. So $\{p_n\}$ is also a Cauchy sequence and thus converges in $\mathbb{C}$. $\square$

If all the a_n are positive. Then we have

Proposition 4.1.4. *Suppose all the $a_n > 0$. Then $\prod(1+a_n)$ converges if and only if $\sum a_n$ converges.*

Proof. Suppose $\prod(1 + a_n)$ converges. By

$$a_1 + \cdots + a_n \leq (1 + a_1) \cdots (1 + a_n)$$

we conclude immediately that $\sum a_n < \infty$.

Conversely, since $1+a < e^a$ for all $a > 0$, hence $(1+a_1) \cdots (1+a_n) < e^{a_1+\cdots+a_n}$. Thus $\prod(1 + a_i)$ converges. $\square$

Definition 4.1.5. The infinite product $\prod(1+a_n)$ is said to be *absolutely convergent* if the product $\prod(1 + |a_n|)$ converges.

We recall from Example 4.1.2 that the infinite product

$$(1+1)(1-\frac{1}{2})(1+\frac{1}{3})\cdots$$

is not absolutely convergent. The converse, however, is definitely true.

Theorem 4.1.6. *If* $\prod(1+|a_n|)$ *converges, then* $\prod(1+a_n)$ *converges.*

Proof. Method I: The result follows immediately from the observation that

$$|(1+a_{n+1})\cdots(1+a_m)-1| \le (1+|a_{n+1}|)\cdots(1+|a_m|)-1,$$

and by Theorem 4.1.3.

Method II: Let $p_n = \prod_1^n(1+a_i)$ and $P_n = \prod_1^n(1+|a_i|)$. Then

$$\begin{aligned} p_n - p_{n-1} &= (1+a_1)\cdots(1+a_{n-1})a_n, \\ P_n - P_{n-1} &= (1+|a_1|)\cdots(1+|a_{n-1}|)|a_n|, \end{aligned}$$

and

$$|p_n - p_{n-1}| \le P_n - P_{n-1}.$$

Since $P_n \to \prod_1^\infty(1+|a_i|)$, we have $\sum_2^n(P_i - P_{i-1}) = P_n - P_1$ converges. But then $\sum_2^\infty(p_i - p_{i-1})$ converges absolutely by the above inequality. Hence the limit $\prod(1+a_n)$ exists. □

Theorem 4.1.7. *A product* $\prod(1+a_n)$ *is absolutely convergent if and only if* $\sum a_n$ *converges absolutely.*

Proof. If $\prod_1^\infty(1+|a_n|)$ converges then $\sum|a_n|$ must converges by Proposition 4.1.4. The converse also follows from Proposition 4.1.4. □

We deduce immediately that

Proposition 4.1.8. $\prod_1^\infty(1+a_n)$ *converges if* $\sum_1^\infty a_n$ *converges absolutely.*

We next turn to the study whether the statement "if $\prod(1+a_n) = p$, then $\sum \log(1+a_n) = \log p$" holds? Here $\log p$ is the principal logarithm.

Proposition 4.1.9. *If* $\sum \log(1+a_n)$ *converges, then* $\prod(1+a_n)$ *converges. If* $\prod(1+a_n)$ *converges, then* $\sum \log(1+a_n)$ *converges to a branch of* $\log(\prod (1+a_n))$ *.*

Proof. Let $s_n = \sum_1^n \log(1+a_i)$ then the hypothesis implies that $s_n \to \sum_1^\infty \log(1+a_i) = s$, say, as $n \to \infty$. That is,

$$\prod_1^n (1+a_i) = e^{s_n} \to e^s, \quad n \to \infty$$

i.e.

$$\prod_1^\infty (1+a_i) = e^s.$$

Suppose now $p = \prod_1^\infty (1+a_i)$ converges. Let $p_n = \prod_1^n (1+a_i)$. Then we must have $\log \frac{p_n}{p} \to 0$ as $n \to \infty$. We decompose it as $\log \frac{p_n}{p} = S_n - \log p + h_n(2\pi i)$. Then

$$\log \frac{p_{n+1}}{p} - \log \frac{p_n}{p} = \log(1+a_{n+1}) + (h_{n+1} - h_n)2\pi i.$$

But the left side tends to zero as $n \to \infty$. Also $\log(1+a_{n+1}) \to 0$ as $n \to \infty$. Thus $h_{n+1} - h_n = 0$ for all n sufficiently large. Let it be h. Then

$$s_n - \log p + h(2\pi i) = \log \frac{p_n}{p} \to 0.$$

That is $s_n \to S = \log p - h(2\pi i)$ answering the question raised before the proposition. $\square$

Finally, we give a criterion for the absolutely convergent product $\prod(1+a_i)$ in terms of $\sum \log(1+a_i)$.

Theorem 4.1.10. $\prod_1^\infty (1+a_i)$ *converges absolutely if and only if* $\sum_1^\infty \log(1+a_i)$ *converges absolutely.*

Proof. The result follows immediately from Theorem 4.1.7 and the limit

$$\lim_{z\to 0} \frac{\log(1+z)}{z} = 1.$$

It suffices to show $\sum |\log(1+a_i)|$ and $\sum |a_i|$ converges and diverges together. The details is left to the reader. $\square$

Example 4.1.11. 1. $\prod_1^\infty \left(1+\frac{1}{n^\alpha}\right)$ converges whenever $\alpha > 1$.

2. $\prod_1^\infty \left(1-\frac{2}{n(n+1)}\right) = \frac{1}{3}$.

3. $\prod_1^\infty \left(1+\frac{x}{n}\right) = \begin{cases} +\infty, & x > 0 \\ 0, & x < 0. \end{cases}$

4. $\prod(1+z^n)$ is absolutely convergent for every $|z| < 1$.

5. If $\sum a_n$ converges absolutely, then $\prod(1+a_n z)$ converges absolutely for every z. For example $\prod\left(1+\frac{z}{n^2}\right)$ converges absolutely.

6. If $\sum a_n$ and $\sum |a_n|^2$ are convergent, then $\prod(1+a_n)$ is convergent (Hint: $\log(1+a_n) = a_n + O(|a_n|^2)$).

7. Suppose $a_{2n-1} = \frac{-1}{\sqrt{n+1}}$, $a_{2n} = \frac{1}{\sqrt{n+1}} + \frac{1}{n+1} + \frac{1}{(n+1)\sqrt{n+1}}$. Then $\prod(1+a_n)$ converges, but $\sum a_n$ and $\sum a_n^2$ both diverge.

8. If a_n is real and $\sum a_n$ is convergent, then the product $\prod(1+a_n)$ converges or diverges to zero according to $\sum a_n^2$ converges or diverges respectively.

4.2 Infinite Product of Functions

It is not difficult to see that the main results from the previous section can be generalized to infinite product of functions.

Let G be a region in $\mathbb{C}$, and $\{f_n\}$ be a sequence of analytic functions defined on G.

Theorem 4.2.1. *Let $\{f_n\} \subset H(G)$ and $\sum_1^\infty |f_n|$ converges uniformly on every compact subsets of G. Then the infinite product $\prod(1 + f_n(z))$ converges uniformly to an analytic function f on G , i.e. $\prod_1^\infty(1+f_n) = f \in H(G)$.*

Moreover, f has a zero at those, and only those points of G at which at least one of the factors is equal to zero. The order of such a zero is finite and is equal to the sum of the orders to which those factors vanish there.

Proof. Let K be any compact subset of G. Since $\sum |f_n|$ converges uniformly on K, there exists a $M > 0$ such that $\sum_1^\infty |f_n(z)| < M$ for all $z \in K$. Thus, for any $n \in \mathbb{N}$, we have

$$(1 + |f_1(z)|)\cdots(1 + |f_n(z)|) \leq e^{|f_1(z)|+\cdots+|f_n(z)|} < e^M$$

for all $z \in K$. Set $P_n(z) = \prod_1^n(1 + |f_i(z)|)$. Then

$$\begin{aligned} P_n(z) - P_{n-1}(z) &= (1 + |f_1(z)|)\cdots(1 + |f_{n-1}(z)|)|f_n(z)| \\ &< e^M|f_n(z)| \end{aligned}$$

for all $n \geq 2$ and all $z \in K$. Hence

$$\begin{aligned} \left|\prod_1^n(1 + f_i(z)) - (1 + f_1(z))\right| &\leq \sum_{i=2}^n (P_i(z) - P_{i-1}(z)) \\ &< e^M \sum_{i=2}^n |f_i(z)| < e^{2M} \end{aligned}$$

for all $z \in K$. So we deduce that $\prod_1^\infty(1 + f_i(z)) - (1 + f_1(z))$ converges uniformly on K. But $H(G)$ is complete, so $\prod_1^\infty(1+f_i(z))-(1+f_1(z))$ is analytic and hence $\prod_1^\infty(1 + f_i(z))$ is analytic on K. But K is arbitrary, so $\prod_1^\infty(1 + f_i(z))$ is analytic on G.

Since $\sum |f_n(z)| < \infty$ for each $z \in K$, there exists an $N \in \mathbb{N}$ such that $\sum_n^\infty |f_i(z)| < \dfrac{1}{2}$ for all $n > N$. Suppose now that $z \in K$ and

$f(z) = 0$. It follows that $\Pi_{N+1}^{\infty}(1 + f_i(z)) \neq 0$ on K and hence the order of the zero is equal to the sum of orders of those factors (i.e. $\Pi_1^N(1 + f_i(z))$ vanishes there). $\square$

Remark. It is clear from the proof of Theorem 4.1.10 that $\sum |f_i(z)|$ and $\sum |\log(1 + f_i(z))|$ converge and diverge together. So we could rephrase Theorem 4.2.1 such that the hypothesis $\sum |f_i| < \infty$ is replaced by $\sum |\log(1 + f_i)| < \infty$. It turns out that both conditions are useful in applications.

4.3 Weierstrass Factorization Theorem

Suppose f is entire and non-vanishing. Then we can write f as e^g where g is an entire function (see Theorem 1.10.13). If f has only a finite number of zeros (can be repeated) $z_1, z_2, \ldots, z_n$, say, then $\dfrac{f}{(z - z_1)\cdots(z - z_n)}$ is entire and non-vanishing. Thus we have $f(z) = \Pi_{i=1}^n (z - z_i) e^g$. A natural question is for an representation for f as above when f has an infinite number of zeros. We can also view the above question as an interpolation problem: Given $z_1, z_2, \ldots, z_n, \ldots$ and $w_1, w_2, \ldots, w_n, \ldots$, find an entire function f such that $f(z_i) = w_i$ for $i = 1, 2, 3, \ldots$. If $w_i = 0$ for $i = 1, 2, \ldots$, then our question become a special case of the interpolation problem.

Thus a natural guess of an answer of the interpolation is the *function*

$$f(z) = z^m e^{g(z)} \prod_1^{\infty} \left(1 - \frac{z}{z_i}\right).$$

But it is unclear of whether such a *function* exists since the infinite product may diverge. According to Proposition 4.1.8, $\displaystyle\prod_1^{\infty}\left(1 - \frac{z}{z_i}\right)$ converges if $\displaystyle\sum \left|\frac{z}{z_i}\right| = |z| \sum \left|\frac{1}{z_i}\right|$ is convergent for every z. Thus if $z_n = n^2$, then $\displaystyle\sum \left|\frac{1}{z_n}\right| = \sum \frac{1}{n^2} < \infty$ and so f has the above factorized form.

Weierstrass was able to construct a convergent-producing factor called *primary factor* so that a factorization of f always exist regardless of the given sequence $\{z_n\}$.

Definition 4.3.1. Let $p \geq 0$ be an integer. We define the *Weierstrass primary factor* by

$$E_p(z) = E(z,p) = \begin{cases} (1-z)\exp\left(z + \dfrac{z^2}{2} + \dfrac{z^3}{3} + \cdots + \dfrac{z^p}{p}\right), & p \geq 1; \\ 1-z, & p = 0. \end{cases}$$

Theorem 4.3.2 (Weierstrass Factorization Theorem)**.** *Let $\{a_n\}$ be a sequence of complex numbers with $\lim a_n = \infty$. Then there exists an entire function f with $f(a_n) = 0$ for all n and f has a zero at $z = 0$ of order $m \geq 0$. In fact, f is given by*

$$\begin{aligned} f(z) &= z^m e^{g(z)} \prod_{n=1}^{\infty} E_{p_n}\Big(\frac{z}{a_n}\Big) \\ &= z^m e^{g(z)} \prod_{n=1}^{\infty} \Big(1 - \frac{z}{a_n}\Big) \exp\Big[\frac{z}{a_n} + \cdots + \frac{1}{p_n}\Big(\frac{z}{a_n}\Big)^{p_n}\Big], \end{aligned}$$

where $g(z)$ is an entire function and $\{p_n\}$ is any non-negative integer sequence for which

$$\sum_1^{\infty} \Big(\frac{r}{|a_n|}\Big)^{p_n+1} < \infty$$

for each $r > 0$.

Remark. (i) $p_n = n - 1$ always satisfy the hypothesis. The idea is to choose $\{p_n\}$ as simple as possible.

(ii) The above *factorization* has already taken care of the multiplicity of $\{a_n\}$.

Lemma 4.3.3. *Let p be a non-negative integer. Then*

(i) $|E_p(z) - 1| \leq |z|^{p+1}$ *if* $|z| \leq 1$*;*

(ii) $|\log E_p(z)| < \frac{k}{k-1}|z|^{p+1}$ *if* $|z| < \frac{1}{k}$ *and* $k > 1$*;*

(iii) $|E_p(z) - 1| < 6|z|^{p+1}$*, if* $|z| < \frac{1}{2}$*.*

Proof. (i) We expand E_p into a power series:

$$E_p(z) = 1 + \sum_1^\infty a_k z^k$$

where all the a_k are real. Differentiating both sides yields

$$E_p'(z) = \sum_1^\infty k a_k z^{k-1}. \tag{4.1}$$

But the left side is equal to

$$\begin{aligned} E_p'(z) &= [(1-z)(1+z+z^2+\cdots+z^{p-1}-1]\exp\left(z+\frac{z^2}{2}+\cdots+\frac{z^p}{p}\right) \\ &= [(1-z^p)-1]\exp\left(z+\frac{z^2}{2}+\cdots+\frac{z^p}{p}\right). \end{aligned} \tag{4.2}$$

By comparing the coefficients of (4.1) and (4.2), we deduce $a_1 = \cdots = a_p = 0$ and $a_k \le 0$ for the rest of k. Thus for $|z| \le 1$,

$$\begin{aligned} |E_p(z) - 1| &= \left|\sum_{p+1}^\infty a_k z^k\right| = |z^{p+1}|\left|\sum_0^\infty a_{p+k+1} z^k\right| \\ &\le |z|^{p+1}\sum_0^\infty |a_{p+k+1}| = -|z|^{p+1}\sum_0^\infty a_{p+k+1} = |z|^{p+1} \end{aligned}$$

since $0 = E_p(1) = 1 + \sum_{p+1}^\infty a_k$ and so $\sum |a_k| = -\sum a_k = 1$.

(ii) Since

$$\begin{aligned}
|\log E_p(z)| &= \left|\log(1-z) + z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right| \\
&= \left|\left(-z - \frac{z^2}{2} - \cdots\right) + z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right| \\
&= \left|-\frac{1}{p+1}z^{p+1} - \frac{1}{p+2}z^{p+2} - \cdots\right| \\
&\le |z|^{p+1}\left(\frac{1}{p+1} + \frac{1}{p+2}|z| + \frac{1}{p+3}|z|^2 + \cdots\right) \\
&\le |z|^{p+1}(1 + |z| + |z|^2 + \cdots) \\
&< |z|^{p+1}\left(1 + \frac{1}{k} + \frac{1}{k^2} + \cdots\right) = \frac{k}{k-1}|z|^{p+1}.
\end{aligned}$$

(iii) By the definition of $E_p(z)$,

$$\begin{aligned}
|E_p(z) - 1| &= \left|(1-z)\exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right) - 1\right| \\
&\le \exp\left(\frac{|z|^{p+1}}{p+1} + \frac{|u|^{p+2}}{p+2} + \cdots\right) - 1 \\
&\quad \left(\text{by } 1 - z = \exp\left(-z - \frac{z^2}{2} - \cdots\right)\right) \\
&\le \exp[|z|^{p+1}(1 + |z| + |z|^2 + \cdots)] - 1 \\
&= \exp\left(|z|^{p+1}\frac{1}{1-|z|}\right) - 1 \\
&< \exp(2|z|^{p+1}) - 1 \\
&\le 2|z|^{p+1}\exp(2|z|^{p+1}) \quad \because e^x - 1 \le xe^x \text{ for } x \ge 0 \\
&< 2|z|^{p+1}e^1 < 6|z|^{p+1}.
\end{aligned}$$

□

Now we can prove Theorem 4.3.2.

Proof of Theorem 4.3.2. Let $\{a_n\}$ be the given sequence of complex numbers such that $a_n \to \infty$ as $n \to +\infty$. Thus given any $|z| = r$, we can find an $N > 0$ such that $|a_n| > 2r$, or $\left|\dfrac{z}{a_n}\right| < \dfrac{1}{2}$ for $|z| < r$. Thus Lemma 4.3.3(iii) gives

$$\left|E_{p_n}\left(\frac{z}{a_n}\right) - 1\right| < 6\left|\frac{z}{a_n}\right|^{p_n+1} < \left(\frac{r}{|a_n|}\right)^{p_n+1}$$

for $n > N$ and $|z| < r$. It follows from the hypothesis that the sum $\sum(E_{p_n}(z/a_n) - 1)$ converges uniformly and absolutely on any compact subset of $B(0, r)$. Theorem 4.2.1 implies that the infinite product $\prod_1^\infty E_{p_n}(z/a_n)$ converges to an analytic functions in $B(0, r)$. But r is arbitrary, so it is actually an entire function.

Suppose f is an entire function with zeros given by $\{a_n\}$, then $f / \prod_1^\infty E_{p_n}(z/a_n)$ is zero-free. Hence we can find an entire function g such that

$$f(z) = z^m e^{g(z)} \prod_1^\infty E_{p_n}\left(\frac{z}{a_n}\right)$$

where $m \geq 0$ is an integer.

It is easy to see that we can always find the sequence $\{p_n\}$ by choosing $p_n = n - 1$. Since $\sum\left|\dfrac{r}{a_n}\right|^{p_n+1} < \sum\left(\dfrac{1}{2}\right)^n < +\infty$ for each r. This completes the proof of the theorem.

Alternatively, we can prove the theorem by applying Lemma 4.3.3(ii). We choose $k > 1$ and N so large that $|a_n| > kr$ for $n > N$ and $|z| < r$. Thus

$$\left|\log E_{p_n}\left(\frac{z}{a_n}\right)\right| < \frac{k}{k-1}\left|\frac{z}{a_n}\right|^{p_n+1} < \frac{k}{k-1}\left(\frac{1}{k}\right)^{p_n+1}.$$

Choose $p_n = n - 1$ again implies $\sum|\log E_{p_n}(z/a_n)|$ converges uniformly. The discussion in the remark after Theorem 4.2.1 shows that $\prod_1^\infty E_{p_n}(z/a_n)$ converges to an entire function. You may fill in the details as an exercise. □

Remark. Note that some authors will phrase Theorem 4.3.2 as: *Let* $\{a_n\}$ *be a sequence of complex numbers with* $\lim a_n = \infty$*, then there exists* $\{p_n\}$ *such that the following* f *is an entire function*

$$f(z) = z^m e^{g(z)} \prod_1^\infty E_{p_n}\left(\frac{z}{a_n}\right)$$

where g *is an entire function.*

This is because we can always obtain the estimate, as in the proof,

$$\left|E_{p_n}\left(\frac{z}{a_n}\right) - 1\right| < 6\left(\frac{1}{2}\right)^{p_n+1}.$$

Hence any increasing non-negative integer sequence $\{p_n\}$ will make $\sum |E_{p_n}(z/a_n) - 1|$ converges uniformly.

Proposition 4.3.4. *Suppose* G *is an open set and* $\{f_n\} \subset H(G)$ *such that* $f = \prod f_n$ *converges in* $H(G)$*. Then*

(a) $f' = \sum\limits_{k=1}^\infty \left(f'_k \prod\limits_{n\neq k} f_n\right)$

(b) $\dfrac{f'}{f} = \sum\limits_{k=1}^\infty \dfrac{f'_k}{f_k}$

on any compact subset $K \subset G$ *provided* $f \neq 0$ *on* K*. (See Conway p.174)*

Proof. (*Sketch*) For (a), Consider $F'_k = \sum_{i=1}^k \left(f'_i \prod_{n\neq i} f_n\right) = \left(\prod_1^k f_i\right)'$. By Theorem 3.3.1, since we have $F_k \to f$, then $\left|f' - \sum_{i=1}^k \left(f'_i \prod_{n\neq i} f_n\right)\right|$ converges in $H(G)$ and $f' = \sum_{i=1}^\infty \left(f'_i \prod_{n\neq i} f_n\right)$ as required.

For (b), let K be an arbitrary compact set. Hence $|f| > a > 0$ for all $z \in K$. Then

$$\left|\frac{f'}{f} - \frac{F'_k}{F_k}\right| = \left|\frac{f'F_k - fF'_k}{fF_k}\right| \to 0 \quad \text{as } k \to \infty$$

since $F'_k \to f'$ and $F_k \to f$ in $H(G)$. □

4.4 Factorization of Sine Function

We define

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots + (-1)^n \frac{z^{2n-1}}{(2n-1)!} + \cdots .$$

Since this series is convergent uniformly and absolutely on any closed disk centred at the origin, we could rearrange the terns so that

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}).$$

It follows that each zero of $\sin(\pi z)$ is simple. In fact, the zeros are real and equal to $0, \pm 1, \pm 2, \ldots, \pm n, \ldots$. Let a_k be the non-zero zeros. Then

$$\sum_{k=1}^{\infty} \left(\frac{r}{|a_k|} \right)^2 = \sum_{\substack{-\infty \\ (n\neq 0)}}^{\infty} \left(\frac{r}{n} \right)^2 = r^2 \sum_{\substack{-\infty \\ (n\neq 0)}}^{\infty} \frac{1}{n^2}$$

always converge for each $r > 0$ by choosing $\{p_n\} = \{1\}$. It follows from the Weierstrass factorization theorem (Theorem 4.3.2) that

$$\begin{aligned} \sin \pi z &= z e^{g(z)} \prod_{-\infty}^{\infty} \left(1 - \frac{z}{n}\right) e^{z/n} \\ &= z e^{g(z)} \prod_{1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) \end{aligned}$$

for some entire function $g(z)$. We deduce from Proposition 4.3.4 that

$$\pi \cot \pi z = \pi \frac{\cos \pi z}{\sin \pi z} = \frac{1}{z} + g'(z) + \sum_{1}^{\infty} \frac{2z}{z^2 - n^2}$$

converges uniformly on compact subsets of $\mathbb{C} \setminus \mathbb{Z}$.

We now need a standard contour integration result which can be found p.122 in Conway :

$$\pi \cot \pi z = \frac{1}{z} + \sum_{1}^{\infty} \frac{2z}{z^2 - n^2} \quad \text{for } z \in \mathbb{Z}.$$

Hence g is identically a constant. In fact $g(z) = \log \pi$ because $\dfrac{\sin \pi z}{\pi z} \to 1$ as $z \to 0$. We finally obtain

$$\sin \pi z = \pi z \prod_1^\infty \left(1 - \frac{z^2}{n^2}\right).$$

4.5 Introduction to Gamma Function

We shall only introduce the definition of Gamma function and leave its more difficult asymptotic expansion to a later chapter when time allows. To begin with, let us consider the following entire function defined by

$$G(z) = \prod_1^\infty \left(1 + \frac{z}{n}\right) e^{-z/n}.$$

The infinite product G converges to an entire function in $H(\mathbb{C})$ with only negative zeros $-1, -2, -1, \ldots$. Similarly the function $G(-z)$ has similar properties except the zeros are $1, 2, 3, \ldots$. It is readily seen that $\pi z G(z) G(-z) = \sin \pi z$.

Consider now $G(z-1)$ which has the same zeros as $G(z)$ plus a new zero at the origin. Hence there exists an entire function $\gamma(z)$ such that

$$G(z-1) = z e^{\gamma(z)} G(z).$$

We shall determine $\gamma(z)$. To do so, we take the logarithmic derivative on both sides:

$$\sum_1^\infty \left(\frac{1}{z-1+n} - \frac{1}{n}\right) = \frac{1}{z} + \gamma'(z) + \sum_1^\infty \left(\frac{1}{z+n} - \frac{1}{n}\right).$$

Rewrite

$$\begin{aligned}
\sum_1^\infty \left(\frac{1}{z-1+n} - \frac{1}{n}\right) &= \frac{1}{z} - 1 + \sum_2^\infty \left(\frac{1}{z-1+n} - \frac{1}{n}\right) \\
&= \frac{1}{z} - 1 + \sum_1^\infty \left(\frac{1}{z+n} - \frac{1}{n+1}\right) \\
&= \frac{1}{z} - 1 + \sum_1^\infty \left(\frac{1}{z+n} - \frac{1}{n}\right) + \sum_1^\infty \left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= \frac{1}{z} - 1 + \sum_1^\infty \left(\frac{1}{z+n} - \frac{1}{n}\right) + 1 = \frac{1}{z} + \sum_1^\infty \left(\frac{1}{z+n} - \frac{1}{n}\right).
\end{aligned}$$

This implies that $\gamma'(z) = 0$ and $\gamma(z) = \gamma$ is a constant. Putting $z = 1$ into $G(z-1) = e^\gamma z G(z)$ gives $1 = G(0) = e^\gamma G(1)$. That is

$$e^{-\gamma} = G(1) = \prod_1^\infty \left(1 + \frac{1}{n}\right) e^{-1/n}.$$

The nth-partial product is

$$(n+1)e^{-(1+1/2+1/2+\cdots+1/n)}$$

and this implies

$$\begin{aligned}
\gamma &= \lim \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log(n+1)\right) \\
&= \lim \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n - \log\left(1 + \frac{1}{n}\right)\right) \\
&= \lim \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\right) - \lim \log\left(1 + \frac{1}{n}\right) \\
&= \lim \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\right).
\end{aligned}$$

The number $\gamma (\approx 0.57722)$ is called *Euler's constant* whose numerical value is still unknown. In fact, it is still undecided whether γ is rational or irrational.

Using $H(z) = e^{\gamma z} G(z)$ on $G(z-1) = e^\gamma z G(z)$ gives us a new relation:

$$H(z-1) = zH(z).$$

A further change of notation $\Gamma(z) = \dfrac{1}{zH(z)}$ gives us the right order: $\Gamma(z-1) = \dfrac{\Gamma(z)}{z-1}$ or

$$\Gamma(z+1) = z\Gamma(z) \quad \text{for } z \neq -1, -2, \ldots.$$

Of course

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_1^\infty \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}$$

is now an infinite product of meromorphic functions. The convergence can easily be justified by considering compact sets K in $\mathbb{C} \setminus \{-1, -2, \ldots\}$. $\Gamma(z)$ is call *(Euler's) gamma function.* Clearly $\Gamma(1) = 1$ and we deduce from the functional equation above that $\Gamma(2) = \Gamma(1) = 1$, $\Gamma(3) = 2\Gamma(2) = 2 \cdot 1 = 2!, \ldots, \Gamma(n) = (n-1)!$. Thus the gamma function can be considered as a generalization of the factorial. Also

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

which gives $\Gamma(1/2) = \sqrt{\pi}$.

One can show that

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt. \quad \text{(Mellin transform)}$$

We shall extend Weierstrass factorization theorem to an arbitrary region.

Theorem 4.5.1. *Let G be a region and $\{a_j\} \subset G$ is a sequence of points without a limit point in G. Then there exists an analytic function $f : G \to \mathbb{C}$ such that $f(a_j) = 0$ and f has no other zeros in G.*

Proof. We first show that it is possible to simplify the problem by considering G unbounded and $\lim_{z\to\infty} f(z) = 1$. More precisely, we consider G such that $\{z : |z| > R\} \subset G$ and $|a_j| < R$ for all j.

For suppose this is true, then given any region G_1 and an arbitrary sequence $\{\alpha_j\}$ such that $\{\alpha_j\}$ does not have a limit point in G_1. We choose $a \in G_1$, $r > 0$ such that $\overline{B}(a, r) \subset G_1$ and $\alpha_j \notin B(a, r)$ for all j. Let $T(z) = \dfrac{1}{z-a}$, then $G := T(G_1) \setminus \{\infty\}$ is such that $\{z : |z| > R\} \subset G$ and $|a_j| = |1/(\alpha_j - a)| < R$ for all j and some $R > 0$.

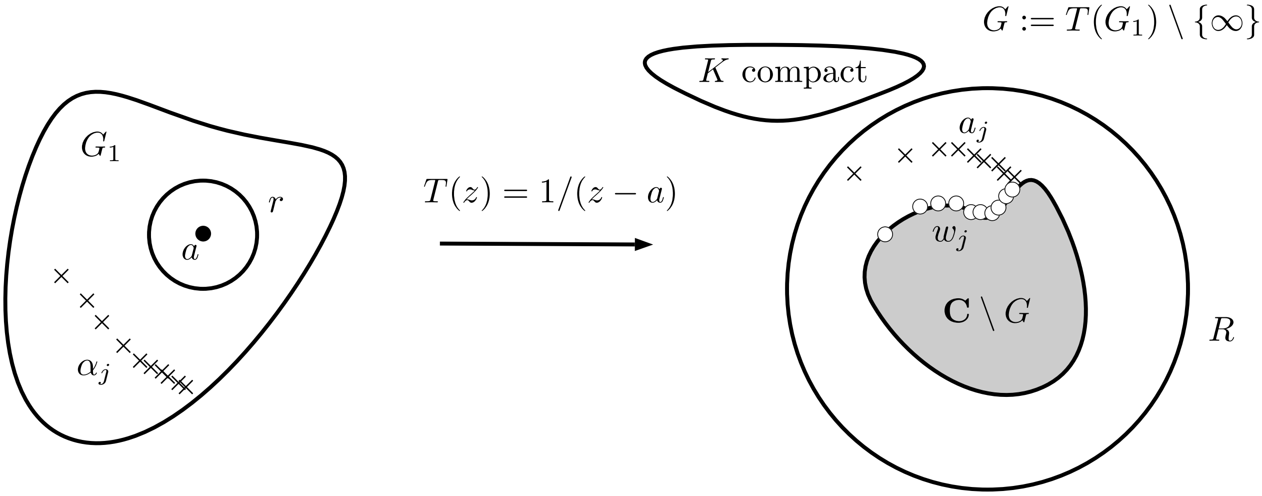

Figure 4.1: $\mathbb{C} \setminus G$

Since $\lim_{z\to\infty} f = 1$, $f(T(z))$ has a removable singularity at a and f has zeros at precisely $a_j = \dfrac{1}{\alpha_j - a}$ for all j. Then according to the definition, there exists an analytic function g on G_1 such that $g = f(T(z))$ on $G_1 \setminus \{a\}$. Clearly g has the zeros precisely on $\{\alpha_j\}$. It remains to prove the special case mentioned above.

Since G is open, so for each a_n we can find $w_n \in \mathbb{C} \setminus G$ such that

$$|w_n - a_n| = d(a_n, \mathbb{C} \setminus G)$$

and

$$\lim_{n\to\infty} |w_n - a_n| = 0$$

We aim to show that the infinite product $\prod E_n\left(\dfrac{a_n - w_n}{z - w_n}\right)$ converges in $H(G)$. So let K be any compact subset of G, and hence $d(K, \mathbb{C}\setminus G) > 0$.

Then, for $z \in K$,

$$\left|\frac{a_n - w_n}{z - w_n}\right| \le \frac{|a_n - w_n|}{d(w_n, K)} \le \frac{|a_n - w_n|}{d(K, \mathbb{C} \setminus G)}.$$

Hence given δ, $0 < \delta < 1$, there exists an N such that

$$\left|\frac{a_n - w_n}{z - w_n}\right| < \delta$$

for all $n > N$ and all $z \in K$. Thus Lemma 4.3.3(i) implies that

$$\sum_{N+1}^{\infty} \left|E_n\left(\frac{a_n - w_n}{z - w_n}\right) - 1\right| < \sum_{N+1}^{\infty} \delta^{n+1}.$$

That is, $\sum_1^\infty \left|E_n\left(\frac{a_n - w_n}{z - w_n}\right) - 1\right|$ converges uniformly, and Theorem 4.2.1 implies that $f := \prod_1^\infty E_n\left(\frac{a_n - w_n}{z - w_n}\right)$ converges to an analytic function in $H(G)$.

The only remaining fact to verify is that $\lim_{z\to\infty} f(z) = 1$. Given $\epsilon > 0$, let $R_1 > R$ so that if $|z| > R_1$, $|a_n| < R$, we have

$$\left|\frac{a_n - w_n}{z - w_n}\right| \le \frac{2R}{R_1 - R}.$$

In particular, we can choose R_1 sufficiently large such that $\dfrac{2R}{R_1 - R} < \delta$ for any $0 < \delta < 1$. Thus by Lemma 4.3.3(i) again,

$$\left|E_n\left(\frac{a_n - w_n}{z - w_n}\right) - 1\right| \le \left(\frac{2R}{R_1 - R}\right)^{n+1} < \delta^{n+1}$$

for all $|z| > R_1 > R$. Recall that $\lim_{z\to 0} \dfrac{\log(1+z)}{z} = 1$. Thus we may choose R_1 so large that when $|z| > R_1$, there exists $C > 0$ such that

$$\begin{aligned}\left|\sum_1^\infty \log E_n\left(\frac{a_n - w_n}{z - w_n}\right)\right| &\le \sum_1^\infty \left|\log E_n\left(\frac{a_n - w_n}{z - w_n}\right)\right| \\ &\le C\sum_1^\infty \left|E_n\left(\frac{a_n - w_n}{z - w_n}\right) - 1\right| \\ &\le C\sum_1^\infty \delta^{n+1} \\ &= C\frac{\delta^2}{1-\delta}.\end{aligned}$$

Thus by choose δ sufficiently small and hence R_1 sufficiently large that,

$$\begin{aligned}|f(z) - 1| &= \left|\exp\left(\sum \log E_n\left(\frac{a_n - w_n}{z - w_n}\right)\right) - 1\right| \\ &< \epsilon\end{aligned}$$

for all $|z| > R_1$. This completes the proof. □

4.6 Jensen's Formula

We shall derive a useful formula called Jensen's formula. It is a special case of the more general Poisson-Jensen formula. Jensen's formula will be used again in later sections.

Theorem 4.6.1. *Let f be analytic on a region containing $\overline{B(0,r)}$ and that $a_1, \ldots, a_n$ are the zeros of f in $B(0,r)$. Suppose in addition that $f(z) \neq 0$ on $|z| = r$ and $f(0) \neq 0$, then*

$$\log|f(0)| = -\sum_{k=1}^n \log\frac{r}{|a_k|} + \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta.$$

Alternatively, Jensen's formula can be written as

$$|f(0)|\prod_1^n \frac{r}{|a_k|} = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta\right).$$

Proof. We first prove Jensen's formula when f is non-vanishing on $\overline{B(0,r)}$. Hence we may find an analytic branch of $\log f(z)$. We then have, by Cauchy's integral formula

$$\log f(z) = \frac{1}{2\pi i}\int_\gamma \frac{\log f(\zeta)}{\zeta - z}\, d\zeta$$

where $\gamma = \partial B(0,r)$ and $\zeta = re^{i\theta}$. Thus

$$\log f(0) = \frac{1}{2\pi}\int_0^{2\pi} \log f(re^{i\theta})\, d\theta,$$

and we obtain the Jensen formula by taking the real parts on both sides.

We next consider f to have a finite number of zeros in $B(0,r)$. Let $b \in \Delta = \{z : |z| < 1\}$, then it is known that the map $\dfrac{z-b}{z-\bar{b}z}$ is an automorphism of Δ with $|z| = 1$ being mapped to $|z| = 1$. Based on this automorphism, it is not difficult to check that

$$\frac{r(z-a_k)}{r^2 - \overline{a_k}z}$$

maps $B(0,r)$ onto $B(0,1)$ in an one-to-one manner with $|z| = r$ mapped to $|z| = 1$ and $a_k \mapsto 0$. So the function defined by

$$F(z) = \frac{f(z)}{\Pi_1^n \dfrac{r(z-a_k)}{r^2 - \overline{a_k}z}} = f(z)\prod_1^n \frac{r^2 - \overline{a_k}z}{r(z-a_k)}$$

is non-vanishing on $B(0,r)$, and $|F(z)| = |f(z)|$ on $|z| = r$.

We now apply the result in the first part to $F(z)$ to obtain

$$\begin{aligned}\log|F(0)| &= \frac{1}{2\pi}\int_0^2 \pi \log|F(re^{i\theta})|\, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\, d\theta.\end{aligned}$$

But $\log|F(0)| = \log|f(0)| + \Sigma_1^n \log\dfrac{r}{|a_k|}$. This completes the proof. $\square$

Remark. (i) If f has a finite number of poles $b_1, \ldots, b_m$ except at the origin, then the Jensen formula becomes

$$\log|f(0)| = -\sum_1^n \frac{r}{|a_k|} + \sum_1^m \frac{r}{|b_v|} + \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta.$$

(ii) The Jensen formula in Theorem 4.6.1 still holds even if there are finite number of zeros on $|z| = r$. It suffices to show that f has only a simple zero $a = re^{i\varphi}$ on $|z| = r$. Let us recall that the function $F(z)$ defined in the proof of Theorem 4.6.1. Now the function $\dfrac{F(z)}{z-a}$ is zero-free on $\overline{B(0,r)}$ and hence

$$\begin{aligned}\log\left|\frac{F(0)}{0-a}\right| &= \frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{F(re^{i\theta})}{re^{i\theta}-a}\right| d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log|F(re^{i\theta})|\,d\theta - \frac{1}{2\pi}\int_0^{2\pi} \log(r|1-e^{i(\theta-\varphi)}|)\,d\theta.\end{aligned}$$

Hence

$$\log|F(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|F(re^{i\theta})|\,d\theta - \frac{1}{2\pi}\int_0^{2\pi} \log|1-e^{i(\theta-\varphi)}|\,d\theta.$$

The above equation will become the Jensen formula if the second integral on the right hand side vanishes. This will be done in the next lemma.

Lemma 4.6.2.

$$\frac{1}{2\pi}\int_0^{2\pi} \log|1-e^{i\theta}|\,d\theta = 0.$$

Proof. Consider the simply connected region $\Omega = \{z : \Re(z) < +1\}$. Hence we may define an analytic branch $\log(1-z)$ in Ω since $1-z \neq 0$. In particular, the branch is unique if we choose $\log(1-0) = 0$. Notice that $\Re(1-z) > 0$, so we have

$$\Re(\log(1-z)) = \log|1-z| \quad \text{and} \quad |\arg(1-z)| < \frac{\pi}{2}.$$

We then consider two paths:

$$\Gamma(t) = e^{it},\ \delta \le t \le 2\pi - \delta \quad \text{and} \quad \gamma(t) = 1 + \rho e^{it},\ \text{joining } e^{-i\delta} \text{ to } e^{i\delta}$$

We apply Cauchy's integral formula to $\log(1-z)$ to obtain

$$\frac{1}{2\pi}\int_{\delta}^{2\pi-\delta} \log|1-e^{i\theta}|\, d\theta = \Re\left[\frac{1}{2\pi i}\int_{\Gamma} \log(1-z)\frac{dz}{z}\right] = \Re\left[\frac{1}{2\pi i}\int_{-\gamma} \log(1-z)\frac{dz}{z}\right].$$

But on $\gamma(t)$,

$$\begin{aligned}\frac{\log(1-z)}{z} &= \log(-\rho e^{it})(1-\rho e^{it} + \frac{(\rho e^{it})^2}{2!} - \cdots)\\ &= \log(-\rho e^{it})(1+O(\rho))\\ &= -\log\frac{1}{\rho}(1+O(\rho)) + i(\text{imaginary part})(1+O(\rho)).\end{aligned}$$

Hence

$$\left|\Re\left(\frac{1}{2\pi i}\int_{-\gamma}\frac{\log(1-z)}{z}\,dz\right)\right| \le C\delta\log\frac{1}{\delta} \to 0$$

as $\delta \to 0$; thus proving the lemma. □

We shall study Weierstrass factorization type problem for the unit disc in this section (briefly).

Definition 4.6.3. Let $\Delta = \{z : |z| < 1\}$. Then we define a subset of $H(\Delta)$ as

$$H^\infty = H^\infty(\Delta)$$

where $\sup\{|f(z)| : z \in \Delta\} < +\infty$ for all $f \in H^\infty$, i.e. *the set of all bounded analytic functions on* Δ.

Definition 4.6.4. We also define

$$B(z) = z^k \prod_{n=1}^{\infty} \frac{\alpha_n - z}{1 - \overline{\alpha_n} z}\frac{|\alpha_n|}{\alpha_n} \tag{4.3}$$

on Δ, which is called *Blaschke product* provided the infinite product converges. Here the sequence $\{\alpha_n\}$ consisting of complex numbers in the unit disc.

Thus the natural questions is under what condition on $\{\alpha_n\}$ will (4.3) converges. We shall give an answer to this question in the following theorem. In fact we shall give a characterization to the existence of Blaschke product.

Theorem 4.6.5. *Let $\{\alpha_n\}$ be a sequence in Δ without limit points. Then (4.3) converges uniformly to an analytic function if and only if $\sum_1^\infty (1-|\alpha_n|) < +\infty$.*

Lemma 4.6.6. *Suppose $0 \le a_n < 1$. Then $\prod_1^\infty (1-a_n) > 0$ exists if and only if $\sum_1^\infty a_n < \infty$.*

Proof. Since $\sum a_n < \infty$, there exists N such that $\sum_{N+1}^\infty a_n < 1/2$. Note that

$$\begin{aligned}(1-a_{N+1})(1-a_{N+2}) &\ge 1 - a_{N+1} - a_{N+2}\\ &\cdots\cdots\\ (1-a_{N+1})\cdots(1-a_{N+k}) &\ge 1 - a_{N+1} - \cdots - a_{N+k} \quad \text{for all } k\\ &> 1/2.\end{aligned}$$

Hence $p_n = (1-a_1)\cdots(1-a_n)$ is monotonic decresing and bounded below by a positive number. Thus $\prod_1^\infty (1-a_n) > 0$ exists.

Conversely, suppose $\prod_1^\infty (1-a_n) = p > 0$. Then

$$0 < p \le p_n = \prod_1^\infty (1-a_k) \le \exp\left(-\sum_1^n a_k\right)$$

and if we assume $\sum_1^\infty a_k = +\infty$ then $\exp\left(-\sum_1^\infty a_k\right) \to 0$. A contradiction. □

Proof of Theorem 4.6.5. Suppose $\sum_1^\infty (1-|\alpha_n|) < \infty$. According to Theorem 4.2.1, it is sufficient to show

$$\sum_1^\infty \left|1 - \frac{\alpha_n - z}{1-\overline{\alpha_n}z}\frac{|\alpha_n|}{\alpha_n}\right| < \infty.$$

Notice that

$$1 - \frac{\alpha_n - z}{1-\overline{\alpha_n}z}\frac{|\alpha_n|}{\alpha_n} = \frac{\alpha_n + |\alpha_n|z}{(1-\overline{\alpha_n}z)\alpha_n}(1-|\alpha_n|).$$

For each $|z| \le r < 1$, we have

$$\left|\frac{(1-|\alpha_n|)(\alpha_n + |\alpha_n| z)}{(1-\overline{\alpha_n}z)\alpha_n}\right| \le \frac{1+r}{1-r}(1-|\alpha_n|)$$

since $|1-\overline{\alpha_n}z| \ge 1 - |\overline{\alpha_n}|r \ge 1-r$. Hence (4.3) converges if $\sum_1^\infty (1 - |\alpha_n|) < \infty$.

Suppose now the Blaschke product converges. Then $|B(z)| < 1$ for all $z \in \Delta$. We may assume $B(0) \neq 0$, since the factor z^k does not affect the convergence of $\sum(1-|\alpha_n|)$. By Jensen's formula we have for $r < 1$ and n zeros in $|z| < r$,

$$|B(0)| \prod_1^n \frac{r}{|\alpha_n|} = \exp\left[\frac{1}{2\pi}\int_0^{2\pi} \log|B(re^{i\theta})|\, d\theta\right].$$

But $|B(z)| < 1$ for all $z \in \Delta$. Hence the right hand side of the Jensen's formula is bounded by a constant $C > 0$ for all $0 < r < 1$. Thus

$$\prod_1^\infty |\alpha_n| \ge C^{-1}|B(0)| > 0$$

as $r \to 1$. Lemma 4.6.6 implies $\sum(1-|\alpha_n|)$ must converges. □

4.7 Hadamard's Factorization Theorem

We have applied Weierstrass factorization theorem to obtain an infinite product representation of $\sin \pi z$:

$$\begin{aligned}\sin \pi z &= z e^{g(z)} \prod_{-\infty}^{\infty} \left(1 - \frac{z}{n}\right) e^{z/n} \\ &= \pi z \prod_1^\infty \left(1 - \frac{z^2}{n^2}\right).\end{aligned}$$

It is perhaps difficult to see at the beginning that $g(z)$ reduces to a constant $\log \pi$ and thus $2z \prod_1^\infty \left(1 - \dfrac{z^2}{n^2}\right)$ behaves as $e^{i\pi z} - e^{-i\pi z}$

(which is the original definition) in the above representation for $\sin \pi z$. A question comes into our minds immediately is that how will the growth of e^g and $\prod_1^\infty E_{p_n}(z/a_n)$ relate to the growth of the function f. We shall study this question when g is taken as a polynomial and $p_n = p$ for all n in this section. This line of research has dominated the development of function theory of one complex variable for the past seventy years. This area of research is related to subharmonic functions ($\log |f(z)|$ is harmonic away from the zeros of f; see next chapter for harmonic functions) and potential theory. Most easier problems have been solved, with the remaining open problems exceedingly difficult.

Let $\{a_n\}$ be a sequence of numbers in $\mathbb{C}$ such that $a_n \to \infty$ as $n \to \infty$ and that there exists a non-negative integer p such that $\sum \dfrac{1}{|a_n|^{p+1}} < \infty$. Then according to Weierstrass factorization theorem that

$$\prod_1^\infty E_p\left(\frac{z}{a_n}\right)$$

converges to an entire function on $\mathbb{C}$.

Definition 4.7.1. If the integer p described above is chosen so that $\sum 1/|a_n|^p = +\infty$ and $\sum 1/|a_n|^{p+1} < +\infty$, then the integer is called the *genus* of $\{a_n\}$, and the infinite product is said to be *canonical (standard).* We also call p the genus of the canonical product.

Example 4.7.2. The infinite product

$$\sin \pi z = \pi z \prod_{-\infty}^{\infty} \left(1 - \frac{z}{n}\right) e^{z/n} = \pi z \prod_1^\infty \left(1 - \frac{z^2}{n^2}\right)$$

has genus one, since $\sum_1^\infty \dfrac{1}{n} = +\infty$ and $\sum_1^\infty \dfrac{1}{n^2} < +\infty$.

So if a function has a representation

$$f(z) = z^m e^{g(z)} \prod_1^\infty E_p\left(\frac{z}{a_n}\right), \quad p = \text{ genus}$$

where g is a polynomial, then the growth is determined by that of $\Pi_1^\infty E_p(z/a_n)$ and e^g. Note that the above infinite product representation of f is unique if p is chosen to be the genus. We now define the *genus of* f to be $\mu = \max\{\deg g, p\}$. We next show the genus of f determines its growth.

Theorem 4.7.3. *Let f be an entire function of genus μ (refered to its Weierstrass factorization). Then given $\alpha > 0$, there exists $R > 0$ such that*

$$|f(z)| \le \exp(\alpha |z|^{\mu+1})$$

for $|z| > R$.

We first obtain a result for canonical product.

Theorem 4.7.4. *Let $P(z)$ be a canonical product with genus p. Then given $\alpha > 0$, there exists $R > 0$ such that*

$$|P(z)| < \exp(\alpha |z|^{p+1})$$

for $|z| > R$.

Proof. We need some elementary estimates for the primary factors. Since

$$|E_p(z)| \le (1 + |z|) \exp\left(|z| + \frac{|z|^2}{2} + \cdots + \frac{|z|^p}{p}\right),$$

thus

$$\log |E_p(z)| \le \log(1 + |z|) + |z| + \cdots + \frac{|z|^p}{p}.$$

Thus given any $A > 0$, there exists $R > 0$ such that

$$\log |E_p(z)| < A|z|^{p+1} \quad \text{for } |z| > R. \tag{4.4}$$

We also recall from Lemma 4.3.3(ii) that for $k > 1$,

$$\log |E_p(z)| \le |\log E_p(z)| \le \frac{k}{k-1} |z|^{p+1}, \quad \text{for } k < 1, \quad |z| < 1/k.$$

Without loss of generality, we may assume $\dfrac{1}{k} < R$. For $\dfrac{1}{k} \le |z| \le R$, the function

$$\frac{\log |E_p(z)|}{|z|^{p+1}}$$

is easily seen to be continuous there except when $z = 1$ where $\log |E_p(z)| = -\infty$. In any case an absolute upper bound exists. Thus there exists $B > 0$ such that

$$\log |E_p(z)| \le B|z|^{p+1}$$

for $\dfrac{1}{k} \le |z| \le R$. Let $M = \max\{A, B, \dfrac{k}{k-1}\}$, we have

$$\log |E_p(z)| \le M|z|^{p+1}$$

for all $z \in \mathbb{C}$.

Since $\sum_1^\infty \dfrac{1}{|a_n|^{p+1}} < \infty$, we choose N so large that

$$\sum_{N+1}^{\infty} \frac{1}{|a_n|^{p+1}} < \frac{\alpha}{4M}.$$

Thus

$$\begin{aligned} \sum_{N+1}^{\infty} \log \left| E_p\left(\frac{z}{a_n}\right) \right| &\le M|z|^{p+1} \sum_{N+1}^{\infty} \frac{1}{|a_n|^{p+1}} \\ &\le \frac{\alpha |z|^{p+1}}{4}. \end{aligned} \tag{4.5}$$

To estimate $\sum_1^N \log |E_p(z/a_n)|$, we note that $|z/a_n|$ are large for $1 \le n \le N$, hence we may assume the constant $A > 0$ in (4.4) is chosen such that $A = \dfrac{\alpha}{4N} \min_{1\le i\le N} |a_i|^{p+1}$ for $|z| > R_1 > R$, say. Thus

$$\begin{aligned} \sum_1^N \log |E_p(z/a_n)| &< \frac{\alpha}{4N} |z|^{p+1} \left(\sum_1^N \frac{1}{|a_n|^{p+1}} \right) \min_{1\le i\le N} |a_i|^{p+1} \\ &\le \frac{\alpha}{4N} |z|^{p+1} \left(N \max_{1\le i\le N} \frac{1}{|a_i|^{p+1}} \right) \min_{1\le i\le N} |a_i|^{p+1} \\ &\le \frac{\alpha}{4} |z|^{p+1} \end{aligned} \tag{4.6}$$

for $|z| > R_1 > R$. Combining (4.5) and (4.6) yields

$$\log |P(z)| = \sum_1^\infty \log |E_p(z/a_n)| < \alpha |z|^{p+1}$$

for all $|z| > R_1$. This completes the proof. $\square$

Proof of Theorem 4.7.3. It is now easy to complete the proof of Theorem 4.7.3. For $\deg g < \mu + 1$, so $|z|^m e^{|g|}/\exp(\alpha|z|^{\mu+1}) \to 0$ as $|z| \to \infty$. But $p + 1 \leq \mu + 1$. The required estimate follows from Theorem 4.7.4. $\square$

Example 4.7.5. If $\sum 1/|a_n|^2 < \infty$ and $\sum 1/|a_n| = +\infty$, then

$$f = \prod_1^\infty \left(1 - \frac{z}{a_n}\right) e^{z/a_n}$$

has genus 1. So $|f| < \exp(\alpha|z|^2)$. It also follows that $\sin \pi z$ has genus 1.

Suppose $\sum_1^\infty 1/|a_n| < \infty$. Then $\prod \left(1 - \dfrac{z}{a_n}\right)$ has genus zero. Hence

$$f = e^z \prod \left(1 - \frac{z}{a_n}\right)$$

also has genus 1.

The above theorems show that we can know the growth of f provided we know the function g and the growth (genus) of the zeros of f. We shall study the converse problem in what follows. Namely, what can we say about g and the zeros of f if we know the growth of f.

Definition 4.7.6. Let $S(r)$ be a positive and monotonic increasing function of $r > 0$. The *order* λ of $S(r)$ is defined to be

$$\limsup_{r \to +\infty} \frac{\log S(r)}{\log r}.$$

We say $S(r)$ has infinite order if no finite λ can be found.

Remark. The above definition is equivalent to: given any $\epsilon > 0$, there exists a $r_0 > 0$ such that

(i) $S(r) < r^{\lambda+\epsilon}$ for $r > r_0$, and

(ii) $S(r) > r^{\lambda-\epsilon}$ holds for infinitely many $r > r_0$.

Example 4.7.7. The order r and $r(\log r)^3$ where $s \neq 0$ are both equal to 1. The order of e^r is infinite.

Definition 4.7.8. Let f be an entire function and $M(r) = M(r, f) = \max_{|z|=r} |f(z)|$. Then the *order of f* is defined to be the real number:

$$\lambda = \limsup_{r\to\infty} \frac{\log\log M(r,\ f)}{\log r}.$$

Example 4.7.9. (i) $\lambda(e^z) = 1$.

(ii) $\lambda(e^{p(z)}) = n$, where $p(z)$ is a polynomial of degree n.

(iii) $\lambda(\exp(e^z)) = \infty$.

Definition 4.7.10. Let $n(r)$ be the *number of zeros of f in $|z| < r$* (counted according to multiplicities).

Proposition 4.7.11. *The order of $n(r)$ does not exceed that of f, i.e. $\lambda(n(r)) \leq \lambda(f)$.*

Proof. We may assume $f(0) \neq 0$. Given $\epsilon > 0$, there exists $r_0 > 0$ such that

$$\log M(r, f) < r^{\lambda(f)+\epsilon} \quad \text{for } r > r_0.$$

Putting $2r$ into Jensen's formula yields

$$\begin{aligned} 2^{n(r)}|f(0)| \leq |f(0)| \prod_1^{n(r)} \frac{2r}{|a_n|} \leq |f(0)| \prod_1^{n(2r)} \frac{2r}{|a_n|} &= \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|f(2re^{i\theta})|\,d\theta\right) \\ &\leq M(2r, f). \end{aligned}$$

Hence

$$\begin{aligned}\log|f(0)| + n(r)\log 2 \le \log M(2r, f)\\ < 2^{\lambda+\epsilon} r^{\lambda+\epsilon} \quad \text{for } r > r_0.\end{aligned}$$

Thus, proving the proposition. □

Proposition 4.7.12. *Suppose $\lambda(f) = \lambda < +\infty$ and $\{r_i\}$ are the moduli of the zeros of f. Then the series $\sum r_n^{-\alpha} < +\infty$ whenever $\alpha > \lambda$.*

Proof. Let $\lambda < \beta < \alpha$. It follows from Proposition 4.7.11 that $n(r) < Ar^\beta$ for $r > r_0$, say. Suppose $r = r_n$, then $n = n(r_n) < Ar_n^\beta$. Hence $r_n^{-\beta} < An^{-1}$, and so $r_n^{-\alpha} < An^{-\alpha/\beta}$. Thus $\sum r_n^{-\alpha} < A\sum n^{-\alpha/\beta} < +\infty$ since $\alpha/\beta > 1$. □

Definition 4.7.13. The real number

$$v = \inf\left\{\alpha : \sum_1^\infty \frac{1}{r_n^\alpha} < +\infty\right\}$$

is called the *exponent of convergence* of the sequence $\{r_n\}$.

If $\{a_n\}$ is a sequence of the zeros of $P(z)$ and $|a_n| = r_n$, then

$$p \le v < p + 1$$

where p is the genus of $\{a_n\}$. Also, it was proved in Theorem 4.7.4 that $|P(z)| < \exp(\alpha|z|^{p+1})$. We can prove a more precise result.

Theorem 4.7.14. *The order of a canonical product is equal to the exponent of convergence of its zeros.*

Proof. See exercise/homework. □

It follows that if v is the exponent of convergence of $P(z)$, then

$$|P(z)| < \exp(\alpha|z|^{v+\epsilon})$$

for all $|z|$ sufficiently large. It also follows that

$$\text{genus } p \le \text{order of a canonical product} < p + 1.$$

Note also that $p = [v]$, $p \le v \le \lambda$ (for an entire function).

Lemma 4.7.15. *Let f be an entire function of order $\lambda < +\infty$ and $f(0) = 1$. Suppose $\{a_i\}$ are the zeros of f and an integer $p > \lambda - 1$. Then*

$$\frac{d^p}{dz^p}\left(\frac{f'(z)}{f(z)}\right) = -p!\sum_{n=1}^{\infty}\frac{1}{(a_n - z)^{p+1}}$$

for $z \neq a_1, a_2, \ldots$.

The proof of this lemma will be given after the Poisson-Jensen formula.

Theorem 4.7.16 (Hadamard's Factorization Theorem)**.** *Let f be an entire function of order $\lambda < +\infty$, and suppose $\{a_i\}$ are the zeros of f where $f(0) = 1$. Then*

$$f(z) = e^{g(z)}P(z)$$

where g is a polynomial of degree $\leq \lambda$, and $P(z)$ is the canonical product form from the zeros of f.

Proof. Let p be an integer such that $p \leq \lambda < p + 1$. Since the order of f, $\lambda(f) < \infty$, Proposition 4.7.12 implies that the zeros $a_1, a_2, \ldots$ of f satisfy $\sum \dfrac{1}{|a_n|^{p+1}} < +\infty$ since $p + 1 > \lambda$. Let $P(z)$ be the canonical product forms from the zeros of f, and v be its exponent of convergence of zeros.

Weierstrass factorization theorem implies that there exists an entire function $g(z)$ such that $f(z) = e^g P(z)$.

It remains to show that g is a polynomial. But it is easy to check that

$$\frac{d^p}{dz^p}\left(\left[E_p\left(\frac{z}{a_n}\right)\right]' \Big/ E_p\left(\frac{z}{a_n}\right)\right) = -p!\frac{1}{(a_n - z)^{p+1}}.$$

Combining this and by using Lemma 4.7.15, we obtain

$$g^{(p+1)}(z) - p!\sum_{1}^{\infty}\frac{1}{(a_n - z)^{p+1}} = \frac{d^p}{dz^p}\left(\frac{f'(z)}{f(z)}\right) = -p!\sum_{1}^{\infty}\frac{1}{(a_n - z)^{p+1}}.$$

Hence g must be a polynomial of degree at most p. $\square$

4.8 Poisson-Jensen Formula

Theorem 4.8.1. *Let f be analytic on $\overline{B(0,r)}$ and let $a_1, \ldots, a_n$ be the zeros of $f(z)$ in $B(0,r)$. Suppose $f(z) \neq 0$ for $z \in B(0,r)$, then*

$$\begin{aligned}\log|f(z)| &= -\sum_{k=1}^{n} \log\left|\frac{r^2 - a_n z}{r(z-a_k)}\right| + \frac{1}{2\pi}\int_0^{2\pi} \Re\left(\frac{re^{i\theta}+z}{re^{i\theta}-z}\right)\log|f(re^{i\theta})|\,d\theta \\ &= -\sum_{k=1}^{n} \log\left|\frac{r^2 - a_n z}{r(z-a_k)}\right| + \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2-\rho^2}{r^2 - 2r\rho\cos(\phi-\theta)+\rho^2}\log|f(re^{i\theta})|\,d\theta\end{aligned}$$

where $z = \rho e^{i\theta}$.

Proof. We need to quote the quote following result from Chapter 6: Suppose g is *analytic* on $\overline{B}(0,r)$ and that $\Re(g(z) = U(z)$. Then for $z = \rho e^{i\phi}$, $\rho < r$, we have

$$\begin{aligned}U(\rho e^{i\phi}) &= \frac{1}{2\pi}\int_0^{2\pi} \Re\left(\frac{re^{i\theta+z}}{re^{i\theta}-z}\right) U(re^{i\theta})\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2-\rho^2}{r^2-2r\rho\cos(\phi-\theta)+\rho^2} U(re^{i\theta})\,d\theta.\end{aligned}$$

Let $g(z) = f(z)\prod_1^n \dfrac{r^2 - \overline{a_k}z}{r(z-a_k)}$ then it is zero-free on $B(0,r)$. Hence $\log g(z)$ is analytic on $\overline{B}(0,r)$. We thus obtain

$$\begin{aligned}\log|g(z)| &= \frac{1}{2\pi}\int_0^{2\pi} \Re\left(\frac{re^{i\theta+z}}{re^{i\theta}-z}\right)\log|g(re^{i\theta})|\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \Re\left(\frac{re^{i\theta+z}}{re^{i\theta}-z}\right)\log|f(re^{i\theta})|\,d\theta.\end{aligned}$$

But

$$\log|g(z)| = \log|f(z)| + \sum_{k=1}^{n}\log\left|\frac{r^2-\overline{a_k}z}{r(z-a_k)}\right|$$

and the required formula now follows. □

Remark. The Poisson-Jensen formula recovers the Jensen's formula after putting $z = 0$.

Now we can give a proof of Lemma 4.7.15.

Proof of Lemma 4.7.15. The easiest way to obtain $(f'/f)^{(p)}$ is to differentiate the Poisson-Jensen formula. Suppose $f(z) \neq 0$, then $\log f(z)$ exists and Cauchy-Riemann equations gives $f'/f = \dfrac{d}{dz}\log f(z) = \dfrac{\partial}{\partial x}\Re(\log f(z)) - i\dfrac{\partial}{\partial y}\Re(\log f(z)) = \dfrac{\partial}{\partial x}\log|f(z)| - i\dfrac{\partial}{\partial y}\log|f(z)|$. We apply this formula and differentiation under the integral (see Conway p.69),

$$\log|f(z)| = -\sum_1^n \log\left|\frac{r^2 - \overline{a_k}z}{r(z-a_k)}\right| + \frac{1}{2\pi}\int_0^{2\pi} \Re\left(\frac{re^{i\theta}+z}{re^{i\theta}-z}\right)\log|f(re^{i\theta})|\,d\theta.$$

We obtain

$$\frac{f'(z)}{f(z)} = \sum_1^n \frac{1}{z-a_k} + \sum_1^n \frac{\overline{a_k}}{r^2 - \overline{a_k}z} + \frac{1}{2\pi}\int_0^{2\pi}\frac{2re^{i\theta}}{(re^{i\theta}-z)^2}\log|f(re^{i\theta})|\,d\theta.$$

Differentiating this formula p time yields:

$$\begin{aligned}\frac{d^p}{dz^p}\left(\frac{f'(z)}{f(z)}\right) &= -p!\sum_1^n \frac{1}{(a_k-z)^{p+1}} + p!\sum_1^n \frac{\overline{a_k}^{p+1}}{(r^2-\overline{a_k}z)^{p+1}} \\ &\quad + (p+1)!\frac{1}{2\pi}\int_0^{2\pi}\frac{2re^{i\theta}}{(re^{i\theta}-z)^{p+2}}\log|f(re^{i\theta})|\,d\theta.\end{aligned}$$

It remains to show the last two terms tend to zero as $r \to \infty$ ($n \to \infty$). We consider the integral of the last expression first, and note that the integral

$$\int_0^{2\pi}\frac{re^{i\theta}}{(re^{i\theta}-z)^{p+2}}\,d\theta = 0.$$

Hence

$$-\int_0^{2\pi}\frac{re^{i\theta}}{(re^{i\theta}-z)^{p+2}}\log|f(re^{i\theta})|\,d\theta = \int_0^{2\pi}\frac{re^{i\theta}}{(re^{i\theta}-z)^{p+2}}\log\frac{M(r,f)}{|f(re^{i\theta})|}\,d\theta.$$

Suppose $r > 2|z|$, then

$$\begin{aligned}
\left|\int_0^{2\pi} \frac{re^{i\theta}}{(re^{i\theta}-z)^{p+2}} \log|f(re^{i\theta})|\,d\theta\right| &\le \frac{2r}{(r-r/2)^{p+2}} \int_0^{2\pi} \log\frac{M(r,f)}{|f(re^{i\theta})|}\,d\theta \\
&= 2^{p+3} r^{-p-1} \int_0^{2\pi} \log\frac{M(r,f)}{|f(re^{i\theta})|}\,d\theta \\
&= 2^{p+3} r^{-p-1} \int_0^{2\pi} (\log M(r,f) - \log|f(re^{i\theta})|)\,d\theta \\
&\le 2^{p+2} \int_0^{2\pi} \frac{\log M(r,f)}{r^{p+1}}\,d\theta \\
&\quad \left(\text{by } \int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta \ge 0\right)
\end{aligned}$$

by Jensen's formula. But $\log M(r,f)/r^{p+1} \to 0$ as $r \to \infty$ since $\lambda < p+1$ and this proves the integral tends to zero as $r \to \infty$.

We now consider an individual term in the second summand: we assume again $r > 2|z|$,

$$\left|\frac{\overline{a_k}}{r^2 - \overline{a_k} z}\right|^{p+1} \le \frac{|a_k|^{p+1}}{(r^2 - r^2/2)^{p+1}} = \frac{(2r)^{p+1}}{r^{2(p+1)}} = \left(\frac{2}{r}\right)^{p+1}.$$

Hence

$$\sum_1^n \left|\frac{\overline{a_k}}{r^2 - \overline{a_k} z}\right|^{p+1} \le 2^{p+1} \sum_1^n \frac{1}{r^{p+1}} \le 2^{p+1} \frac{n(r)}{r^{p+1}} \to 0$$

as $r \to \infty$, $n(r) \ge n(r_n)$ by Proposition 4.7.11. Thus

$$\frac{d^p}{dz^p}\left(\frac{f'(z)}{f(z)}\right) \to -p! \sum_1^\infty \frac{1}{(z-a_k)^{p+1}}$$

as $r \to \infty$ and this completes the proof. □

We can rewrite the above lemma as $p \le \lambda < p+1$. We also note the following:

Theorem 4.8.2. *Let f be an entire function of finite order, then f assumes each complex number with at most one exception.*

Proof. Suppose $f(z) \neq \alpha, \beta$ for all $z \in \mathbb{C}$. Since $f - \alpha$ has the same order of growth of f and never vanish. By Theorem 4.7.16, there exists a polynomial g such that $f - \alpha = \exp(g)$. Thus $\exp(g)$ never assume $\beta - \alpha$ and so $g(z)$ never assume $\log(\beta - \alpha)$, a contradiction to the Fundamental theorem of algebra. $\square$

Theorem 4.8.3. *Suppose the order of an entire function is finite and not equal to an integer. Then the function must have an infinite number of zeros.*

Theorem 4.8.4. *Let α be a real number. Then the function*

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n!)^\alpha}$$

has order $1/\alpha$.

Proof. Suppose z is real and positive. By considering

$$\frac{z}{1^\alpha} \cdot \frac{z}{2^\alpha} \cdot \cdots \cdot \frac{z}{(n-1)^\alpha} \cdot \frac{z}{n^\alpha},$$

we clearly deduce $z^n/(n!)^\alpha$ is increasing when $|z| > n^\alpha$ and it starts to decrease when $|z| < n^\alpha$. Hence the maximum of $z^n/(n!)^\alpha$ occurs when $z = n^\alpha$. Thus

$$\begin{aligned} \frac{z^n}{(n!)^\alpha} = \frac{n^{\alpha n}}{(n!)^\alpha} &= \frac{n^{n\alpha}}{(n^{n+(1/2)}e^{-n}\sqrt{2\pi}(1+o(1)))^\alpha} = \frac{e^{n\alpha}}{n^{\alpha/2}(2\pi)^{\alpha/2}(1+o(1))} \\ &= \frac{e^\alpha z^{1/\alpha}}{z^{1/2}(2\pi)^{\alpha/2}(1+o(1))} \end{aligned}$$

by Stirling formula (Titchmarsh, p.58).

But the order of growth of $f(z)$ must be greater than its individual term when $z > 0$. Hence $\lambda \geq 1/\alpha$.

On the other hand, $|f(z)| \leq f(|z|)$ when z is real,

$$\begin{aligned} f(z) &= \sum_{n=0}^{N} \frac{z^n}{(n!)^\alpha} + \sum_{N+1}^{\infty} \frac{z^n}{(n!)^\alpha} \\ &< \sum_{n=0}^{N} \frac{z^n}{(n!)^\alpha} + \sum_{N+1}^{\infty} \frac{z^n}{[(N+1)!N^{n-N-1}]^\alpha}. \end{aligned}$$

Suppose $N^\alpha > z$, then

$$\begin{aligned}
\sum_{N+1}^{\infty} \frac{z^n}{[(N+1)!N^{n-N-1}]^\alpha} &= \frac{(N^{N+1})^\alpha}{[(N+1)!]^\alpha} \sum_{N+1}^{\infty} \frac{z^n}{(N^n)^\alpha} \\
&= \frac{(N^{N+1})^\alpha}{[(N+1)!]^\alpha} \left(\frac{z^{N+1}}{(N^{N+1})^\alpha} + \frac{z^{N+2}}{(N^{N+2})^\alpha} + \cdots \right) \\
&= \frac{z^{N+1}}{[(N+1)!]^\alpha} \left(1 + \frac{z}{N^\alpha} + \cdots \right) \\
&= \frac{z^{N+1}}{[(N+1)!]^\alpha (1 - (z/N^\alpha))}.
\end{aligned}$$

Thus we have

$$f(z) < Az^N + \frac{z^{N+1}}{[(N+1)!]^\alpha (1 - (z/N^\alpha))}$$

whenever $N^\alpha > z$. Hence, by taking $N = [(2z)^{1/\alpha}]$,

$$f(z) = O(z^N) = O(z^{(2z)^{1/\alpha}}) = O(\exp(z^{(1/\alpha)+\epsilon})),$$

and we deduce the order of f does not exceed $1/\alpha$. Hence $\lambda = 1/\alpha$. $\square$

Remark. *Stirling formula:*

$$\Gamma(z) = z^{z-(1/2)} e^{-z} \sqrt{2\pi}(1 + o(1))$$

where $\Gamma(n+1) = n!$.

Exercise. What is the order of $\sum_{n=1}^{\infty} \dfrac{z^n}{n^{\alpha n}}$ for $\alpha > 0$?

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