Dunkl operators and Clifford algebras

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Lecture series HKUST

Abstract

In the first part we discuss the theory of Dunkl operators. In the second part, we introduce Clifford algebras, define a class of Dunkl Dirac operators and study its properties, with emphasis on the related Fourier transform.

Contents

Introduction

The basic reference to the field of Dunkl operators is the book [15] by Dunkl and Xu. This book focusses mostly on the orthogonal polynomials and special functions related to the theory and also gives many explicit examples.

A short (yet quite complete) introduction to the subject can be found in the lecture notes by Rösler, [23], available on the arXiv.

The lectures are divided in 5 different topics. Most topics will take about one (1-hour) lecture. Topic 3 and 5 will probably take 2 lectures.

1 The rank one Dunkl operator

I'll be following mostly the notes of Cherednik, see [4]. Here, an elementary introduction is given to the Dunkl operator on the real line and the connection is made with the double affine Hecke algebra. Somewhat related work (especially concerning $\mathfrak{sl}_2$, Hankel transforms and special functions) can be found in [19].

Topics include: definition, Dunkl transform, $\mathfrak{sl}_2$ relations, relation with the double affine Hecke algebra, etc.

I will also briefly discuss the complex version of the 1-dimensional Dunkl operator (following [16]), to emphasize the difference with the real case. In the next lectures however, I will restrict myself to the real case.

Finally, I will compare the Dunkl operator with the classical derivative and explain the plan of the subsequent lectures.

2 Dunkl operators in higher rank

First, some basic notions on root systems and reflection groups will be given (see [18]). Using these concepts, I will give the general definition of the Dunkl operators related to such a reflection group.

I will prove that Dunkl operators are commutative (as in the original paper by Dunkl, [12]). I will show that the Dunkl Laplacian (defined similarly as the ordinary Laplacian but with Dunkl operators instead of derivatives) generates a simple algebraic structure, namely the Lie algebra $\mathfrak{sl}_2$ (as first observed by Heckman in [17]). This algebraic structure turns out to be crucial for the sequel, as it allows to transfer several results of classical harmonic analysis to Dunkl analysis, especially those related to orthogonal polynomials.

Finally, there is an important operator, called *intertwining operator*, that allows to transform partial derivatives into Dunkl operators and vice versa. I will discuss its existence and some of the difficult questions related to it.

3 Orthogonal polynomials and special functions in Dunkl theory

I will start with a discussion of Dunkl harmonics, which are generalized spherical harmonics (i.e. polynomial null-solutions of the Dunkl Laplacian), including the Fischer decomposition, orthogonality on the sphere and the related measure, the generalized Funk-Hecke theorem, integration of the intertwining operator, the reproducing kernel etc. (see e.g. [25, 26]).

Next, I will define generalized Hermite polynomials for Dunkl operators (see [21]). In particular I will show that their definition only relies on the $\mathfrak{sl}_2$ relations. I will give basic properties, such as recursion relations, orthogonality, Rodrigues formula. I will also discuss special cases, that include classical Hermite and Laguerre polynomials, types of Jack polynomials etc.

Finally, I will apply these Hermite polynomials to the solution of a type of Calogero-Moser-Sutherland quantum systems. I will determine the spectrum explicitly.

The results will be compared with what happens in the classical case. For a more detailed comparison, including also the symplectic situation, the reader is referred to [5].

4 The Dunkl transform

The Dunkl transform is a generalization of the Fourier transform to an integral transform invariant under a finite reflection group. I will first explain the ideas behind this generalization.

Then I will give several explicit definitions of the transform (direct definition, definition via $\mathfrak{sl}_2$, series approach). I will discuss the eigenfunctions and spectrum of the transform, give Bochner relations (i.e. relation with the one dimensional Hankel transform). This summarizes results obtained in [13, 10, 2].

Next, I will show how one can define a generalized translation related to the Dunkl transform (see e.g. [22, 24]). In particular I will compute the translation of a radial function in terms of classical translation and the intertwining operator (running ahead of some results obtained in [9]).

5 Clifford algebras and Dunkl operators

In this lecture, I will first introduce Clifford algebras. I will give the definition and show how it is possible to construct finite reflection groups using Clifford algebras. I will discuss the relation with $(s)pin$ groups. References are [11] and, from a more algebraic point of view, [1].

Next, I will define the Dunkl Dirac operator, which factorizes the Dunkl Laplacian. I will show that the $\mathfrak{sl}_2$ relations are now further refined to $\mathfrak{osp}(1|2)$ relations (a so-called Lie superalgebra). This was first observed for the Dunkl case in [20]. I will also give some results on Dunkl monogenics (a refinement of Dunkl harmonics, which are polynomial null-solutions of the Dunkl Dirac operator).

Then I will discuss recent work on further deformations of Dunkl operators. I will start by explaining the radial deformations introduced in [3], where the Dunkl Laplacian is weighted with a radial factor. I will explain how in this situation again a Fourier-type transform arises and how its kernel can be obtained.

After that, I will construct similar radial deformations of the Dunkl Dirac operator, following [7, 8]. I will show how we can associate a weight to this deformation and how the associated L_2 space has a nice basis consisting of generalized Hermite functions (which are further generalizations of the Hermite functions introduced in lecture 3). I'll continue by discussing the Hermite semigroup associated to this deformation. I will also show how to obtain the master formula (in the sense of Cherednik) for this semigroup.

If time allows, some other topics can be studied, such as: more results on generalized Fourier transforms and translation operators in relation to Clifford algebras (see [9, 6]), relation with the Dirac operator in the Hecke algebra and Vogan's conjecture (see [1]), etc.

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