Chapter 7

Picard's theorem

7.1 Monodromy

The terminologies below are used to handle multi-valued functions.

An analytic function f defined on a region Ω that constitute (f, Ω) is called a **function element**. A **global analytic function** is a collection of function elements (f, Ω). Two function elements (f_1, Ω_1), (f_2, Ω_2) are **direct analytic continuations** of each other if $\Omega_1 \cap \Omega_2 \neq \emptyset$ and $f_1(z) = f_2(z)$ over $\Omega_1 \cap \Omega_2$. There need not be any direct analytic continuation of f_1 from Ω_1 to Ω_2. But the continuation must be unique if there is such a continuation (by the identity theorem since $\Omega_1 \cap \Omega_2 \neq \emptyset$).

Suppose the chain (f_1, Ω_1), (f_2, Ω_2), $\cdots (f_n, \Omega_n)$ are analytic continuations of each other so that $\Omega_{k-1} \cap \Omega_k \neq \emptyset$ for each k. Then we say (f_n, Ω_n) is **an analytic continuation** of (f_1, Ω_1). This is an equivalence relation. The equivalence classes are called **global analytic functions**. We label the global analytic determined by the function element by **f**. However, a global analytic function **f** can have many function element (f, Ω) on Ω. In fact, we call each function element (f, Ω) **a branch** of **f**.

We now replace a region Ω by a single pont ζ, and we say that two function elements (f_1, ζ_1) and (f_2, ζ_2) are **equivalent** if and only if $\zeta_1 = \zeta_2$ and $f_1 = f_2$ in a neighbourhood of ζ_1 $(= \zeta_2)$. This is again an equivalence relation. In this case, the equivalence classes are called **germs** or **germs of analytic functions**. Each germ determines a unique ζ, called the **projection** of the germ, which we denote by $\mathbf{f}_\zeta$. Thus a function element (f, Ω) gives raise to a germ $\mathbf{f}_\zeta$ for each $z \in \Omega$.

Theorem 7.1.1 (Monodromy theorem)**.** *If the two arcs γ_1, γ_2 are homotopic in a region Ω, and if a given germ $\boldsymbol{f}$ at the initial point can be continued along all arcs in Ω, then the continuations of this germ along γ_1 and γ_2 lead to the same germ at the end point.*

We refer the reader to Ahlfors for its proof.

7.2 Picard's theorem

A value $a \in \mathbb{C}$ is called a **lacunary value** of an analytic function f if $f(z) \neq a$ for all z in the region Ω where f is defined. The exponential function e^z has $z = 0$ as the only lacunary value.

Theorem 7.2.1 (Picard (1879))**.** *An entire function that has at least two finite lacunary values reduces to a constant.*

Proof. WLOG, we may assume that f has two lacunary values a, b and that $a = 0$ and $b = 1$. For we could consider the function

$$F(z) = \frac{f(z) - a}{b - a},$$

otherwise. The main idea is to construct a global analytic function $\mathbf{h}$ such that its function elements (h, Ω) satisfies

$$\Im(h(z)) > 0, \quad \lambda(h(z)) = f(z), \quad z \in \Omega,$$

where $\lambda(z)$ is the elliptic modular function constructed in the last chapter. Then we want to show that $\mathbf{h}$ can be continued to along all paths. Since the $\mathbb{C}$ is simply connected, so the monodromy theorem asserts that $\mathbf{h}$ defines an entire function.

Theorem 6.3.1 asserts that there is a $\tau_0 \in \Omega$ such that

$$\lambda(\tau_0) = f(0).$$

Since the $\lambda(\tau)$ is conformal, so $\lambda'(\tau_0) \neq 0$. Therefore we can find a local inverse λ_0^{-1} of λ over a neighbourhood Δ_0 of $f(0)$ where

$$\lambda(\lambda_0^{-1}(w)) = w, \quad w \in \Delta_0$$

and

$$\lambda_0^{-1}(f(0)) = \tau_0.$$

By continuity there is a neighbourhood Ω_0 of $z = 0$ where $f(z) \in \Delta_0$. Hence we can define

$$h(z) = \lambda_0^{-1}(f(z)), \quad z \in \Omega_0.$$

Hence we have a function element (h, Ω_0). We next show that the germ $\mathbf{h}$ that the function element determines can be continued in all possible ways to become an entire function and that the continuation of (h, Ω_0) has

$$\Im(h(z)) > 0$$

throughout the continuation. If this continuation is not possible, then we can find a path $\gamma[0, t_1]$ such that h can be continued and $\Im(h)$ remains positive up the $t < t_0$, where

- either the h cannot be continued to t_1 (which is not possible),
- or

$$\Im(h(z)) \to 0, \quad t \to t_1.$$

Let us take a closer look at $t = t_1$. Then there is a $\tau_1 \in \mathbb{H}$ and $\lambda(\tau_1) = f(\gamma(t_1))$. There is a local inverse λ_1^{-1} over a neighbourhood Δ_1 of $f(\gamma(t_1))$ such that

$$\lambda_1^{-1}(f(\gamma(t_1)) = \tau_1.$$

Let Ω_1 be a neighbourhood of $\gamma(t_1)$ (in the $z-$plane) so that $f(z) \in \Delta_1$ when $z \in \Omega_1$.

Let $t_2 < t_1$ be so chosen that $\gamma(t_2) \in \Omega_1$ for $t \in [t_2, t_1]$. But $\lambda(\tau_2) = f(\gamma(t_2))$ can simultaneously be computed by

- $$\tau = h(\gamma(t_2))$$
- and
$$\tau = \lambda_1^{-1} f(\gamma(t_2))$$

indicating that the τ is "pull-back" from two different branches of $\mathbf{h}$. So Theorem 6.3.2 asserts that there is a elliptic modular transformation S that belongs to congruence subgroup mod 2 such that

$$S[\lambda_1^{-1} f(\gamma(t_2))] = h(\gamma(t_2)).$$

Thus we could define a continuation function element (h_1, Ω_1) by

$$h_1(z) = S[\lambda_1^{-1} f(z))], \quad z \in \Omega_1$$

so that h_1 and hence $\mathbf{h}$ can be continued to $\gamma(t_1)$ with

$$\lambda(h_1(z)) = f(z), \quad \Im(h_1)(z) > 0.$$

As we have anticipated that we have constructed a global analytic function $\mathbf{h}$ so that

$$\lambda(h(z)) = f(z)$$

for all function element (h, Ω). Now consider the funcion

$$e^{ih(z)},$$

which is an entire function with $|e^{ih}| \leq 1$ bounded since $\Im(h) > 0$. Liouville's theorem implies that the h in

$$\lambda(h(z)) = f(z)$$

is a constant. Hence f must also reduce to a constant. □