Chapter 6

Modular functions

This chapter is a brief introduction to modular functions.

We recall that the **modular group** consists of the set of all Möbius transformations of the from

$$\tau' = \frac{a\tau + b}{c\tau + d}$$

where a, b, c, d are integers such that (WLOG) $ad - bc = 1$. This group is denoted by Γ. Such a Möbius transformation can be represented in a matrix form:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1.$$

Definition 6.0.1. An analytic function λ which satisfies

$$\lambda\Big(\frac{az + b}{cz + d}\Big) = \lambda(z),$$

where the Möbius transformation belongs to the modular group is called an **automorphic function**.

Recall that for a given Weierstrass elliptic function $\wp(z)$, we have

$$e_1 = \wp(\omega_1/2), \quad e_2 = \wp(\omega_2/2), \quad e_3 = \wp(\omega_3/2),$$

where $\omega_1 + \omega_2 + \omega_3 = 0$.

6.1 The function $\lambda(\tau)$

We observe that scaling of the periods ω_k ($k = 1, 2, 3$) by $t\omega_k$ results in

$$\wp\Big(t\frac{\omega_k}{2}\Big) = \frac{1}{t^2}\wp\Big(\frac{\omega_k}{2}\Big) = \frac{1}{t^2}e_k, \quad k = 1, 2, 3.$$

Thus the function

$$\lambda(\tau) = \frac{e_3 - e_2}{e_1 - e_2}, \tag{6.1}$$

is a function of $\tau := \omega_2/\omega_1$. Since the $e_j \neq e_k$ whenever $j \neq k$, so the $\lambda(\tau)$ is an analytic function in the upper half-plane $\Im(\tau) > 0$. Moreover,

$$\lambda(\tau) \neq 0, 1$$

since $e_2 \neq e_3$ and $e_1 \neq e_3$ respectively.

Applying knowledge from theta function, one can actually write

$$\lambda(\tau) = \frac{e_3 - e_2}{e_1 - e_2} = 16q\prod_{k=1}^{\infty}\Big(\frac{1+q^{2k}}{1+q^{2k-1}}\Big)^8,$$

where $q = e^{i\pi\tau}$.

Congruent subgroup of mod 2

Suppose our initial ω_1, ω_2 is replaced by

$$\begin{aligned} \omega_2' &= a\omega_2 + b\omega_1, \\ \omega_1' &= c\omega_2 + d\omega_1. \end{aligned} \tag{6.2}$$

But since the $\wp(z)$ is invariant with respect to any modular transformation, so it follows from the differential equation

$$\begin{aligned} \wp'(z)^2 &= 4\wp^3(z) - g_2\wp(z) - g_3 \\ &= (\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3). \end{aligned}$$

the corresponding e_k ($k = 1, 2, 3$) are permuted (and so changing the value of λ) under a unimodular transformation.

The identity map for the e_k $(k = 1, 2, 3)$ from the following unimodular transformation. If we choose the a, b, c, d such that $a \equiv 1 \equiv d \mod 2$ and $b \equiv 0 \equiv c \mod 2$, then this imply

$$\frac{\omega_1'}{2} \equiv \frac{\omega_1}{2}, \quad \frac{\omega_2'}{2} \equiv \frac{\omega_2}{2}. \qquad \mod M$$

So the e_k $(k = 1, 2, 3)$ remain fixed. We may rephrase the above by writing

$$\lambda\Big(\frac{a\tau + b}{c\tau + d}\Big) = \lambda(\tau), \quad \text{when} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mod 2. \tag{6.3}$$

The collection of unimodular transformations can easily be seen to form a group, called the **congruence subgroup** mod 2 of the modular group. In general a function f that satisfies the equation $f(M\tau) = f(\tau)$ is called *automorphic*. An automorphic function with respect to a subgroup of the full modular group is called a **(elliptic) modular function**.

Incongruent subgroup of mod 2

It is sufficient to consider

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \mod 2 \tag{6.4}$$

since the other ones can be composed from these two. The equation (6.3) would therefore be violated. Indeed, in the first case above

$$\frac{\omega_2'}{2} \equiv \frac{\omega_1 + \omega_2}{2}, \quad \frac{\omega_1'}{2} \equiv \frac{\omega_1}{2}, \qquad \mod M$$

we have, so that $e_2 \leftrightarrow e_3$ (they are interchanged), e_1 remains fixed. We have the $\lambda(\tau)$ becomes

$$\lambda(\tau) = \frac{e_3 - e_2}{e_1 - e_2} \longmapsto \frac{\lambda(\tau)}{\lambda(\tau) - 1} = \frac{e_2 - e_3}{e_1 - e_3}.$$

But the corresponding unimodular transformation is $\tau \to \tau + 1$. Hence

$$\lambda(\tau + 1) = \frac{\lambda(\tau)}{\lambda(\tau) - 1}.$$

The second transformation corresponds to

$$\frac{\omega_2'}{2} \equiv \frac{\omega_1}{2}, \quad \frac{\omega_1'}{2} \equiv \frac{\omega_2}{2}, \qquad \text{mod } M \tag{6.5}$$

so that $e_1 \leftrightarrow e_2$ and e_3 remains unchanged. We see that

$$\lambda(\tau) = \frac{e_3 - e_2}{e_1 - e_2} \longmapsto 1 - \lambda(\tau) = \frac{e_3 - e_1}{e_2 - e_1}$$

the corresponding unimodular transformation is $\tau \to -1/\tau$. Hence

$$\lambda\Big(-\frac{1}{\tau}\Big) = 1 - \lambda(\tau).$$ [1]

Remark. We note that the choice of the matrices representations (6.4) are far from unique. For example, if we rewrite (6.5) with

$$\frac{\omega_2'}{2} \equiv \frac{\omega_1}{2}, \quad \frac{\omega_1'}{2} \equiv -\frac{\omega_2}{2}, \qquad \text{mod } M \tag{6.6}$$

then we would have matrix representation

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{mod } 2$$

instead of

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{mod } 2.$$

[1] This formula is called Jacobi's imaginary transformation formula (1828).

6.2 Growth properties of $\lambda(\tau)$

We normalise the choice $\omega_1 = 1$ and $\omega_2 = \tau$ for ease of later discussion. We observe that

Theorem 6.2.1. *The elliptic modular function* (6.1) $\lambda(\tau)$ *is real when* τ *is purely imaginary.*

Proof. This essentially follows from the definition of the e_k, namely

$$e_3 - e_2 = \sum_{m,\,n=-\infty}^{\infty} \Big(\frac{1}{(m - \frac{1}{2} + (n - \frac{1}{2})\tau)^2} - \frac{1}{(m + (n - \frac{1}{2})\tau)^2} \Big),$$

and

$$e_1 - e_2 = \sum_{m,\,n=-\infty}^{\infty} \Big(\frac{1}{(m - \frac{1}{2} + n\tau)^2} - \frac{1}{(m + (n - \frac{1}{2})\tau)^2} \Big),$$

where the double series are absolutely convergent. If $\tau = it$ $(t > 0)$, then clearly, the above sums remain unchanged with τ is replaced by $-\tau = \bar{\tau}$. This establishes the theorem. □

Theorem 6.2.2. *The elliptic modular function* (6.1) $\lambda(\tau)$ *satisfies*

1. $\lambda(\tau) \to 0$ *as* $\Im(\tau) \to +\infty$ *uniformly with respect to the* $\Re(\tau)$,

2. *more precisely,*

$$\lambda(\tau)/e^{i\pi\tau} \to 16, \quad \Im(\tau) \to +\infty, \tag{6.7}$$

3. $\lambda(\tau) \to 1$ *as* $\tau \to 0$ *along the imaginary axis.*

Proof. Let us quote the elementary Mittag-Leffler expansion formula:

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{m=-\infty}^{\infty} \frac{1}{(z - m)^2}.$$

Applying this expansion in the definition of e_k summing first over m yields

$$e_3 - e_2 = \pi^2 \sum_{n=-\infty}^{\infty} \Big(\frac{1}{\cos^2 \pi(n-\frac{1}{2})\tau} - \frac{1}{\sin^2 \pi(n-\frac{1}{2})\tau} \Big)$$

and

$$e_1 - e_2 = \pi^2 \sum_{n=-\infty}^{\infty} \Big(\frac{1}{\cos^2 \pi n\tau} - \frac{1}{\sin^2 \pi(n-\frac{1}{2})\tau} \Big).$$

Notice that the terms $|\sin n\pi\tau|$ and $|\cos n\pi\tau|$ are comparable to $e^{|n|\pi\Im(\tau)}$ so that the above sums are uniformly convergent as $n \to \pm\infty$ when $\Im(\tau) \geq \delta > 0$ (for some $\delta > 0$). This also means that we could take limit on individual terms of the above sum as $\Im(\tau) \to +\infty$. This yields

$$e_3 - e_2 \to 0, \quad e_1 - e_2 \to \pi^2, \quad \Im(\tau) \to +\infty,$$

and hence $\lambda(\tau) \to 0$ as $\Im(\tau) \to +\infty$ as asserted. If we let $\tau \to 0$ along the imaginary axis, then we easily deduce from the equation $\lambda(-1/\tau) = 1 - \lambda(\tau)$ that $\lambda(\tau) \to 1$.

We note that the leading terms (i.e., $n = 0, 1$) of the above sum for $e_3 - e_2$ are given by

$$2\pi^2 \Big(\frac{4e^{\pi i\tau}}{(1+e^{\pi i\tau})^2} + \frac{4e^{\pi i\tau}}{(1-e^{\pi i\tau})^2} \Big).$$

This concludes the part (2). □

6.3 Covering property of $\lambda(\tau)$

Let

$$\Omega := \big\{ z : \ 0 < \Re(z) < 1, \ \Im(z) > 0 \big\} \cap \big\{ z : \ |z-1| \geq 1/2 \big\} \tag{6.8}$$

We are ready to deal with