Chapter 5

Periodic functions

An (analytic) function $f(z)$ is said to be **periodic** if there is a non-zero constant ω such that

$$f(z+\omega) = f(z), \quad z \in \mathbb{C}.$$

We call the number ω a **period** of the function $f(z)$.

Definition 5.0.5. We call ω a **fundamental (primitive) period** of f if $|\omega|$ is the smallest amongst all periods.

5.1 Simply periodic functions

The simplest periodic function of period ω is $e^{2\pi iz/\omega}$. Suppose Ω is a region such that if $z \in \Omega$ then $z + k\omega \in \Omega$ for all $k \in \mathbb{Z}$.

Theorem 5.1.1. *Given a meromorphic function f defined on a region Ω (as discussed above). Then there exists a unique meromorphic function F in Ω' which is the image of Ω under $e^{2\pi iz/\omega}$, such that*

$$f(z) = F\Big(e^{2\pi iz/\omega}\Big).$$

Proof. Suppose f is meromorphic in Ω in the $z-$plane with period ω. Let $\zeta = e^{2\pi iz/\omega}$. Then we define F by

$$f(z) = f(\log\zeta) = F(\zeta).$$

Then clearly F is meromorphic in the $\zeta-$plane whenever $f(z)$ is meromorphic in the $z-$plane. □

Example 5.1.2. Let $0 < |q| < 1$. Consider the function

$$f(z) = \sum_{k=-\infty}^{\infty} (-1)^k q^{k^2/2} e^{kiz}$$

which represents a $2\pi-$periodic entire function in $\mathbb{C}$. In fact, this is a complex form of a Fourier series. Let $\zeta = e^{iz}$.

$$F(\zeta) = \sum_{k=-\infty}^{\infty} (-1)^k q^{k^2/2} \zeta^k$$

which can be shown to converge in the punctured plane $0 < |\zeta| < +\infty$. Thus we have

$$f(z) = F(\zeta)$$

as asserted by the last theorem. Here we have $\omega = 2\pi$. Thus, the function F is analytic in $0 < |\zeta| < +\infty$.

More generally, if the series

$$F(\zeta) = \sum_{k=-\infty}^{\infty} c_n \zeta^k$$

converges in an annulus $r_1 < |\zeta| < r_2$, then

$$f(z) := F(\zeta) = \sum_{k=-\infty}^{\infty} c_n e^{2\pi kiz/\omega}, \quad \zeta = e^{2\pi iz/\omega},$$

is a $\omega-$periodic analytic function in the infinite horizontal strip $\{\zeta : e^{r_1} < \Im(\zeta) < e^{r_2}\}$. We can represent the coefficient

$$\begin{aligned} c_k &= \frac{1}{2\pi i}\int_{|\zeta|=r} \frac{F(\zeta)}{\zeta^{k+1}}\, d\zeta, \quad r_1 < |\zeta| < r_2 \\ &= \frac{1}{\omega}\int_a^{a+\omega} f(z) e^{-2\pi kiz/\omega}\, dz, \end{aligned}$$

where a is an arbitrary in the infinite strip $\{\zeta : e^{r_1} < \Im(\zeta) < e^{r_2}\}$ and the integration is taken along any path lying in the strip.

5.2 Period module

Let M denote that set of all periods of a meromorphic function f in $\mathbb{C}$. If $\omega \neq 0$ is a period, then $n\omega$, for any integer n, obviously belongs to M. If, however, there are two distinct periods ω_1 and ω_2, then $m\omega_1 + n\omega_2$ is also a period for any integral multiples m, n. That is, $m\omega_1 + n\omega_2 \in M$. This shows that M is a **module** over the ring of integers.

We also note that the set of periods must be discrete since if there is a sequence of periods with a limit point, then this would contradicts the identity theorem for analytic functions. We are ready to answer Jacobi's first question.

Theorem 5.2.1. *A discrete module M consists of either $n\omega$ for an arbitrary integer n and $\omega \neq 0$, or $m\omega_1 + n\omega_2$ for arbitrary integers n, m and non-zero ω_1, ω_2 with $\Im(\omega_2/\omega_1) \neq 0$.*

Proof. Without loss of generality, we may assume $M \neq \emptyset$. Let $\omega = \omega_1 \in M$ and there are at most a finite number of $n\omega_1$ belong to M in a fixed $|z| \leq r$. Amongst all these ω_1, we choose the one with the smallest $|\omega_1|$.

If however, there is a period $\omega \in M$ that is not of the form $n\omega_1$ for $n \in \mathbb{Z}$. Then again we call ω_2 with $|\omega_2|$ the smallest (but not less than $|\omega_1|$). We claim that $\Im(\omega_2/\omega_1) \neq 0$. For if it were, then there is an integer n such that

$$n < \frac{\omega_2}{\omega_1} < n + 1,$$

or

$$0 < \left|\frac{\omega_2}{\omega_1} - n\right| < 1$$

or $|n\omega_1 - \omega_2| < |\omega_1|$. But $n\omega_1 - \omega_2$ is a period which is smaller than $|\omega_1|$. This contradicts the assumption that ω_1 is the "smallest" period.

It remains to show that any period ω must be of the form $m\omega_1 + n\omega_2$ for some integers n, m. Without loss of generality, we may assume $\Im(\omega_2/\omega_1) > 0$. Hence any complex number ω can be written as $\omega =$

$\lambda_1\omega_1+\lambda_2\omega_2$ for constants λ_1, λ_2. We claim that λ_1, λ_2 are real. Suppose

$$\omega = \lambda_1\omega_1 + \lambda_2\omega_2,$$
$$\bar{\omega} = \lambda_1\bar{\omega}_1 + \lambda_2\bar{\omega}_2.$$

Then one can can find unique solutions λ_1, λ_2 since $\omega_1\bar{\omega}_2 - \omega_2\bar{\omega}_1 \neq 0$. But then $\bar{\lambda}_1$, $\bar{\lambda}_2$ are also solutions. So they are real.

Clearly we find integers m_1 and m_2 such that

$$|\lambda_1 - m_1| \leq \frac{1}{2}, \quad |\lambda_2 - m_2| \leq \frac{1}{2}.$$

If $\omega \in M$, then so does

$$\omega' = \omega - m_1\omega_1 - m_2\omega_2.$$

But then

$$\begin{aligned} |\omega'| &< |\lambda_1 - m_1||\omega_1| + |\lambda_2 - m_2||\omega_2| \\ &\leq \frac{1}{2}|\omega_1| + \frac{1}{2}|\omega_2| \\ &\leq |\omega_2| \end{aligned}$$

where the first inequality is strict since ω_2 is not a real multiple of ω_1. That is, the $|\omega'| < |\omega_2|$ while $\omega' \in M$. We conclude that ω' is an integral multiple of ω_1. This gives ω the desired form. □

5.3 Unimodular transformations

We consider the case that M is generated by two distinct ω_1 and ω_2 such that $\Im(\omega_2/\omega_1) > 0$. We recall that M consists of discrete points $n\omega_1 + m\omega_2$ where m, , n are integers. Suppose ω_1' and ω_2' is another pair of distinct points that also generate M. Then we must havw

$$\omega_1' = a\omega_2 + b\omega_1,$$
$$\omega_2' = c\omega_2 + d\omega_1$$

for some integers a, b, c, d. We can rewrite this in a matrix form:

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}. \tag{5.1}$$

There is a similar matrix equation for complex conjugates:

$$\begin{pmatrix} \bar{\omega}_1' \\ \bar{\omega}_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \bar{\omega}_1 \\ \bar{\omega}_2 \end{pmatrix}. \tag{5.2}$$

We can combine the above two matrix equations into one:

$$\begin{pmatrix} \omega_1' & \bar{\omega}_1' \\ \omega_2' & \bar{\omega}_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 & \bar{\omega}_1 \\ \omega_2 & \bar{\omega}_2 \end{pmatrix}. \tag{5.3}$$

Similarly, we can find integers a', b', c', d'

$$\begin{pmatrix} \omega_1 & \bar{\omega}_1 \\ \omega_2 & \bar{\omega}_2 \end{pmatrix}. = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} \omega_1' & \bar{\omega}_1' \\ \omega_2' & \bar{\omega}_2' \end{pmatrix} \tag{5.4}$$

The determinant $\omega_1\bar{\omega}_2 - \omega_2\bar{\omega}_1 \neq 0$ since ω_2/ω_1 would have real ratio. Hence

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence the determinants equal

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a' & b' \\ c' & d' \end{vmatrix} = \pm 1$$

Definition 5.3.1. The set of all such 2×2 linear transformations with determinant ± 1 is called **unimodular**. When we restrict to determinant begin 1, it is also recognised as a subgroup of the **projective special linear group** $PSL(2, \mathbb{C})$ which we label as $\Gamma = PSL(2, \mathbb{Z})$ or just **modular group**.

It turns out that the modular group has generators

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We label the lattice generated by ω_1, ω_2 by $\Omega(\omega_1, \omega_2)$. Thus if ω_1, ω_2 by $\Omega(\omega_1', \omega_2')$ is another lattice, then two latices are connected by a unimodular transformation.

We can make the basis ω_1, ω_2 by a suitable restriction.

Theorem 5.3.2. *Let* $\tau = \omega_2/\omega_1$. *If*

1. $\Im(\omega_2/\omega_1) > 0$,
2. $-\frac{1}{2} < \Re(\tau) \leq \frac{1}{2}$,
3. $|\tau| \geq 1$,
4. $\Re(\tau) \geq 0$ *when* $|\tau| = 1$,

then the τ *is uniquely determined.*

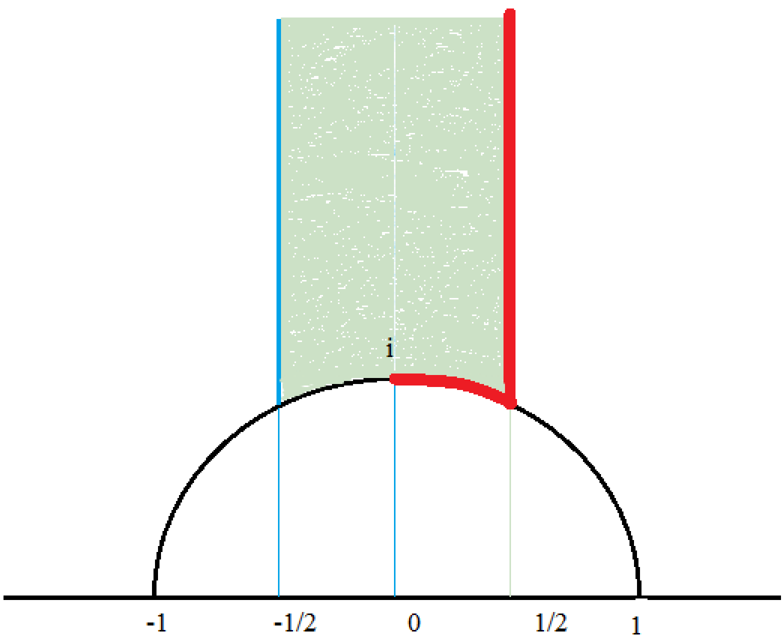

Figure 5.1: Fundamental region of modular function

It is clear that the region defined by the criteria (1-4) in the theorem is not an open. But it still call it a **fundamental region**. If it happens that $\Im(\omega_2/\omega_1) < 0$, then we could replace (ω_1, ω_2) by $(-\omega_1, \omega_2)$

without changing the assumption $-\frac{1}{2} < \Re(\tau) \le \frac{1}{2}$. The assumption (2) is also arbitrary in the sense if $-\frac{1}{2} \le \Re(\tau) < \frac{1}{2}$, then we could use $(\omega_1, \omega_1 + \omega_2)$. Finally, if the last assumption (4) is replaced by $\Re(\tau) < 0$ when $|\tau| = 1$, then we consider $(-\omega_2, \omega_1)$ instead.

Proof. Let τ' be

$$\tau' = \frac{a\tau + b}{c\tau + d},$$

where a, b, c, d are integers and such that $ad - bc = \pm 1$. We recall that the above Möbius transformation that maps the upper half $\tau-$plane onto itself if the determinant is $+1$ and onto the lower half $\tau-$plane if he determinant is -1. A simple calculation gives

$$\Im(\tau') = \frac{\pm \Im(\tau)}{|c\tau + d|^2} \tag{5.5}$$

where the $\pm$ accords to that of $ad - bc$.

Suppose both the τ', τ situate inside the fundamental region. We want to show that $\tau' = \tau$. Without loss of generality, we may assume that $ad - bc = 1$, and $\Im(\tau') \ge \Im(\tau)$. This means that

$$|c\tau + d| \le 1.$$

Since c, d are integers, so there are not too many cases to check.

If $c = 0$, then $d = \pm 1$. The condition $ad - bc = 1$ implies $ad = 1$. So either $a = d = 1$ or $a = d = -1$, so that the equation (5.5) becomes

$$\tau' = \tau \pm b.$$

But both τ', τ satisfy the assumption (2) which implies that

$$|b| = |\Re(\tau') - \Re(\tau)| < 1.$$

Thus $b = 0$ and $\tau' = \tau$.

Suppose now that $c \ne 0$. We have

$$|\tau + d/c| \le 1/|c|.$$

We claim that $|c| = 1$. For suppose $|c| \geq 2$, then $|\tau + d/c| \leq \frac{1}{2}$ meaning that τ is closer to the d/c (real axis) than $1/2$. This contradicts the assumption (3) that $|\tau| \geq 1$. Thus $c = \pm 1$ and

$$|\tau \pm d| \leq 1.$$

But since τ situates in the fundamental region, so either $d = 0$ or $d = \pm 1$. In the latter, the $|\tau + 1| \leq 1$ has no solution there (the only point being $e^{2\pi i/3}$ is outside the fundamental region). The other inequality $|\tau - 1| \leq 1$ has the only one solution $e^{i\pi/3}$ and it becomes an equality and $|c\tau + d| = 1$. We deduce from (5.5) that $\Im(\tau') = \Im(\tau)$ and hence $\tau' = \tau$. Suppose $d = 0$ and $|c| = 1$. So $|\tau| \leq 1$. This together with the assumption (3) $|\tau| \geq 1$ imply that $|\tau| = 1$. Hence

$$\tau' = \frac{a\tau + b}{c\tau} = \frac{a}{c} + \frac{b}{c\tau} = \frac{a}{c} + \frac{-1}{\tau}$$

since $bc = -1$. Hence

$$\tau' = \pm a - \frac{1}{\tau} = \pm a - \bar{\tau}.$$

But then $\Re(\tau') = \pm a - \Re(\bar{\tau}) = \pm a - \Re(\tau)$ so that

$$\Re(\tau' + \tau) = \pm a$$

which is an integer. This is possible only if $a = 0$. Thus $\tau' = -1/\tau$ and the only solution for this equation in the fundamental region is when $\tau' = \tau = i$ (since $|\tau| = 1$). □

5.4 Doubly periodic functions

Definition 5.4.1. Let ω_1 and ω_2 be two distinct non-zero complex numbers such that $\Im\omega_1/\omega_2 > 0$. An **elliptic function** f is a meromorphic function on $\mathbb{C}$ such that

$$f(z + \omega_1) = f(z), \quad f(z + \omega_2) = f(z)$$

for any two distinct periods ω_1 and ω_2.

That is, $f(z+\omega) = f(z)$ whenever $\omega = n\omega_1 + m\omega_2$ for any integers n, m. Thus we may shift the vertex $z = 0$ of the parallelogram to any point a and the above statement still hold. We denote such parallelogram by P_a with vertices a, $a+\omega_1$, $a+\omega_2$, $a+\omega_1+\omega_2$.

We note that the $\tau = \omega_2/\omega_1$ when restricted to the fundamental region described in the Theorem 5.3.2 is unique.

Theorem 5.4.2. *An elliptic function without poles must be a constant.*

Proof. Being without poles, so an elliptic function f is bounded on the period spanned by $\{0, \omega_1, \omega_2, \ \omega_1+\omega_2\}$ which is a compact set. Hence f is a bounded entire function. Thus f is constant by Liouville's theorem. □

Theorem 5.4.3. *The sum of the residues of an elliptic function is zero.*

Proof. Without loss of generality, we may choose a so that none of the poles falls on the boundary of P_a. Hence

$$\sum \operatorname{Res} f(\text{poles}) = \frac{1}{2\pi i}\int_{P_a} f(z)\, dz = 0$$

since the integral along the opposite sides of the parallelogram have equal magnitudes but with opposite signs. □

Definition 5.4.4. The sum of orders of the poles of an elliptic function in its period parallelogram is called **the order of the function**.

We deduce that the order of an elliptic function in a period parallelogram is at least two. That is, an elliptic function cannot have a single simple pole in a period parallelogram.

Theorem 5.4.5. *A non-constant elliptic function has an equal number of poles and zeros in its period parallelogram.*

Proof. Note that the quotient $f'(z)/f(z)$ is an elliptic of function of the same periods as f. But then the last theorem asserts that

$$\begin{aligned} 0 &= \frac{1}{2\pi i}\int_{P_a} \frac{f'(z)}{f(z)}\,dz \\ &= (\text{no. of zeros}) - (\text{no. of poles}) \end{aligned}$$

in the period parallelogram P_a. □

By considering the function

$$\frac{f(z)}{f(z)-a},$$

we deduce immediately that

Theorem 5.4.6. *An elliptic function of order $m \geq 2$ assumes every value m times in the period parallelogram (counted according to multiplicities).*

Theorem 5.4.7. *Let $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ be the number of zeros and poles of an elliptic function f in a period parallelogram respectively. Then*

$$\sum_{k=1}^{n} (a_k - b_k) = n\omega_1 + m\omega_2$$

for some integers n, m.

We sometime write the above conclusion in the abbreviated form $\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} b_k \pmod{M}$.

Proof. By choosing a suitable a we assume that there is no zeros and poles of f that lie on the boundary of the period parallelogram P_a. It follows from the residue theorem that

$$\frac{1}{2\pi i}\int_{\partial P_a} \frac{zf'(z)}{f(z)}\,dz = \sum_{k=1}^{n} (a_k - b_k).$$

However, the integral has another interpretation: consider the integrations of the integrand from a to $a+\omega_1$ and then from $a+\omega_2$ to $a+\omega_1+\omega_2$. So

$$\begin{aligned}&\frac{1}{2\pi i}\Big(\int_a^{a+\omega_1}-\int_{a+\omega_2}^{a+\omega_1+\omega_2}\Big)\frac{zf'(z)}{f(z)}\,dz\\&\quad=\frac{1}{2\pi i}\int_a^{a+\omega_1}\frac{zf'(z)}{f(z)}\,dz-\frac{1}{2\pi i}\int_a^{a+\omega_1}\frac{(\zeta+\omega_2)f'}{f}\,d\zeta\\&\quad=-\frac{\omega_2}{2\pi i}\int_a^{a+\omega_1}\frac{f'(z)}{f(z)}\,dz\end{aligned}$$

where the last integral is the winding number of f along the path from a to $a+\omega_1$. Hence the integral is an integral multiple of ω_2. Similar calculation over the second and the fourth sides gives an integral multiple of ω_1. This completes the proof. □

5.5 Weierstrass elliptic functions

We start to construct doubly periodic functions. Since there is no non-constant doubly periodic function with a single pole. Otherwise, such an elliptic function would contradict the sum of residues in a period parallelogram is zero. Thus a simplest elliptic function f has a double pole or at least two simple poles with opposite residues in a period parallelogram. Without loss of generality, we may assume in the former that this double pole locates at the origin $z=0$ (so that the function has zero residue at $z=0$). Moreover, we see that the function

$$f(z)-f(-z)$$

has no pole in a period parallelogram. So it must be a constant. But since $f(\omega_1/2)-f(-\omega_1/2)=0$ so that $f(-z)=f(z)$ implying that f must be an even function. We denote such an elliptic function by $\wp(z)$. Hence we have the following expansion

$$\wp(z)=\frac{1}{z^2}+a_1z^2+a_2z^4+\cdots$$

around the origin. To actually construct such an elliptic function, we could resort to **Mittag-Leffler** theorem. But in this special case we could construct such functions directly. That is, we have

Theorem 5.5.1. *Let* ω_1, ω_2 *be such that* $\Im(\omega_2/\omega_1) \neq 0$. *Then the function*

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega\neq 0}\Big(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\Big) \tag{5.6}$$

where $\omega = n\omega_1 + n\omega_2$ *for all integers* n, m *with* $(n, m) \neq (0, 0)$, *is an elliptic function with fundamental periods* ω_1, ω_2.

Proof. We first show that the infinite sum

$$\sum_{\omega\neq 0}\Big(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\Big)$$

does converges away from the poles. So let $|\omega| > 2|z|$. Then

$$\Big|\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\Big| = \Big|\frac{z(2\omega - z)}{\omega^2(z-\omega)^2}\Big| \leq \frac{10|z|}{|\omega|^3}.$$

It remains to consider the sum

$$\sum_{\omega\neq 0}\frac{1}{|\omega|^3} = \sum_{(n,\, m)\neq(0,\, 0)}\frac{1}{|n\omega_1 + m\omega_2|^3} \tag{5.7}$$

converges.

We let S_1 to denote the part of the infinite sum that runs through the points

$$\pm\omega_1, \quad \pm(\omega_1+\omega_2), \quad \pm\omega_2, \quad \pm(\omega_1-\omega_2)$$

over the lattice that are closest to the origin $(0, 0)$. There are exactly eight points. Let D and d be the longest and shortest distances of the eight points to the origin $(0, 0)$. Then we have

$$\frac{8}{D^3} \leq S_1 \leq \frac{8}{d^3}.$$

The sum S_2 over the second layer has $2 \times 8 = 16$ lattice points. But then

$$\frac{16}{(2D)^3} \le S_2 \le \frac{16}{(2d)^3}.$$

Similarly, the sum S_3 is over $3 \times 8 = 24$ lattice points. Hence

$$\frac{24}{(3D)^3} \le S_3 \le \frac{24}{(3d)^3}.$$

For S_n, we have $8n$ lattice points so that

$$\frac{8}{D^3 n^2} = \frac{8n}{(nD)^3} \le S_n \le \frac{8n}{(nd)^3} = \frac{8}{d^3 n^2}.$$

The above analysis is sufficient to guarantee that the $\wp$ converges uniformly in any compact subset of $\mathbb{C}$ with the lattice points ω and 0 removed.

Then

$$\wp'(z) = -\frac{2}{z^3} - \sum_{\omega \neq 0} \frac{2}{(z-\omega)^3} = -2\sum_{\omega} \frac{1}{(z-\omega)^3}.$$

This shows that the $\wp'$ is doubly periodic. We deduce that

$$\wp(z+\omega_1) - \wp(z), \quad \wp(z+\omega_2) - \wp(z)$$

are both constants. We further note that the $\wp(z)$ as defined above is an even function. Substitute $z = -\omega_1$ and $z = -\omega_2$ into the above formulae shows that the two constants can only be zero. We deduce that $\wp(z)$ is doubly periodic. □

5.6 Weierstrass's Sigma and Zeta functions

Since the sum (5.7)

$$\sum_{\omega \neq 0} \frac{1}{|\omega|^3}$$

converges, so we can form a Hadamard product

$$\sigma(z) = \sigma(z\,|\omega_1,\,\omega_2) := z\prod_{\omega\neq 0}\Big(1-\frac{z}{\omega}\Big)\exp\Big(\frac{z}{\omega}+\frac{z^2}{2\omega^2}\Big). \tag{5.8}$$

Thus the infinite product converges uniformly in any compact subset of $\mathbb{C}$, so it represents an entire function (of order 2). It is not an elliptic function, for it would reduce to a constant otherwise. The function is called **Weiestrass's Sigma function**.

We further note that

$$\begin{aligned}\sigma(z) = z\prod_{m,\,n>0}&\Big(1-\frac{z}{\omega}\Big)\exp\Big(\frac{z}{\omega}+\frac{z^2}{2\omega^2}\Big)\\ \times\prod_{m,\,n>0}&\Big(1+\frac{z}{\omega}\Big)\exp\Big(-\frac{z}{\omega}+\frac{z^2}{2\omega^2}\Big)\end{aligned}$$

Hence

$$\sigma(-z) = -\sigma(z)$$

showing that the Sigma function $\sigma(z)$ is an odd function.

We now take logarithmic derivative on both sides of the Sigma function. This gives

$$\frac{\sigma'(z)}{\sigma(z)} = \frac{1}{z}+\sum_{\omega\neq 0}\Big(\frac{1}{z-\omega}+\frac{1}{\omega}+\frac{z}{\omega^2}\Big)$$

We define the **Weiestrass's Zeta function** to be

$$\zeta(z) = \frac{d}{dz}\log\sigma(z).$$

Notice that

$$\zeta(z) = \frac{1}{z}+\sum_{\omega\neq 0}\Big(\frac{1}{z+\omega}-\frac{1}{\omega}+\frac{z}{\omega^2}\Big),$$