hence

$$\begin{aligned}\zeta(-z) &= -\frac{1}{z} + \sum_{\omega\neq 0}\Big(\frac{1}{-z+\omega} - \frac{1}{\omega} + \frac{z}{\omega^2}\Big)\\ &= \frac{1}{z} + \sum_{\omega\neq 0}\Big(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2}\Big)\\ &= -\zeta(z).\end{aligned}$$

So $\zeta(z)$ is an odd function. Although the Zeta function is meromorphic, it is not an elliptic function. For it has a residue 1 at the only pole in each period parallelogram.

We now connect the Weierstrass's Sigma function and the elliptic function $\wp(z)$. It should be self-evident that

$$\wp(z) = -\frac{d}{dz}\zeta(z) = \frac{1}{z^2} + \sum_{\omega\neq 0}\Big(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\Big).$$

Pseudo-periodicity of Zeta function

Since $-\zeta'(z) = \wp(z) = \wp(z+\omega_1) = -\zeta'(z+\omega_1)$. So

$$\zeta(z+\omega_1) = \zeta(z) + 2\eta_1, \tag{5.9}$$

for a suitable η_1. Let $z = -\omega_1/2$ in the above relation. We deduce

$$2\eta_1 = \zeta(\omega_1/2) - \zeta(-\omega_1/2) = 2\zeta(\omega_1/2)$$

because $\zeta(z)$ is odd. Hence $\eta_1 = \zeta(\omega_1/2)$. Similarly, if

$$\zeta(z+\omega_2) = \zeta(z) + 2\eta_2, \tag{5.10}$$

then $\eta_2 = \zeta(\omega_2/2)$. We also observe that $(\eta_1, \eta_2) \neq (0, 0)$ for if it were, then $\zeta(z)$ being doubly periodic would be an elliptic function, contradicting to our earlier conclusion.

The above relations (5.9) and (5.10) are called **pseudo-periodicity** of ζ.

Theorem 5.6.1. *Let* η_1, η_2 *be defined by* $\eta_j = \zeta(\omega_j)$ $(j = 1, 2)$. *Then*

$$\eta_1\omega_2 - \eta_2\omega_1 = \pi i.$$

Proof. We consider a contour following the parallelogram defined by

$$P := \big[-\frac{\omega_1}{2} - \frac{\omega_2}{2},\ \frac{\omega_1}{2} - \frac{\omega_2}{2},\ \frac{\omega_1}{2} + \frac{\omega_2}{2},\ -\frac{\omega_1}{2} + \frac{\omega_2}{2},\ -\frac{\omega_1}{2} - \frac{\omega_2}{2}\big]$$

Because the $\zeta(z)$ has a residue 1 at the only simple pole $z = 0$ inside the contour P, so Residue's theorem implies

$$\begin{aligned}
2\pi i &= \int_P \zeta(z)\, dz \\
&= \int_{[-\frac{\omega_1}{2}-\frac{\omega_2}{2},\ \frac{\omega_1}{2}-\frac{\omega_2}{2}]} \zeta(z)\, dz + \int_{[\frac{\omega_1}{2}-\frac{\omega_2}{2},\ \frac{\omega_1}{2}+\frac{\omega_2}{2}]} \zeta(z)\, dz \\
&\quad + \int_{[\frac{\omega_1}{2}+\frac{\omega_2}{2},\ -\frac{\omega_1}{2}+\frac{\omega_2}{2}]} \zeta(z)\, dz + \int_{[-\frac{\omega_1}{2}+\frac{\omega_2}{2},\ -\frac{\omega_1}{2}-\frac{\omega_2}{2}]} \zeta(z)\, dz \\
&= \int_{[-\frac{\omega_1}{2}-\frac{\omega_2}{2},\ \frac{\omega_1}{2}-\frac{\omega_2}{2}]} \zeta(z)\, dz + \int_{[\frac{\omega_1}{2}-\frac{\omega_2}{2},\ \frac{\omega_1}{2}+\frac{\omega_2}{2}]} \zeta(z)\, dz \\
&\quad - \int_{[-\frac{\omega_1}{2}+\frac{\omega_2}{2},\ \frac{\omega_1}{2}+\frac{\omega_2}{2}]} \zeta(z)\, dz - \int_{[-\frac{\omega_1}{2}-\frac{\omega_2}{2},\ -\frac{\omega_1}{2}+\frac{\omega_2}{2}]} \zeta(z)\, dz \\
&= \int_{[-\frac{\omega_1}{2}-\frac{\omega_2}{2},\ \frac{\omega_1}{2}-\frac{\omega_2}{2}]} [\zeta(z) - \zeta(z+\omega_2)]\, dz \\
&\quad + \int_{[\frac{\omega_1}{2}-\frac{\omega_2}{2},\ \frac{\omega_1}{2}+\frac{\omega_2}{2}]} [\zeta(z) - \zeta(z-\omega_1)]\, dz \\
&= (\omega_1)(-2\eta_2) + (\omega_2)(2\eta_1)
\end{aligned}$$

as required. □

The above relationship

$$\eta_1\omega_2 - \eta_2\omega_1 = \pi i.$$

is known as **Legendre's relation**.

Pseudo-periodicity of Sigma function

It follows from integrating

$$\zeta(z+\omega_1) = \zeta(z) + 2\eta_1,$$

that

$$\sigma(z+\omega_1) = Ae^{2\eta_1 z}\sigma(z).$$

for some non-zero A. Putting $z = -\frac{\omega_1}{2}$ in the above equation yields

$$A = e^{\eta_1\omega_1}\,\frac{\zeta(\omega_1/2)}{\zeta(-\omega_1/2)} = -e^{\eta_1\omega_1}$$

since $\sigma(z)$ is an odd function. Hence

$$\sigma(z+\omega_1) = -e^{\eta_1\omega_1}e^{2\eta_1 z}\sigma(z) = -e^{\eta_1(\omega_1+2z)}\sigma(z).$$

Similarly, we have

$$\sigma(z+\omega_2) = -e^{\eta_2(\omega_2+2z)}\sigma(z).$$

Exercise 5.6.1. Let ω_3 be the period of $\wp(z)$ defined by $\omega_1+\omega_2+\omega_3 = 0$. Show that

1. $\eta_1+\eta_2+\eta_3 = 0$,
2. $\sigma(z+\omega_3) = -e^{\eta_3(\omega_3+2z)}\sigma(z)$,
3. $\pi i = \eta_2\omega_3 - \eta_3\omega_2 = \eta_3\omega_1 - \eta_1\omega_3 = \eta_1\omega_2 - \eta_2\omega_1$.

5.7 The differential equation satisfied by $\wp(z)$

We recall the following expansion

$$\wp(z) = \frac{1}{z^2} + \sum_{k=0}^{\infty} a_k z^k = \frac{1}{z^2} + a_2 z^2 + a_4 z^4 + \cdots$$

around the origin since $\wp$ is an even function, so there are no odd coefficients in the Laurent expansion. Notice that for z sufficiently small,

$$\begin{aligned}\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} &= \frac{1}{\omega^2(1-z/\omega)^2} - \frac{1}{\omega^2} \\ &= \frac{1}{\omega^2}\sum_{k=1}^{\infty} k\left(\frac{z}{\omega}\right)^{k-1} - \frac{1}{\omega^2} \qquad (5.11) \\ &= 2\,\frac{z}{\omega^3} + 3\,\frac{z^2}{\omega^4} + 4\,\frac{z^3}{\omega^5} + 5\,\frac{z^4}{\omega^6} + \cdots\end{aligned}$$

This implies that

$$a_2 = 3\sum_{\omega\neq 0}\frac{1}{\omega^4}, \quad a_4 = 5\sum_{\omega\neq 0}\frac{1}{\omega^6},$$

and so on. So

$$\wp(z) = \frac{1}{z^2} + a_2 z^2 + a_4 z^4 + O(z^6)$$

where the $O(z^6)$ represents a function analytic at $z = 0$ with a zero of order 6. Hence

$$\wp'(z) = -\frac{1}{z^3} + 2a_2 z + 4a_4 z^3 + O(z^5).$$

Notice that,

$$\wp^3(z) = \frac{1}{z^6} + 3\,\frac{a_2}{z^2} + 3a_4 + O(z^2)$$

$$\wp'(z)^2 = \frac{4}{z^4} - \frac{8a_z}{z^2} - 16a_4 + O(z^2)$$

so that

$$\begin{aligned}\wp'(z)^2 - 4\wp^3(z) &= -20\frac{a_2}{z^2} - 28a_4 + O(z^2)\\ &= -20a_2\wp(z) - 28a_4 + O(z^2).\end{aligned}$$

This shows that the function

$$\Phi(z) := \wp'(z)^2 - 4\wp^3(z) + 20a_2\wp(z) + 28a_4$$

has a double zero around the origin $z = 0$ and hence analytic there. Moreover, the construction of the function Φ asserts that it is also an elliptic function with periods ω_1 and ω_2. That is, the $\Phi(z)$ is analytic at every ω which are the only potential singularities. So the $\Phi(z)$ is an entire function in $\mathbb{C}$. So it must reduce to a constant which mush equals to 0 (because the function has a double zero at $z = 0$.).

Let us summarise the above discussion into a theorem.

Theorem 5.7.1. *The elliptic function $\wp(z)$ with periods ω_1 and ω_2 satisfies the differential equation*

$$y'(z)^2 = 4y^3(z) - g_2y(z) - g_3 \tag{5.12}$$

where

$$g_2 := 20a_2 = 60\sum_{\omega\neq 0}\frac{1}{\omega^4}, \quad g_3 = 28a_4 = 140\sum_{\omega\neq 0}\frac{1}{\omega^6}.$$

We actually can have

Theorem 5.7.2. *$\wp(z)$ has Laurent expansion of the form*

$$\wp(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty}(2k+1)G_{2k+2}z^{2k},$$

where

$$G_k = \sum_{\omega\neq 0}\frac{1}{\omega^k}, \quad k \geq 3$$

is called the Eisenstein series of order n.

Proof. Exercise. □

That is, $g_2 = 60G_4$ and $g_3 = 140G_6$.

Exercise 5.7.1. Show that

1. $$\wp''(z) = 6\wp^2 - \frac{1}{2}g_2.$$

2. $$\wp^{(3)} = 12\wp\wp'$$

3. $$\wp^{(4)} = 120\wp^3 - 18g_2\wp - 12g_3.$$

Exercise 5.7.2. Recall the Taylor expansion

$$\wp(z) - \frac{1}{z^2} = \sum_{k=1}^{\infty} c_1 z^2 + c_2 z^4 + \cdots + c_n z^{2n} + \cdots .$$

Show that

$$(n-2)(2n+3)c_n = 3(c_1c_{n-2} + c_2c_{n-3} + \cdots c_{n-2}c_1), \quad n \geq 3.$$

Hence prove that each c_n is a polynomial in g_2 and g_3 with positive rational coefficients.

Exercise 5.7.3. Show that

1. $\sigma(\lambda z\,|\lambda\omega_1,\, \lambda\omega_2) = \lambda\sigma(z\,|\omega_1,\, \omega_2)$,
2. $\zeta(\lambda z\,|\lambda\omega_1,\, \lambda\omega_2) = \lambda^{-1}\zeta(z\,|\omega_1,\, \omega_2)$,
3. $\wp(\lambda z\,|\lambda\omega_1,\, \lambda\omega_2) = \lambda^{-2}\wp(z\,|\omega_1,\, \omega_2)$.

Three roots of $\wp'(z)$

We shall revisit the differential equation

$$\wp'(z)^2 = 4\wp^3(z) - g_2\wp(z) - g_3$$

obtained above.

We also recall that

$$\wp'(z) = -2\sum_{\omega} \frac{1}{(z-\omega)^3},$$

and it is therefore clear that the $\wp'$ is an odd elliptic function. Hence

$$\wp'(\omega_1/2) = \wp'(-\omega_1/2) = -\wp'(\omega_1/2)$$

and this immediately implies that $\wp'(\omega_1/2) = 0$. Similarly,

$$\wp'(\omega_2/2) = 0.$$

Notice that

$$\wp'(\omega_1/2 + \omega_2/2) = \wp'(-\omega_1/2 - \omega_2/2) = -\wp'(\omega_1/2 + \omega_2/2).$$

Hence $\wp'(\omega_1/2 + \omega_2/2) = 0$. Recall that $\omega_1 + \omega_2 + \omega_3 = 0$. Then

$$-\wp'(\omega_3/2) = \wp'(-\omega_3/2) = \wp'(\omega_1/2 + \omega_2/2) = 0.$$

Since the $\omega_3/2$ are incongruent module to $\omega_1/2$ and $\omega_2/2$ within a period parallelogram, so we have shown that all the three simple roots of $\wp'(z)$ there (since $\wp'$ has order 3 there).

Let

$$\wp(\omega_1/2) = e_1, \quad \wp(\omega_2/2) = e_2, \quad \wp(\omega_3/2) = e_3.$$

Since $\wp'(\omega_1/2) = 0$, so the elliptic function $\wp(z) - e_1$, which is of order 2, has a double root at $\omega_1/2$. So it cannot vanish at any other point in the period parallelogram. This implies that $e_1 \neq e_2$, and $e_1 \neq e_3$. Similarly, $e_2 \neq e_3$ so that all three numbers e_1, e_2, e_3 are distinct. It follows that

$$\wp'(z)^2/[(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3)]$$

has no zero in any period parallelogram and hence in $\mathbb{C}$. Thus the quotient is a constant C, say. Hence

$$\wp'(z)^2 = C(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3).$$

Comparing with the lowest term above with that in (5.12) implies that $C = 4$ which gives the desired

$$\wp'(z)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3).$$

Moreover, the e_1, e_2, e_3 are three roots of the algebraic equation $y^2 = 4x^3 - g_2 x - g_3$.

Exercise 5.7.4. Verify

1. $e_1e_2 + e_2e_3 + e_3e_1 = -\frac{1}{4}g_2$,
2. $e_1e_2e_3 = \frac{1}{4}g_3$,
3. $e_1^2 + e_2^2 + e_3^2 = \frac{1}{2}g_2$.

5.8 Elliptic integrals

The differential equation

$$\wp'(z)^2 = 4\wp^3(z) - g_2\wp(z) - g_3$$

gives the solution $w = \wp(z)$. We can invert the z by

$$z = \int^w \frac{dw}{\sqrt{4w^3 - g_2 w - g_3}}.$$

More precisely,

$$z - z_0 = \int_{\wp(z_0)}^{\wp(z)} \frac{dw}{\sqrt{4w^3 - g_2 w - g_3}},$$

where the path of integration is the path of $\wp$ on a path from z_0 to z avoiding the zeros and poles of $\wp'(z)$.

There is already a similar elliptic integral we encountered earlier under the conformal mapping of the upper half-plane $\mathbb{H}$ onto a rectangle:

$$f(z) = \alpha \int_0^z \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} + \beta.$$

The Jacobian sine elliptic function is $w = \mathrm{sn}(z)$ is the function behind.