PRELIMINARIES FOR HYPERGEOMETRIC EQUATION

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Abstract. We give a brief introduction to some preliminaries for Gauss hypergeometric equations.

We will only consider differential equations with regular singularities in this lectures.

1. Regular singular points

We shall primary be interested in second order linear differential equations of the form

$$\frac{d^2w}{dz^2}+p(z)\frac{dw}{dz}+q(z)w=0, \tag{1.1}$$

where $p(z)$ and $q(z)$ are meromorphic in a domain because many important functions that we shall encounter later satisfy second order equations.

1.1. **Ordinary points.**

Definition 1.1. A point ξ in $\mathbb{C}$ of the linear system (1.1) is called an **ordinary point** at which the $p(z)$ and $q(z)$ are analytic at z. All other points are called **singular points**.

Theorem 1.2 (L. I. Fuchs (1866)). *Let z_0 be an ordinary point of the equation (1.1) and let a_0 and a_1 be two arbitrary constants. Then the equation (1.1) has a unique* analytic *solution $w(z)$ that satisfies the* initial condition *$w(z_0)=a_0$ and $w'(z_0)=a_1$.*

Proof. Without loss of generality, we may assume that z_0 in order to simplify the argument. For one can consider $z'=z-z_0$ to recover the general case. Since both $p(z),\ q(z)$ are analytic at z_0, so let us write their Taylor expansions in the forms

$$p(z)=\sum_{k=0}^{\infty}p_kz^k, \qquad q(z)=\sum_{k=0}^{\infty}q_kz^k,$$

and both converge in $|z|<R$. We substitute formally our "solution"

$$w(z)=\sum_{k=0}^{\infty}a_kz^k=a_0+a_1z+a_2z^2+\cdots$$

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into the (1.1)

$$\frac{d^2w}{dz^2}+\Big(\sum_{k=0}^{\infty}p_kz^k\Big)\frac{dw}{dz}+q(z)\Big(\sum_{k=0}^{\infty}q_kz^k\Big)w=0,$$

and equating the coefficients. This yields

$$\begin{aligned}-2a_2&=a_1p_0+a_0q_0,\\ -2\cdot3a_3&=2a_2p_0+a_1p_1+a_1q_0+a_0q_1,\\ &\cdots\\ &\cdots\\ -(k-1)ka_k&=(k-1)a_{k-1}p_0+(k-2)a_{k-2}p_1+\cdots a_1p_{k-2}+\\ &\quad a_{k-2}q_0+a_{k-3}q_1+\cdots+a_1q_{k-3}+a_0q_{k-2}\end{aligned}$$

for all $k\geq2$. The above equations show that one can express any a_n, successively, as linear combination of a_0 and a_1. Notice that the above recurrence on a_n is only formal. That is, we still need to justify if the sum $w(z)=\sum_{k=0}^{\infty}a_kz^k$ really converges.

Let

$$M=M_r=\max_{|z|=r}|p(z)|,\qquad N=N_r=\max_{|z|=r}|q(z)|$$

for $r<R$. Then the Cauchy inequality gives

$$|p_k|\leq\frac{M}{r^k},\qquad |q_k|\leq\frac{N}{r^k}$$

and we may write

$$|p_k|\leq\frac{K}{r^k},\qquad |q_k|\leq\frac{K}{r^{k+1}},$$

where $K=\max\{M,\,Nr\}$. Writing $b_0=|a_0|$ and $b_1=|a_1|$. Then we have

$$2|a_2|\leq b_1|p_0|+b_0|q_0|\leq b_1K+b_0K/r\leq 2b_1K+b_0K/r.$$

We define $2b_2=2b_1K+b_0K/r$. Hence $|a_2|\leq b_2$. Similarly, we have

$$\begin{aligned}2\cdot3|a_3|&\leq 2|a_2||p_0|+|a_1||p_1|+|a_1||q_0|+|a_0||q_1|\\ &\leq 2b_2K+b_1K/r+b_1K/r+b_0K/r^2\\ &=3b_2K+b_1K/r+b_1K/r+b_0K/r^2.\end{aligned}$$

We define

$$\begin{aligned}2\cdot3b_3:&=3b_2K+b_1K/r+b_1K/r+b_0K/r^2\\ &=3b_2K+2b_1K/r+b_0K/r^2\end{aligned}$$

Hence $|a_3|\leq b_3$.

Continuing this process yields $|a_n|\leq b_n$ where

$$(k-1)kb_k:=kb_{k-1}K+(k-1)b_{k-2}K/r+\cdots b_0K/r^{k-1}.$$

Replacing the k by $k-1$ in the above equation and multiplying both sides of the resulting equation by $\frac{1}{r}$ yield

$$(k-2)(k-1)b_{k-1}/r=(k-1)b_{k-2}K/r+(k-2)b_{k-3}K/r^2+\cdots b_0K/r^{k-1}.$$

Combining these two equations yields the recurrence relation

$$(k-1)kb_k=kb_{k-1}K+(k-2)(k-1)b_{k-1}/r,$$

or

$$\frac{b_k}{b_{k-1}} = \frac{K}{k-1} + \frac{k-2}{kr} \to \frac{1}{r}$$

as $k \to \infty$. This shows that the radius of convergence of $\sum b_k r^k$ is r. However, since, $|a_k| \le b_k$, so it follows that the radius of convergence of $w(z) = \sum a_k z^k$ cannot be less than r. But $r < R$ is arbitrary, this implies that $w(z) = \sum a_k z^k$ has radius of convergence at least R. It follows that $w(z) = \sum a_k z^k$ is analytic function at the origin. Since the power series $w(z) = \sum a_k z^k$ converges uniformly and absolutely in $|z| < R$, so one may differentiate it term by term and series multiplication and rearrangements are all valid. So one can substitute the series into the equation (1.1) to verify that the series $w(z) = \sum a_k z^k$ is indeed a unique analytic solution in the neighbourhood of the origin. In particular, $w(0) = a_0$ and $w'(0) = a_1$. □

Exercise 1.3 (Whittaker & Watson)**.** Show that the equation

$$(1-z^2)u'' - 2zu' + \frac{3}{4}u = 0,$$

has the origin as an ordinary point and that both the series

$$u_1 = 1 - \frac{3}{8}z^2 - \frac{21}{128}z^4 - \cdots ,$$

and

$$u_2 = z + \frac{5}{24}z^3 + \frac{15}{128}z^5 + \cdots ,$$

are two linearly independent solutions of the above equation. Find the general coefficient in each series and show that their radii of convergence are both 1.

1.2. **Regular Singular Points.** We now turn to scalar equation

$$\frac{d^2w}{dz^2} + p(z)\frac{dw}{dz} + q(z)w = 0, \tag{1.2}$$

where the coefficients are analytic functions.

Definition 1.1. A point z_0 is said to be a **regular singular point** of the equation (1.1) if the functions $(z-z_0)p(z)$ and $(z-z_0)^2q(z)$ are both analytic at z_0 . Points that are not regular singular points are called **irregular singular points**.

Example 1.4. Show that the **Euler equation**:

$$z^2y'' + (1-a-b)zy' + aby = 0,$$

has $z = 0$ as a regular singularity.

Again without loss of generality, we may assume that $z_0 = 0$. That is, $zp(z)$ and $z^2q(z)$ have Taylor expansions

$$zp(z) = \sum_{k=0}^{\infty} p_k z^k, \qquad z^2q(z) = \sum_{k=0}^{\infty} q_k z^k, \tag{1.3}$$

where the coefficients p_0, q_0 and q_1 are not all zero.

Suppose we write the solution of (1.1) in the form

$$f(z)=\sum_{k=0}^{\infty} a_k z^{\alpha+k}, \tag{1.4}$$

where α is a constant to be determined. Multiply z^2 on both sides of the equation (1.1) and substitute this expansion $f(z)$ into (1.1) yields

$$\sum_{k=0}^{\infty}(\alpha+k)(\alpha+k-1)a_k z^{\alpha+k}+\Big(\sum_{k=0}^{\infty}p_k z^k\Big)\Big(\sum_{k=0}^{\infty}(\alpha+k)a_k z^{\alpha+k}\Big)$$
$$+\Big(\sum_{k=0}^{\infty}q_k z^k\Big)\Big(\sum_{k=0}^{\infty}a_k z^{\alpha+k}\Big)=0.$$

Collecting the coefficients of $z^{k+\alpha}$ yields:

$$\begin{aligned}(\alpha+k)(\alpha+k-1)a_k+\sum_{j=0}^{k}(\alpha+k)p_{k-j}a_j&\\ +\sum_{j=0}^{k}q_{k-j}a_j=0,&\end{aligned} \tag{1.5}$$

or

$$F(\alpha+k)\,a_k+\sum_{j=0}^{k-1}a_j\big((\alpha+j)p_{k-j}+q_{k-j}\big)=0, \qquad k\geq 1, \tag{1.6}$$

where we have set

$$F(\alpha+k)=(\alpha+k)(\alpha+k-1)+p_0(\alpha+k)+q_0 \tag{1.7}$$

and

$$F(\alpha)=\alpha(\alpha-1)+p_0\alpha+q_0 \tag{1.8}$$

when $k=0$. The equation $F(\alpha)=0$ is called the **indicial equation** and its roots the **(characteristic) exponents** of the regular singularity z_0. We see that if we choose α to be one of the two roots of the equation $\alpha(\alpha-1)+p_0\alpha+q_0=0$, then a_0 above can be arbitrarily chosen. If we assume that this quadratic equation has two distinct roots which do not differ by an integer, then the coefficient $F(\alpha+k)$ from (1.7) is non zero for any $k\geq 1$, implying that one can determine the coefficient a_k in terms of a_s for $0\leq s\leq k-1$, and hence in terms of a_0 after applying the recurrence relation (1.6) successively. We will then obtain two different formal expansions with respect to two different exponents. Thus we have

Proposition 1.1. Suppose that z_0 is a regular singular point for the equation (1.1) and that the indicial equation has two distinct roots. Then the equation (1.1) has two different formal power series expansions solutions.

If we have a double root, then it is clear that we will find only one formal series expansion solution for (1.1). If the two roots differs by an integer instead, then the coefficient $F(\alpha+k)$ will vanish at $k=n$, say (assuming that α is the smaller root). This means that the coefficient a_n and indeed the subsequent a_k could be arbitrarily chosen. So we do not obtain a formal series expansion in both these cases discussed. We shall discuss the cases when the indicial equation has double root or two roots that are differed by an integer at a later stage.

Theorem 1.5. *Let the equation (1.1) to have a regular singular point at* $z = 0$, *and that* $zp(z)$ *and* $z^2q(z)$ *are analytic in* $|z| < R$. *Suppose that the indicial equation of (1.1) at the regular singular point* $z = 0$ *to have two distinct roots* α, α' *such that their difference is not zero or an integer. Then the radius of convergence of the solution (1.4)* $w(z) = \sum_{k=0}^{\infty} a_k z^{\alpha+k}$ *at* $z = 0$ *is at least* R.

Proof. We may assume without loss of generality that the series for $w(z)$ does not terminate. Let α, α' be two distinct roots of (1.1) that are not equal and also do not differ by an integer. Substituting the formal solution $w(z)$ into the equation (1.1), then it is easy to show that the recurrence formula (1.5) can be written in the form

$$k(k + \alpha - \alpha')a_k = -\sum_{j=0}^{k-1} a_j[(\alpha + j)p_{k-j} + q_{k-j}].$$

Let $b_k = |a_k|$, $0 \le k < \delta = |\alpha - \alpha'|$. Let $m = [\delta] + 1$ and $|\alpha| = \tau$. Then

$$\begin{aligned} m(m - \delta)|a_m| &\le |m(m + \alpha - \alpha')a_m| \\ &= \Big|\sum_{j=0}^{m-1} a_j[(\alpha + j)p_{m-j} + q_{m-j}]\Big| \\ &\le \sum_{j=0}^{m-1} b_j[(\tau + j)|p_{m-j}| + |q_{m-j}|], \end{aligned}$$

where $b_j = |a_j|$. Let M, N be the maximum values of $|zp(z)|$ and $|z^2q(z)|$ respectively, on $|z| = r = |\alpha|$. Then

$$|p_k| \le \frac{M}{r^n}, \qquad |q_k| \le \frac{N}{r^n}$$

for $r < R$ and so

$$|p_k| \le \frac{K}{r^n}, \qquad |q_k| \le \frac{K}{r^n}$$

where $K = \max\{M, N\}$. Substituting these bounds for $|p_k|$ and $|q_k|$ implies that

$$|a_m| \le b_m,$$

where

$$m(m - \delta)b_m := \sum_{j=0}^{m-1} Kb_j[(\tau + j + 1)/r^{m-j}].$$

Similarly, we can show that $|a_k| \le b_k$ when $k \ge m$, where

$$k(k - \delta)b_k := \sum_{j=0}^{k-1} Kb_j[(\tau + j + 1)/r^{k-j}].$$

Combining this and a similar one with k replaced by $k - 1$ show that b_k satisfies a recurrence formula given by

$$k(k - \delta)b_k - (k - 1)(k - 1 - \delta)b_{k-1}/r = K(k + \tau)b_{k-1}/r.$$

Thus, we have

$$\frac{b_k}{b_{k-1}} = \frac{(k-1)(k-1-\delta)}{k(k-\delta)r} + \frac{K(k+\tau)}{k(k-\delta)r},$$

and so

$$\lim_{k\to\infty} \frac{b_k}{b_{k-1}} = \frac{1}{r},$$

proving that the series $\sum b_k z^k$ has the radius of convergence r. Since $|a_k| \le b_k$ so the comparison test implies that $\sum a_k z^k$ has a radius of convergence at least r. But r is arbitrary but less than R, so the series must be convergent in $|z| < R$. Hence the series $z^\alpha \sum a_k z^k$ converges uniformly and absolutely in $|z| \le R$. Similar argument can be applied to $z^{\alpha'} \sum a_k z^k$ to show this is the second linearly independent solution of (1.1). □

Exercise 1.6. Show that the Euler equation

$$z^2 y'' + zy' + y = 0$$

has linearly independent solutions z^i and z^{-i}. (Note, however, that both of the two solutions are multivalued functions).

We note that at least one of the two linearly independent solutions considered above has the origin to be a branch point.

We now discuss when the regular singular for the equation (1.1) is at infinity. We have

Proposition 1.2. The differential equation

$$\frac{d^2w}{dz^2} + p(z)\frac{dw}{dz} + q(z)w = 0$$

has a regular singular point at ∞ if and only if $zp(z)$ and $z^2q(z)$ are analytic at ∞.

Proof. Let $w(z)$ be a solution of the differential equation. Then the behaviour of $w(1/t)$ near $t = 0$ is equivalent to that of $w(z)$ at ∞. The same applies to $p(z)$, $q(z)$. Without loss of generality, we may write $\tilde{w}(t) = w\big(\frac{1}{t}\big)$. Then one can easily verify that $\tilde{w}(t)$ satisfies the differential equation

Exercise 1.7.

$$\frac{d^2\tilde{w}}{dt^2} + \Big(\frac{2}{t} - \frac{1}{t^2}p\big(\frac{1}{t}\big)\Big)\frac{d\tilde{w}}{dt} + \frac{1}{t^4}q\big(\frac{1}{t}\big)\tilde{w}(t) = 0. \tag{1.9}$$

Thus (1.9) has a regular singular point at $t = 0$ if and only if

$$t\Big(\frac{2}{t} - \frac{1}{t^2}p\big(\frac{1}{t}\big)\Big) = 2 - \frac{1}{t}p\big(\frac{1}{t}\big), \qquad t^2 \cdot \frac{1}{t^4}q\big(\frac{1}{t}\big) = \frac{1}{t^2}\,q\big(\frac{1}{t}\big),$$

are analytic at $t = 0$. That is, if and only if $zp(z)$ and $z^2q(z)$ are analytic at ∞. □

This shows that

$$p(z) = \frac{p_0}{z} + \frac{p_1}{z^2} + \frac{p_3}{z^3} + \cdots, \qquad q(z) = \frac{q_0}{z^2} + \frac{q_1}{z^3} + \frac{q_4}{z^3} + \cdots.$$

We note in this case, and assuming that the difference of the two roots of the indicial equation do not differ by an integer or 0, that the two solutions can be written as

$$w(z) = z^{-\alpha}\sum_{k=0}^{\infty} a_k z^{-k}, \qquad w(z) = z^{-\alpha'}\sum_{k=0}^{\infty} a'_k z^{-k},$$

where $\alpha,\ \alpha'$ are solutions of

$$\alpha^2 - (p_0 - 1)\alpha + q_0 = 0.$$

Exercise 1.8. Show that the equation

$$zw'' + w' + zw = 0$$

has a regular singular point at $z = 0$ and an irregular singular point at ∞.

Exercise 1.9. Show that the Euler equation:

$$z^2y'' + (1 - a - b)zy' + aby = 0,$$

has $z = \infty$ as a regular singularity.

1.3. **Reduction of Order.**

We shall deal with the cases when the difference of the two roots of the indicial equation is an integer (including zero). If the two roots are identical, then we see from the last section that we could obtain at most one series expansion. If, however, that the roots differ by an integer, then the recurrence relation (1.6) will not give useful information of any second linearly independent solution. The main idea for getting a second solution is by the well-known method of **reduction of order** as explained below.

Theorem 1.10 (L. I. Fuchs (1866)). *Let z_0 be a regular singular point of the differential equation*

$$\frac{d^2w}{dz^2} + p(z)\frac{dw}{dz} + q(z)w = 0.$$

Suppose the exponents of the indicial equation, denoted by α and α', are such that $\alpha - \alpha' = s \in \mathbb{N} \cup \{0\}$. Then the equation possesses a fundamental set of solutions in the form

$$w_1(z) = \sum_{k=0}^{\infty} a_k z^{k+\alpha}, \tag{1.10}$$

and

$$w_2(z) = g_0 w_1(z)\log z + z^{\alpha+1}\sum_{k=0}^{\infty} b_k z^k \tag{1.11}$$

if $s = 0$, and

$$w_2(z) = g_s w_1(z)\log z + z^{\alpha'}\sum_{k=0}^{\infty} c_k z^k, \tag{1.12}$$

if $s \neq 0$.

Proof. Let $w_0(z)$ be the first solution (1.10) of the differential equation. The idea of *reduction of order* works as follows. First we introduce a new dependent variable v by setting $w = w_0(z)v$. Then it is easy to verify that v satisfies the differential equation

$$\frac{d^2v}{dz^2} + \Big(\frac{2w_0'}{w_0} + p(z)\Big)\frac{dv}{dz} = 0.$$

Simple integrations of this equation give

$$v(z) = A + B\int^z \frac{1}{w_0(z)^2}\exp\Big(-\int^z p(z)\,dz\Big)\,dz,$$

where A, B are arbitrary constants for which $B \neq 0$. This shows that a second solution around $z = 0$ is given by

$$w(z) = w_0(z)v = w_0(z)\int^z \frac{1}{w_0(z)^2}\exp\Big(-\int^z p(z)\,dz\Big)\,dz.$$

Since α and $\alpha' = \alpha - s$ are roots of the indicial equation

$$\alpha^2 + (p_0 - 1)\alpha + q_0 = 0,$$

so that $p_0 = 1 + s - 2\alpha$. Hence

$$\begin{aligned}
&\frac{1}{w_0(z)^2}\exp\Big(-\int^z p(z)\,dz\Big) = \\
&\qquad \frac{1}{z^{2\alpha}\big(\sum_{k=0}^\infty a_k z^k\big)^2}\exp\Big\{\int^z\Big(\frac{2\alpha-1-s}{z} - p_1 - p_2 z - \cdots\Big)\,dz\Big\} \\
&\qquad = \frac{z^{-s-1}}{\big(\sum_{k=0}^\infty a_k z^k\big)^2}\exp\Big\{-\int^z (p_1 + p_2 z + \cdots)\,dz\Big\} \\
&\qquad = z^{-s-1}g(z),
\end{aligned}$$

where $g(0) = 1/a_0^2$. Since $a_0 \neq 0$, so $\big(\sum_{k=0}^\infty a_k z^k\big)^{-2}$ is also analytic at $z = 0$. Thus, we may write $g(z) = \sum_{k=0}^\infty g_k z^k$. Substituting this series of $z^{-s-1}g(z)$ into $w(z)$ yields

$$\begin{aligned}
w(z) &= w_0(z)\int^z z^{-s-1}\sum_{k=0}^\infty g_k z^k\,dz \\
&= w_0(z)\Big(\sum_{k=0}^{s-1}\frac{g_k z^{k-s}}{k-s} + g_s\log z + \sum_{k=s+1}^\infty \frac{g_k z^{k-s}}{k-s}\Big).
\end{aligned}$$

Thus, we see that if $s = 0$ (double root α), then we have

$$w(z) = g_0 w_0(z)\log z + z^{\alpha+1}\sum_{k=0}^\infty b_k z^k,$$

where $g_0 \neq 0$. The solution has a logarithmic branch point. If, however, $s \neq 0$, then we have

$$\begin{aligned} w(z) &= g_s w_0(z)\log z + z^{\alpha-s}\Big(\sum_{k=0}^{s-1}\frac{g_k z^k}{k-s} + \sum_{k=s+1}^{\infty}\frac{g_k z^k}{k-s}\Big) \\ &= g_s w_0(z)\log z + z^{\alpha'}\sum_{k=0}^{\infty} c_k z^k, \end{aligned}$$

where g_s may be zero. If that is the case then the second solution has no logarithmic term. □

Frobenius later simplified Fuchs's method by introducing his **Frobenius method** in 1873 [2] when he was at the age of twenty-four, even before he presented his *Habilitationsschrift* (thesis). We shall return to its discussion when time permits later.

Exercise 1.11 (Whittaker and Watson)**.**
Show that the equation

$$w'' + \frac{1}{z}w' - m^2 w = 0$$

has power series solutions about $z = 0$ given by

$$w_1(z) = \sum_{k=0}^{\infty}\frac{m^{2k}z^{2k}}{2^{2k}(k!)^2},$$

and

$$w_2 = w_1(z)\log z - \sum_{k=1}^{\infty}\frac{m^{2k}z^{2k}}{2^{2k}(k!)^2}\Big(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{k}\Big),$$

and that these series converge for all z.

Exercise 1.12. Prove that the equation

$$z^2 w'' + (z+1)w' + w = 0,$$

has an irregular singularity at $z = 0$, and show that one can find only one series solution about $z = 0$ with the method discussed in this chapter.

Exercise 1.13. Show that the equation

$$z^3 w'' + z^2 w' + w = 0,$$

has an irregular singularity at $z = 0$, and that it is impossible to find a series solution there.

Exercise 1.14. Show that under what choices of a, b will the Euler equation:

$$z^2 y'' + (1-a-b)zy' + aby = 0,$$

has a double root from its indicial equation at $z = 0$. Can you write down two linearly independent solutions in this case?

Exercise 1.15. Show that the differential equation $y'' + k^2 y = 0$, where k is a constant, has an irregular singularity at ∞.

2. Frobenius's Method

We recall that Fuchs [3] developed a definitive theory of solutions on such equations[1] with finite number of regular singular points in the $\hat{\mathbb{C}}$. There is, however, a shortcoming that Fuchs's method is "long and more difficult than they need be", as quoted from Gray's historical account on Fuchs's theory [4, p. 56]. F. George Frobenius (1849–1917) from University of Berlin proposed an effective method in 1873 to simplify Fuchs's method when he was at the age of twenty-four, even before he presented his *Habilitationsschrift* (thesis). This new method has since been called **Frobenius's method** which is taught in most differential equation courses. The method is particularly effective for dealing with the situation when the indicial equation of the second order ordinary differential equation possesses a double root or two distinct roots but differs by an integer [5].

2.1. **Double roots case.** Let us recall that we write

$$zp(z)=\sum_{k=0}^{\infty}p_kz^k,\qquad z^2q(z)=\sum_{k=0}^{\infty}q_kz^k,$$

where the coefficients p_0, q_0 and q_1 are not all zero. Let

$$w(z)=\sum_{k=0}^{\infty}a_kz^{\alpha+k},$$

where α is a constant to be determined. Multiply z^2 on both sides of the differential equation (1.2) and substitute this expansion $w(z)$ into (1.2) yields, after simplification,

$$L[w]=z^{\alpha}\sum_{k=0}^{\infty}\Big[F(\alpha+k)a_k+\sum_{j=0}^{k-1}a_j\big((\alpha+j)p_{k-j}+q_{k-j}\big)\Big]z^k, \tag{2.1}$$

where we have demanded that

$$F(\alpha+k)a_k+\sum_{j=0}^{k-1}a_j\big((\alpha+j)p_{k-j}+q_{k-j}\big)=0,\qquad k\geq 1 \tag{2.2}$$

holds, and where we have set

$$F(\alpha+k)=(\alpha+k)(\alpha+k-1)+p_0(\alpha+k)+q_0 \tag{2.3}$$

and

$$F(\alpha)=\alpha(\alpha-1)+p_0\alpha+q_0=(\alpha-\alpha_1)(\alpha-\alpha_2) \tag{2.4}$$

when $k=0$. The equation $F(\alpha)=0$ gives the two (characteristic) exponents α_1, α_2 of the indicial equation. Thus, the expression (2.1) becomes

$$L[w]=a_0z^{\alpha}(\alpha-\alpha_1)(\alpha-\alpha_2),$$

so that $L[w]=0$ when $\alpha=\alpha_1$ or $\alpha=\alpha_2$. Hence we have obtained two linearly independent solutions. Suppose that $\alpha=\alpha_1$ is a double root. Then we have

$$L[w]=a_0z^{\alpha}(\alpha-\alpha_1)^2, \tag{2.5}$$

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[1] Fuchs's original theory allows him to deal with higher order linear differential equation of regular singular type.

and one could only find one solution of the form $w(z) = z^\alpha \sum_{k=0}^\infty a_k z^k$. In order to find a second solution, let us consider

$$L\Big[\frac{\partial w}{\partial \alpha}\Big] = a_0 \log z(\alpha - \alpha_1)^2 z^\alpha + a_0 z^\alpha \cdot 2(\alpha - \alpha_1), \tag{2.6}$$

which equals zero when $\alpha = \alpha_1$. Then both $w(z) = z^\alpha \sum_{k=0}^\infty a_k z^k$ and

$$w_2(z) = \Big(\frac{\partial w}{\partial \alpha}\Big)_{\alpha=\alpha_1} = w(z)\log z + z^{\alpha_1} \sum_{k=1}^\infty \Big(\frac{\partial a_k}{\partial \alpha}\Big)_{\alpha=\alpha_1} z^k, \tag{2.7}$$

are solutions to $L[\cdot] = 0$ when $\alpha = \alpha_1$. This is precisely the first case described in Theorem 1.10.

2.2. **Roots differ by an integer.** Suppose that $\alpha_1 = \alpha_2 + s$ for some positive integer s. Then the previous method no longer holds. Note that the coefficient a_0 is a free parameter. We define

$$a_0 = a_0'(\alpha - \alpha_2),$$

where a_0' is now arbitrary. Then we deduce, as before, that

$$L[w] = L[w] = a_0'\, z^\alpha (\alpha - \alpha_1)(\alpha - \alpha_2)^2$$

holds instead. Similarly, we have

$$\begin{aligned} L\Big[\frac{\partial w}{\partial \alpha}\Big] &= a_0'\, z^\alpha (\alpha - \alpha_1)(\alpha - \alpha_2)^2 \log z + a_0' z^\alpha (\alpha - \alpha_2)^2 \\ &\quad + a_0' z^\alpha (\alpha - \alpha_1) \cdot 2(\alpha - \alpha_2). \end{aligned} \tag{2.8}$$

Then it follows that in addition to $(w)_{\alpha=\alpha_1}$ and $(w)_{\alpha=\alpha_2}$, the $\big(\frac{\partial w}{\partial \alpha}\big)_{\alpha=\alpha_2}$ is also a solution to the original differential equation. We first observe that the two power series solutions obtained for $\alpha = \alpha_1$ and $\alpha = \alpha_2$ differ only by a constant multiple. To see this, we note that $a_0 = a_0'(\alpha - \alpha_2)$ by the above construction, so the recurrence relation (2.2) implies that each of $a_1, a_2, \cdots, a_{s-1}$ has a common factor $(\alpha - \alpha_2)$, and thus vanish when $\alpha = \alpha_2$. That is,

$$\big(a_1\big)_{\alpha=\alpha_2}, \big(a_2\big)_{\alpha=\alpha_2}, \cdots, \big(a_{s-1}\big)_{\alpha=\alpha_2} = 0.$$

When $k = s$ in (2.2), then $F(\alpha_2 + s) = F(\alpha_1) = 0$. So a_s is arbitrary . For the a_k when $k > s$, we write $k = k' + s$, $a'_j = a_{j+s}$ are determined by

$$\begin{aligned}
&F(\alpha_2 + k)a_k + \sum_{j=0}^{k-1} a_j \big[(\alpha_2 + j)p_{k-j} + q_{k-j}\big] \\
&= F(\alpha_2 + s + k')a_{s+k'} + \Big(\sum_{j=0}^{s-1} + \sum_{j=s}^{s+k'-1}\Big) a_j \big[(\alpha_2 + j)p_{k-j} + q_{k-j}\big] \\
&= F(\alpha_1 + k')a_{s+k'} + \sum_{j=0}^{k'-1} a_{s+j} \big[(\alpha_1 + j)p_{k-(j+s)} + q_{k-(j+s)}\big] \\
&= F(\alpha_1 + k')a_{s+k'} + \sum_{j=0}^{k'-1} a_{s+j} \big[(\alpha_1 + j)p_{k'-j} + q_{k'-j}\big] \\
&= F(\alpha_1 + k')a'_{k'} + \sum_{j=0}^{k'-1} a'_j \big[(\alpha_1 + j)p_{k'-j} + q_{k'-j}\big],
\end{aligned}$$

for $k' \geq 1$, which is identical to the recurrence relation (2.2). So we have essentially shown that

$$(w)_{\alpha=\alpha_2} = z^{\alpha_2} z^s \sum_{k=0}^{\infty} (a_k)_{\alpha=\alpha_1} z^k = z^{\alpha_1} \sum_{k=0}^{\infty} (a_k)_{\alpha=\alpha_1} z^k$$

holds, where $(a_0)_{\alpha=\alpha_1}$, thus proving that $(w)_{\alpha=\alpha_2}$ is a constant multiple of $(w)_{\alpha=\alpha_1}$.

Thus a second linearly independent solution is given by

$$\begin{aligned}
w_2(z) = \Big(\frac{\partial w}{\partial \alpha}\Big)_{\alpha=\alpha_2} &= \log z \cdot z^{\alpha_2} \sum_{k=s}^{\infty} (a_k)_{\alpha=\alpha_2} z^k + z^{\alpha_2} \sum_{k=0}^{\infty} \Big(\frac{\partial a_k}{\partial \alpha}\Big)_{\alpha=\alpha_2} z^k \\
&= w_1(z) \log z + z^{\alpha_2} \sum_{k=0}^{\infty} \Big(\frac{\partial a_k}{\partial \alpha}\Big)_{\alpha=\alpha_2} z^k \\
&= \Big(z^{\alpha_1} \sum_{k=0}^{\infty} (a_k)_{\alpha=\alpha_1} z^k\Big) \log z + z^{\alpha_2} \sum_{k=0}^{\infty} \Big(\frac{\partial a_k}{\partial \alpha}\Big)_{\alpha=\alpha_2} z^k \\
&= \Big(z^{\alpha_2} \sum_{k=0}^{\infty} (a_k)_{\alpha=\alpha_1} z^{k+s}\Big) \log z + z^{\alpha_2} \sum_{k=0}^{\infty} \Big(\frac{\partial a_k}{\partial \alpha}\Big)_{\alpha=\alpha_2} z^k,
\end{aligned}$$

which is precisely the second case described in Theorem 1.10.

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