Theorem 6.3.1. *The modular function*

$$\lambda(\tau) = \frac{e_3 - e_2}{e_1 - e_2}$$

is a one-one conformal mapping $\lambda : \Omega \to \mathbb{H}$. Moreover, the mapping extends continuously to the boundary of Ω so that

1. *the image of $\partial\Omega$ is real-valued; and*
2. *the boundary points $\tau = 0,\ 1,\ \infty$ correspond to $\lambda = 1,\ \infty,\ 0$;*
3. *the $\lambda(\tau)$ is monotone on $\partial\Omega$ so that $\lambda(\partial\Omega) = (-\infty,\ \infty)$ in such a way that*
    - $\lambda : -\infty \uparrow 0$ *over* $[1,\ 1 + i\infty)$;
    - $\lambda : 0 \uparrow 1 = \lambda(0)$ *over* $(i\infty,\ 0]$;
    - $\lambda : 1 \uparrow +\infty$ *over* $\frac{1}{2} + \frac{1}{2}e^{i\theta}$ *where* $\theta : -\pi \uparrow \pi$.

Proof. We first investigate the behaviour of $\lambda(\tau)$ on the boundary of Ω. We recall from Theorem 6.2.2 that $\lambda(z)$ is real on imaginary axis. So the transformation $\tau + 1$ maps the imaginary axis onto the $\Re(\tau) = 1$. So

$$\lambda(it + 1) = \frac{\lambda(it)}{1 - \lambda(it)}$$

is therefore *real for all* $t > 0$. Moreover, the map $1/\tau$ maps the $\Re(\tau) = 1$, i.e., $\tau = 1 + it$ $(t > 0)$ onto the circle $|\tau - \frac{1}{2}| = \frac{1}{2}$[2] Let $\tau' = \frac{1}{2} + \frac{1}{2}e^{i\theta}$. Let

$$\tau = \frac{1}{1 - \tau'}, \quad \tau' = 1 - \frac{1}{\tau},$$

and

$$\tau = 1 + i\frac{\sin\theta}{1 + \cos\theta}, \quad \Re(\tau) = 1.$$

[2] $\left|1 - \frac{1}{\tau} - \frac{1}{2}\right|^2 = \left|1 - \frac{1}{1+it} - \frac{1}{2}\right|^2 = (\frac{1}{2})^2.$

Then the image of λ on $|\tau - \frac{1}{2}| = \frac{1}{2}$ can "pull-back" by the transformation:

$$\begin{aligned}\lambda(\tau') = \lambda\Big(1 - \frac{1}{\tau}\Big) &= \frac{\lambda(-1/\tau)}{1-\lambda(-1/\tau)} = \frac{1-\lambda(\tau)}{1-(1-\lambda(\tau))} \\ &= \frac{1}{\lambda(\tau)} - 1,\end{aligned}$$

hence showing that λ (where τ lies on the $\Re(\tau) = 1$) is again real on $|\tau - \frac{1}{2}| = \frac{1}{2}$ by the first case. Hence we have established that $\lambda(\tau)$ is real-valued on the whole boundary of Ω.

Since our aim is to prove $\lambda : \Omega \to \mathbb{H}$ is a one-one conformal map, so we choose an arbitrary point w_0 in $\mathbb{H}$. Then Theorem 6.2.2 (1) guarantees that there exists a number $t_0 > 0$ so that

$$w_0 \neq \lambda(\tau) = \lambda(s + it)$$

for $t \geq t_0$.

Let us consider the images of the horizontal line segment

$$L_0 := \{s + it_0 : \ 0 \leq s \leq 1\}$$

under the modular transformations λ:

1. $-1/\tau$: L_0 is mapped onto a circle C_0 tangent to the point $\tau = 0$ in the upper half-plane. Clearly, the "smaller" the circle is when the larger the $t_0 > 0$ is chosen;

2. $1-1/\tau$: L_0 is mapped onto a circle C_1 tangent to the point $\tau = 1$ in the upper half-plane. Clearly, the "smaller" the circle is when the larger the $t_0 > 0$ is chosen again.

We recall that the region of Ω is a "triangle" with all three angles zero (one at ∞). Let us "cut off the three angles" by removing the portions

- $\Im(\tau) > t_0$;

- the whole disc filled from C_0 tangent at $\tau = 0$ constructed above;
- the whole disc filled from C_1 tangent at $\tau = 1$ constructed above.

We write Ω_0 to denote the remaining region of Ω. Since $\lambda(\tau) \to 1$ as $\tau \to 0$ (Theorem 6.2.2 (1)), so $\lambda(-1/\tau) \approx 1$ uniformly on C_0 as $t_0 \to +\infty$.

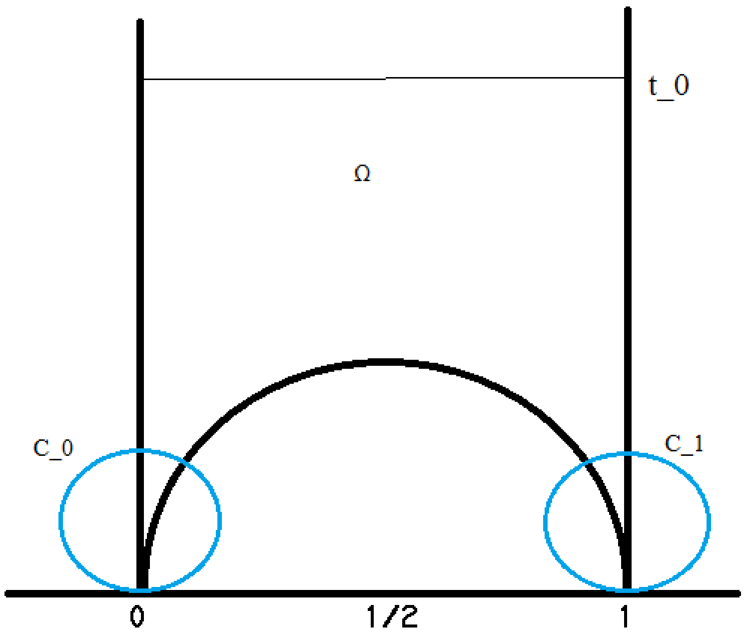

Figure 6.1: Non-Euclidean triangle with three angles 0

On the other hand, Theorem 6.2.2 (2) asserts that when τ' is close to C_1 when $\tau' \approx 1$,

$$\begin{aligned}\lambda(\tau') = \lambda(1 - 1/\tau) &= 1 - 1/\lambda(\tau)\\ &\approx 1 - \frac{1}{16}e^{-i\pi(s+it_0)}\\ &= 1 + \frac{1}{16}e^{\pi t_0 + i\pi(1-s)},\end{aligned}$$

for $0 \leq s \leq 1$, so that this is approximately a semi-circle in the upper half-plane. This together with earlier analysis shows that in the limit as $\Omega_0 \to \Omega$ as $t_0 \to +\infty$ that

$$\begin{aligned} n\big(\lambda(\partial\Omega);\, w_0\big) &= \frac{1}{2\pi i}\int_{\lambda(\partial\Omega)} \frac{d\lambda}{\lambda - w_0} \\ &= \frac{1}{2\pi i}\int_{\partial\Omega} \frac{\lambda'(T)}{\lambda(T) - w_0}\, dT \\ &= 1. \end{aligned}$$

Hence each w_0 in $\mathbb{H}$ has been "taken" once and once only by $\lambda(\tau)$ inside Ω, and none of those points with $\Im(w_0) < 0$ are taken by λ in Ω. It is clear that $\lambda(0) = 1$, $\lambda(1) = \infty$ and $\lambda(\infty) = 0$.

The above analysis shows that $\lambda : \Omega \to \mathbb{H}$ is a one-one conformal map also implies that $\lambda(\tau)$ is *monotone* on $\partial\Omega$. For suppose not, then there would be a boundary point a on $\partial\Omega$ at which $\lambda'(a) = 0$. But then, in a neighbourhood of a in Ω, we have

$$\lambda(z) = \lambda(a) + \frac{\lambda^{(k)}(a)}{k!}(z-a)^k[1 + O(z-a)],$$

where $k \geq 2$, so it is evident that the image of such neighbourhood could not lie entirely within $\mathbb{H}$. A contradiction. □

Corollary 6.3.1.1. *Let Ω' denote the region that is the mirror image of Ω reflected along the imaginary axis in $\mathbb{H}$. Then the modular function maps the Ω' onto the lower half-plane, and $\lambda(\bar{\Omega} \cup \Omega') = \mathbb{C}\backslash\{0,\, 1\}$.*

Remark. We call the modular function λ a *universal cover* of $\mathbb{C}\backslash\{0,\, 1\}$.

Exercise 6.3.1. Show that if $\tau' = \frac{1}{2} + \frac{1}{2}e^{i\theta}$, $\tau' = 1 - \frac{1}{\tau}$, then

$$\tau = 1 + i\frac{\sin\theta}{1 - \cos\theta}.$$

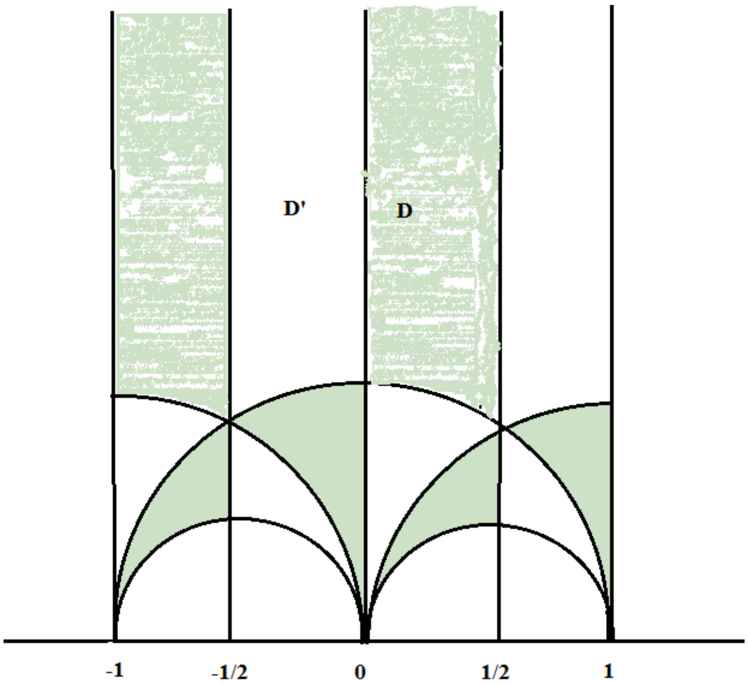

Figure 6.2: D is the "Right-half" of FR, D' its mirror-image. The figure shows regions that are reflections of D and D'

It is routine to check that the six shaded regions in the above figure are images of the Fundamental Region D under the following transformations:

$$\tau, \quad -\frac{1}{\tau}, \quad \tau - 1, \quad \frac{1}{1-\tau}, \quad \frac{\tau-1}{\tau}, \quad \frac{\tau}{1-\tau}. \tag{6.9}$$

which we denote by $S_1,\ S_2, \cdots, S_6$. They form a complete set of incongruent unimodular transformations (i.e., members of modular group) mod 2, in the sense that each unimodular transformation is congruent mod 2 to one of the S_k. Let we denote S_k^{-1} $(k = 1, \cdots, 6)$ to denote the corresponding inverses. Then it can be checked that they map the region D' (the "left-half" of the FD) onto the unshaded regions of the above figure. One see immediately that the union of 12 images of $\bar{D}$ and $\bar{D}'$ covers $\bar{\Omega} \cup \bar{\Omega}'$ (here the closure refers to the $\mathbb{H}$ only).

Let Ω' be the mirror image of Ω reflected along the imaginary axis.

Theorem 6.3.2. *Every point τ in the upper half-plane $\mathbb{H}$ is equivalent under the congruence subgroup* mod 2 *to exactly one point in* $\bar{\Omega} \cup \Omega'$.

Proof. Let τ be an arbitrary point in $\mathbb{H}$. Then according to Theorem 5.3.2 that there is a unimodular transformation S such that $S\tau$ in D, say. But there is a S_k^{-1} such that $S \equiv S_k^{-1} \mod 2$, i.e., $T = S_k S \equiv I \mod 2$. But then $T\tau = S_k(S\tau)$ belongs to one of 12 regions and hence in $\bar{\Omega} \cup \bar{\Omega}'$. A similar reasoning also applies if $S\tau \in D'$. Hence $T\tau \in \bar{\Omega} \cup \bar{\Omega}'$ in either cases. Hence $T\tau \in \bar{\Omega} \cup \Omega'$.

The uniqueness follows from the fact that the $S_1, \cdots, S_6$ as well as $S_1^{-1}, \cdots, S_6^{-1}$ are incongruent mod 2. □

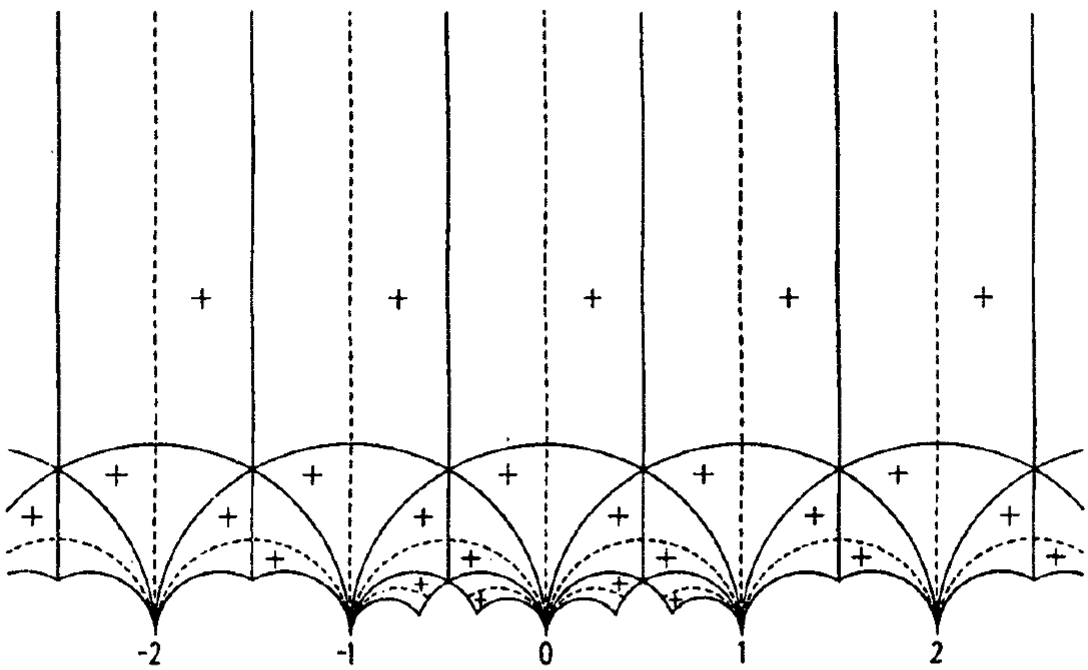

Figure 6.3: Taken from page 426 of E. T. Copson