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# Circle Measurements in Ancient China

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This paper discusses the method of Liu Hui (3rd century) for evaluating the ratio of the circumference of a circle to its diameter, now known as $\pi$. A translation of Liu's method is given in the Appendix. Also examined are the values for $\pi$ given by Zu Chongzhi (429–500) and unsurpassed for a millenium. Although the method used by Zu is not extant, it is almost certain that he applied Liu's method. With the help of an electronic computer, a table of computations adhering to Liu's method is given to show the derivation of Zu's results. The paper concludes with a survey of circle measurements in China. © 1986 Academic Press, Inc.

本文探讨公元263年刘徽如何求得现称为π的圆周率值及结论·刘徽的方法原文，以英文译出，作为附录·祖冲之（429－500）继后所得的圆周率值，其精密性，要等到一千年后才超越它·虽然演算的原文已经遗失，但他采用刘徽的方法，则几乎可以肯定·按照刘徽的方法，我们可以通过电脑得出一个计算表，说明祖率的来源·我们也对中国历代有关圆量法，加以讨论，作为本文的结束· © 1986 Academic Press, Inc.

Dieser Aufsatz erörtert Liu Huis (3. Jahrhundert n. Chr.) Methode, das Verhältnis des Umfanges eines Kreises zu seinem Durchmesser zu berechnen, das heute als $\pi$ bekannt ist. Der Anhang enthält eine Übersetzung von Lius Methode. Ebenso werden die Werte von Zu Chongzhi (425–500) für $\pi$ geprüft, die tausend Jahre lang nicht überboten wurden. Obwohl die von Zu verwandte Methode nicht mehr existiert, ist es fast sicher, daß er sich Lius Methode bediente. Mit Hilfe eines elektronischen Rechners wird eine Berechnungstabelle beigefügt, die sich an Lius Methode anlehnt, um die Ableitung von Zus Ergebnissen zu zeigen. Der Aufsatz schließt mit einem Uberblick über die Kreismessungen in China. © 1986 Academic Press, Inc.

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China, like any other early civilization, had its fair share of men who tried to find as accurately as possible the area or the circumference of a circle. Two men stand out prominently among these: Liu Hui [a][1] of the 3rd century and Zu

[1] Lowercase letters in brackets indicate a Glossary listing (at the end of the paper).



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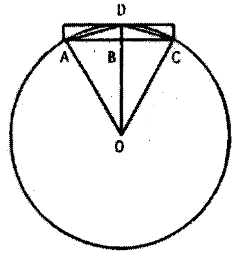


FIGURE 1

Chongzhi [b] of the 5th century. This paper first discusses the contributions of Liu and Zu to calculation of the ratio of the circumference to the diameter, now known as $\pi$, and their significance, and, second, offers a general survey of the evaluation of this ratio in China.

## LIU'S METHOD

Problems 31 and 32 in chapter 1, entitled *fang tian* [c] (mensuration of fields), of the *Jiu zhang suanshu* [d] (*Nine chapters of the mathematical art*) [1] assume the area of a circle is half the circumference times half the diameter. In his commentary on problem 32, Liu Hui explained the derivation of this formula, discussed why the ratio of the circumference to the diameter was generally taken as 3, and then derived a more precise value for the ratio. Liu's commentary—divided into three sections for ease of reference in this paper—is summarized below and offered in translation in the Appendix [2].

*Section 1.* Start with a regular hexagon inscribed in a circle of radius 1 *chi* [e]. The product of one side of the hexagon, the radius, and 3 ($= \frac{1}{2} \times 6$) gives the area of an inscribed dodecagon. Repeat this process by taking the product of one side of the dodecagon, the radius, and 6 ($= \frac{1}{2} \times 12$) to obtain the area of an inscribed polygon of 24 sides. According to Liu's principle of exhaustion, if the process is repeated long enough, eventually a polygon will be reached whose sides are so short that it will coincide with the circle. This explains why the area of the circle is the product of half the circumference and the radius. The value of 3 to 1 for the ratio of the circumference to the diameter is imprecise as this is in fact the ratio of the perimeter of the hexagon to the diameter. However, this inaccuracy was passed down from generation to generation, Liu explains, because of an unwillingness to strive for accuracy.

*Section 2.* This section summarizes the stages of Liu's computation of a more precise value of the ratio. Let the side of an $n$-sided polygon $= a_n$ and its area $= A_n$. In Fig. 1, $n = 6$,

$$OC = \text{radius} = r = 10, \qquad BC = \tfrac{1}{2}a_6 = 5,$$

$$OB = b_6 = \sqrt{r^2 - \tfrac{1}{4}a_6^2} = 8.660254, \qquad DB = c_6 = r - b_6 = 1.339746,$$

$$DC^2 = a_{12}^2 = c_6^2 + \tfrac{1}{4}a_6^2 = 26.7949193445.$$

The same process is repeated for a dodecagon of side $a_{12}$ as follows.

$$b_{12} = \sqrt{r^2 - \tfrac{1}{4}a_{12}^2} = 9.659258, \qquad c_{12} = r - b_{12} = 0.34742,$$

$$a_{24}^2 = c_{12}^2 + \tfrac{1}{4}a_{12}^2 = 6.8148349466.$$

The process continues:

$$b_{24} = 9.914448, \qquad c_{24} = 0.085552, \qquad a_{48}^2 = 1.7110278813,$$

$$a_{48} = 1.30806, \qquad A_{96} = 24ra_{48} = 313\tfrac{584}{625}.$$

$$b_{48} = 9.978589, \qquad c_{48} = 0.021411, \qquad a_{96}^2 = 0.4282154012,$$

$$a_{96} = 0.65438, \qquad A_{192} = 48ra_{96} = 314\tfrac{64}{625}, \qquad A_{192} - A_{96} = \tfrac{105}{625},$$

$$314\tfrac{64}{625} = A_{192} < A < A_{96} + 2(A_{192} - A_{96}) = 314\tfrac{169}{625},$$

where $A$ = the area of the circle.

Hence $A \approx 314$ (to the nearest integer) and $S:A:s = 200:157:100$, where $S$ = the area of the square circumscribing the circle and $s$ = the area of the square inscribed in the circle. Therefore circumference/diameter $= \frac{157}{50}$.

*Section 3.* $A = A_{192} + \frac{36}{625} = 314\frac{4}{25}$. There is no clear explanation of how $\frac{36}{625}$ is derived apart from the fact that it bears some relation to the residual area $A_{192} - A_{96} = \frac{105}{625}$. It follows that $S:A:s = 5000:3927:2500$ and circumference/diameter $= \frac{3927}{1250}$. It is stated that this ratio can be verified by computing $a_{1536}$ and hence deriving $A_{3072}$.

## CONCLUSIONS FROM LIU'S METHOD

1. A problem in the Ahmes Papyrus (from ancient Egypt) gave the numerical area of a circle, and the excavations at Susa in 1936 revealed the old Babylonians' calculation of the circumference of a circle in relation to its inscribed hexagon [Neugebauer 1952, 47]. In instances such as these, historians calculate the value of $\pi$ from the area or other recorded figures. Archimedes [Heath 1897, 93–98] and Liu Hui, however, are the only men from ancient history whose methods for obtaining the ratio of the circumference of a circle to its diameter are known to modern historians.

2. Both Archimedes' and Liu's methods employed regular polygons inscribed in the circle. Archimedes also had polygons circumscribing the circle. Both men assumed the principle of exhaustion, holding that eventually a polygon will be reached whose sides are so short that it will coincide with the circle. This principle was first stated by the Greek philosopher, Antiphon, in the 5th century B.C. when he began with a square inscribed in a circle [Heath 1921 I, 222].

3. The genius of Archimedes is displayed in his method. Without knowledge of subjects such as decimal notation and trigonometry, he was able to devise a method with inscribed and circumscribed polygons of 96 sides which gave the



values of $\pi$ as greater than $3\frac{10}{71}$ but less than $3\frac{1}{7}$. His method of calculating the approximate values of irrational square roots is still in the realm of speculation. Compared with Archimedes' method, then, Liu's method is simple and elegant. He used only inscribed polygons, and each stage of his derivation is clear.

4. Archimedes' method is solely concerned with the evaluation of the perimeters of the inscribed and circumscribed polygons from which the circumference of the circle is deduced. It does not draw any conclusions about the area of the circle in relation to the ratio $\pi$ [3]. By showing that the area of a circle is the product of half the circumference and the radius, Liu's method proved that the ratio of the area of a circle to the square of its radius is identical to the ratio of the circumference to the diameter, or, in other words, $\pi$.

5. One of the reasons for the simplicity of Liu's method is that he inherited a tradition using a decimal number system. The existence of a word–numeral decimal system in China can be traced to the oracle bone characters of the Shang dynasty [Needham 1959, 12–13]. Since the Warring States period (480 to 221 B.C.) counting rods, manifesting the place value of a decimal number system, were used for computation [Ang 1977, 97–98]. Liu's work serves as a fine example of the depth of the ancient Chinese understanding and handling of large numbers and decimal fractions as early as the 3rd century. In his text each numeral has a place name and on the counting board [4] each rod numeral has a place position relative to the other rod numerals. In the text, integral places of order $10, 10^2, 10^3, \ldots, 10^{11}$ are called *shi* [f], *bai* [g], *qian* [h], *wan* [i], *shi wan* [j], *bai wan* [k], *qian wan* [l], *yi* [m], *shi yi* [n], *bai yi* [o], and *qian yi* [p], respectively. For a decimal fraction, the names of the first five decimal places are given as *fen* [q], *li* [r], *hao* [s], *miao* [t], and *hu* [u], respectively. In calculating a value to more than five decimal places, Liu remarked that the "minute numbers" had no place names and so the numeral in the sixth decimal place had to be converted to a fraction. For example, $b_6$ = 8 *cun* [v] [5] 6 *fen* 6 *li* [0 *hao*] 2 *miao* $5\frac{2}{5}$ *hu* (see the Appendix, Section 2). In this fashion, the lengths of all $b_n$'s are truncated at the sixth decimal place. This ensures a certain degree of accuracy for the values of subsequent $a_n^2$'s which are truncated at the tenth decimal place. Without the "modern" notational decimal point, each set of $a_n^2$ numerals is considered by Liu in square *hu* [u] units. For computing the area $A_{2n}$, the value of $a_n$ is obtained from $a_n^2$ by the square root method and is truncated at the fifth decimal place. For example, $a_{96}^2 = 4282154012$ square *hu* and $a_{96}$ = 6 *fen* 5 *li* 4 *hao* 3 *miao* 8 *hu* (see the Appendix, Section 2). The ancient Chinese had devised a method for computing the square root of any number to any degree of accuracy. The earliest record of this method is found in the *Jiu zhang suanshu* [Qian 1963, 150; Wang & Needham 1955, 350–365]. Liu was therefore well aware that a number of $2n$ digits would give a square root with a number of $n$ digits or *vice versa*.

6. Liu's method, written for computation by the counting rod system, illustrates the practicality and immense potential of this computational device. With this system, abstract ideas had to be transformed into concrete ones for the handling of counting rods on the counting board. Like the modern computer, the counting rod system encouraged algorithms such as the one devised by Liu. The first

decimal place was called *fen* [q] and, on the counting board, digits of the first and subsequent decimal places were designated to specified positions. The concept of decimal fractions existed, as it was merely an extension of the integral number system. In the same way the concept of zero in a notational form existed. If there were zero or no digit in a particular place value, the designated position on the counting board for that place value was left blank. P. Beckmann remarks that the Chinese discovery of the equivalence of the digit zero made them "far better equipped for numerical calculations than their western contemporaries" [Beckmann 1970, 27].

7. Both Archimedes and Liu had discovered methods that would enable men of later generations to calculate $\pi$ to any desired degree of accuracy. With these methods, the number of decimal places to which $\pi$ could be calculated was merely a matter of computational ability and perseverance. Thus, in 1593 François Viète and Adriaen van Roomen used Archimedes' method to calculate $\pi$ to 9 and 15 decimal places, respectively. A few years later Ludolph van Ceulen computed $\pi$ to 35 decimal places [Beckmann 1970, 98–99]. There is strong evidence that Zu Chongzhi used Liu's method to obtain his estimates of $\pi$. A mathematician, familiar with the counting rod system and as talented as Zu, would have no difficulty in applying Liu's method to an enlarged number or extending the number of decimal places in the computation and thereby calculating $\pi$ to a higher degree of accuracy than Liu.

## ZU CHONGZHI'S VALUES FOR $\pi$

Zu Chongzhi's values for $\pi$ were not surpassed until a millenium later when al-Kashi evaluated $\pi$ correctly to 16 decimal places [Youschkevitch & Rosenfeld 1973, 258]. It is interesting to note that Zu's fractional value of $\pi$ in the form $\frac{355}{113}$ was also given by the Indians in the 15th century [Gupta, 1975, 3] and by Adriaan Anthoniszoon in the 16th century [Beckmann 1970, 98].

As an addendum to Liu Hui's commentary, Li Chunfeng [w] stated that Zu considered Liu's ratios inaccurate and therefore proposed to compute further [Qian 1963, 106]. Zu's concern for a better approximation for $\pi$ was essentially due to his desire for the compilation of an astronomical system for the empire. There is no doubt that Zu incorporated the method for approximating $\pi$ into his mathematical text, *Zhui shu* [x] (*Method of mathematical composition*), which is not extant. There remain now but quotations in the official histories. For example, the *Sui shu* [y] (*Standard history of the Sui dynasty*) [387–388] relates:

> In ancient mathematics, the ratio of the circumference to the diameter was taken to be 3 to 1 but this was only a rough estimate. Though various efforts had been made by Liu Xin [z], Zhang Heng [aa], Liu Hui, Wang Fan [ab] and Pi Yanzong [ac], the results obtained so far still lacked precision. Towards the end of the [Liu] Song period (420–479), Zu Chongzhi, a historian of Nanxu [ad] district, found further approximations. He took 100000000 [units] as 1 *zhang* [ae] along the diameter of a circle [of length 2 *zhang*] and found an upper value of 3 *zhang* 1 *chi* 4 *cun* [5] 1 *fen* 5 *li* 9 *hao* 2 *miao* 7 *hu* and a lower value of 3 *zhang* 1 *chi* 4 *cun* 1 *fen* 5 *li* 9 *hao* 2 *miao* 6 *hu* for the circumference [saying that] the true value must lie between the upper and lower limits. His "very close" ratio (*mi lu* [af]) was 355 to 113 and the "approximate" ratio (*yue lu* [ag]) was 22 to 7.



It is to be noted from the above account that numbers were enlarged or, in other words, extended to the left rather than the right in order to attain greater accuracy in calculation. Thus Zu used a radius of 1 *zhang*, taken as 100000000 units, while in Liu's figure the radius 1 *chi* equaled 1000000 *hu*. Obviously, it is easier to deal with larger whole units than with decimal fractions extended to more decimal places. From knowledge of Liu's method and Li Chunfeng's [w] statement, it seems logical to infer that Zu's method of finding the value of $\pi$ between 3.1415926 and 3.1415927 was based on Liu's theory. Based on this assumption, the present authors proceeded to do the calculations on an electronic computer. Care was taken to devise a program for the computer which adhered to the method used by Liu Hui [6]. The following findings resulted.

1. According to Liu's method the values of $b_n$, $a_n^2$, and $a_n$ are truncated at the sixth, tenth, and fifth decimal places, respectively. If Liu's procedure is extended and strictly followed, it is impossible to obtain Zu's estimates irrespective of the value of $n$. The simple reason is that Zu's estimates for $\pi$ are up to the seventh decimal place.

2. If an adjustment is made to enlarge the radius from Liu's 10 units to 1000 units as specified in the *Sui Shu*, then the values of $b_n$ and $a_n^2$ can be truncated, as in Liu's method, at the sixth and tenth decimal places, respectively. The procedure terminates when $n = 6144$, and, to obtain Zu's values for $\pi$, the values of $a_{6144}$ and $a_{12288}$ have to be truncated at the eighth and ninth decimal places, respectively. This is a deviation from Liu's method, where values of $a_n$ are truncated at the fifth decimal place. However, an extension of decimal places should not have involved any difficulty for Zu. The figures obtained are shown in the table below.

$b_6 = 866.025403$
$c_6 = 133.974597$
$a_{12}^2 = 267949.1926413124$

$b_{12} = 965.925826$
$c_{12} = 34.074174$
$a_{24}^2 = 68148.3474941103$

$b_{24} = 991.444861$
$c_{24} = 8.555139$
$a_{48}^2 = 17110.2772768368$

$b_{48} = 997.858923$
$c_{48} = 2.141077$
$a_{96}^2 = 4282.1535299291$

$b_{96} = 999.464587$
$c_{96} = 0.535413$
$a_{192}^2 = 1070.8250495627$

$b_{192} = 999.866137$
$c_{192} = 0.133863$
$a_{384}^2 = 267.7241816933$

$b_{384} = 999.966533$
$c_{384} = 0.033467$
$a_{768}^2 = 66.9321654633$

$b_{768} = 999.991633$
$c_{768} = 0.008367$
$a_{1536}^2 = 16.7331113724$

$b_{1536} = 999.997908$
$c_{1536} = 0.002092$
$a_{3072}^2 = 4.1832822195$

$b_{3072} = 999.999477$
$c_{3072} = 0.000523$
$a_{6144}^2 = 1.0458208283$
$a_{6144} = 1.02265381$

$b_{6144} = 999.999869$
$c_{6144} = 0.000131$
$a_{12288}^2 = 0.2614552241$
$a_{12288} = 0.511326924$

From the above table and by Liu's method, where $r$ = the radius, we have

$$A_{2288} = r \times a_{6144} \times \tfrac{1}{2} \times 6144 = r \times 3141.592504$$
$$= r \times 3141.5925 \qquad \text{(truncated at the fourth decimal place)}$$
$$A_{24576} = r \times a_{12288} \times \tfrac{1}{2} \times 12288 = r \times 3141.592621$$
$$= r \times 3141.5926 \qquad \text{(truncated at the fourth decimal place).}$$

Again by Liu's method,

$$A_{24576} < A < A_{12288} + 2(A_{24576} - A_{12288})$$

$$3.1415926 < \frac{A}{r^2} < 3.1415927,$$

where $A$ = the area of the circle.

The degree of accuracy for the approximation of $\pi$ depends on the number of places to which $b_n$ is calculated, and thereafter appropriate extensions of places are performed on $c_n$, $a_n^2$, and $a_n$. All these operations were known to the Chinese and for Zu these would have posed no problem except perhaps perseverance. Moreover, Liu's mathematical works were known to Zu. For instance, in one of the extant fragmentary records, it is noted that Zu and his son Zu Geng [ah] completed the well-known proof of the derivation of the volume of a sphere which was left unfinished by Liu [Lam & Shen 1985].

How Zu obtained what he called his "very close" ratio $\frac{355}{113}$ for $\pi$ is not known. This value is correct to the sixth decimal place, and as for the value of the seventh decimal place a better indication is obtained from his other figures of 3.1415926 and 3.1415927.

## A SURVEY OF CIRCLE MEASUREMENTS IN CHINA

Like all other early civilizations, the ancient Chinese took the value of the ratio of the diameter to the circumference as 3 in their mathematical calculations [7]. As time went by, they realized that this value was a rough approximation, and so tried to improve it. Liu Xin [z], an astronomer and calendar expert of the first century B.C., was said to have been one of the earliest to attempt the improvement. This was first mentioned in the *Sui shu* [y] [387–388], which did not provide any mathematical procedure and hence has led historians of mathematics to look for evidence elsewhere.

When Wang Mang [ai] ascended the throne toward the end of the Western Han (206 B.C.–A.D. 24), he commanded Liu Xin to construct a standard measure for the kingdom. Liu Xin produced a vessel cut from a solid bronze cylinder and called it the *Jia liang hu* [aj]. (For the words of the inscription on it, see the Appendix, Section 3.) It was estimated that about a hundred *Jia liang hu* were made for distribution throughout the entire empire [Sun 1955, 11]. One such vessel is still being kept in the Palace Museum in Beijing. Chinese historians of mathematics who had the opportunity to examine the vessel, including Li Yan [ak], Qian



Baocong [al], Li Naiji [am], Sun Zhifu [an], and Xu Chunfang [ao], thought that Liu Xin had contributed a new value of $\pi$. Their conclusion was drawn from the following procedure:

> The diameter of the measure = $\sqrt{200} + 2 \times 0.095 = 14.332$ cun. Since the area is given as 162 sq. cun, $\pi(14.332/2)^2 = 162$ or $\pi = 3.1547$.

This argument of attributing the new value of $\pi$ to Liu Xin by an inverse operation does not seem convincing. In a recent paper on the same subject Bai Shangshu [ap] [1982, 75–79] pointed out that in his study of six different kinds of standardized vessels for smaller capacities made by Wang Mang [ai], he found four different values of $\pi$, namely 3.1547, 3.1590, 3.1497, and 3.1679. From this inconsistency, Bai thought that it was unreasonable to suggest that Liu Xin assumed $\pi = 3.1547$. Yet Liu Xin, being an astronomer and calendar expert, would certainly not have used the ancient ratio of 3 in his mathematical calculations. His task of constructing a standard measure called for precision, particularly in dealing with solid objects. Furthermore, *Sui shu* [y] [387–388] says that Liu Xin was prompted to find a new value of $\pi$ to replace the old one. Thus, although the theoretical record on his approach to the approximation of $\pi$ is still wanting, it is not too farfetched to suggest that Liu Xin did have some kind of improved value of $\pi$ before constructing the standard vessels.

About a century later, Zhang Heng [aa] (A.D. 78–139) made the first explicit effort to obtain a more accurate figure for $\pi$. The information comes from Liu Hui's commentary on a problem in the *Jiu zhang suanshu* [d] regarding the derivation of a diameter of a sphere from its volume [Qian 1963, 156]. Liu Hui pointed out that during the time of Zhang Heng the ratio of the area of a square to the area of its inscribed circle was taken as 4:3. Following the empirical ratio, it was thought that the volume of the cube to the volume of the inscribed sphere must also be in the ratio $4^2:3^2$, that is, $D^3:V = 16:9$ or $V = \frac{9}{16}D^3$, where $D$ is the diameter of the sphere and $V$, its volume. Hence, the formula for finding the diameter of a sphere from its volume is given in the *Jiu zhang suanshu* as $D = \sqrt[3]{16V/9}$. Zhang Heng realized that the value of the diameter obtained in this way fell short of the real value, and he attributed the discrepancy to the value taken for the ratio. He thought that this error could be corrected by adding an arbitrary value of $\frac{1}{16}D^3$ to the original formula, thus rectifying it to $V = \frac{9}{16}D^3 + \frac{1}{16}D^3 = \frac{5}{8}D^3$. This means that the ratio of the volume of the cube to that of the inscribed sphere is 8:5 and implies that the ratio of the area of the square to that of the circle is $\sqrt{8}:\sqrt{5}$. From this, $\pi$ was calculated as $\sqrt{10}$.

According to a reference by Zu Geng [ah] cited in the *Kaiyuan zhan jing* [aq] (*Kaiyuan treatise on astrology*) [25b, 26a] of the 8th century, Zhang Heng compared the celestial circle to the width (i.e., diameter) of the earth in the proportion of 736 to 232, which gives $\pi$ as 3.1724.

As both of Zhang Heng's values were on the high side, Wang Fan [ab] (217–257) investigated further the value of $\pi$. The *Song shu* [ar] (*Standard history of Liu Song dynasty*) [675] says that, "having tested that 1 was a little long as diameter for the circumference 3, he [Wang Fan] corrected the circumference to 142 and diameter to 45." This gives the value of $\pi$ as $\frac{142}{45}$ or 3.155. While Wang Fan's method of arriving at such a figure was not given in the historical sources, Yan Dunjie [as] [1936a, 39–40] suggests that it could have been obtained soon after Liu Hui proposed the value of 3.14 for $\pi$. Since Wang Fan thought $\pi$ greater than 3, a small fraction $x$ should be added to 3, with $\pi = 3 + x$. As Liu Hui considered 3.14 a slightly low value for $\pi$, a small fraction $y$ should be added to this, giving $\pi = 3.14 + y$. Taking $y = \frac{1}{10}x$, we have $3 + x = 3.14 + \frac{1}{10}x$, resulting in $x = \frac{7}{45}$ or $\pi = \frac{142}{45}$.

Yan Dunjie further suggested that perhaps the value of $\sqrt{2}/0.45$ for $\pi$ given by Zhu Zaiyu [at] toward the end of the Ming dynasty could have been derived from Wang Fan's value. This is only a hypothesis. What appears certain, however, is that after the Han period there was considerable interest in a plausible method for approximating $\pi$ based on theoretical foundation. The mathematician who in A.D. 263 succeeded in giving one was Liu Hui.

Liu strove for precision and refused "to follow the ancients" (*zhong gu* [au]). He aimed at "cutting the circle" continuously until "a limit is reached when the shape of the polygon coincides with that of the circle" so that the exact value might be attained. As a pragmatic mathematician he advocated the value of $\pi = \frac{157}{50}$, but, as a theoretician, he believed that the true value of $\pi$ might be approached as closely as possible by successive approximations. Whether Liu succeeded in "cutting the circle" to the extent of attaining a 1536-sided polygon or not is open to speculation. In *Jiu zhang suanshu xichao tu shuo* [aw] (*Detailed diagrammatic explanations of the "Jiu zhang suanshu"*), Li Huang [av] (d. 1811) was the first to suggest that the ratio 3927 : 1250 was not Liu's contribution but Zu Chongzhi's. This sparked a great controversy involving several eminent historians of mathematics. The more cautious ones, including Li Yan [ak], Yan Dunjie [as], Du Shiran [ax], and He Luo [ay], recognized Liu's derivation of $\pi$ only up to the 96-sided polygon and completely avoided the mention of $\frac{3927}{1250}$. Those who fervently believed that Liu had established $\frac{3927}{1250}$ for $\pi$ were Qian Baocong [al], Xu Chunfang [ao], Wang Shouyi [az], Bai Shangshu [ap], Hua Luogeng [ba], He Shaogeng [bb], Mei Rongzhao [bc], Shen Kangshen [bd], and Li Naiji [am]. Opposing this view and crediting Zu Chongzhi with the invention were the mathematicians Yu Ningsheng [be], Yu Jieshi [bf], Sun Zhifu [an], Li Di [bg] [8], and Donald Wagner [1978, 206–208].

What appears ambivalent in Liu's commentary is the mention of *Jia liang hu* [aj] in the Jin armory. Liu was said to have written his commentary two years before the Wei [bh] kingdom was usurped by Sima Yan [bi], who established the Jin [bj] kingdom (A.D. 265–420). But whether this was a complete commentary on the whole text of *Jiu zhang suanshu* [d] is open to question. In fact elsewhere in *Sui shu* [y] [429], it says that Liu was making a study and a comparison of the *Jia liang hu* and the *hu* measure of his time when he wrote the commentary on chapter 5 (entitled *shang gong* [bk]) of the *Jiu zhang suanshu* in 263. Nevertheless, Liu continued to use $\pi = \frac{157}{50}$ for all the eight problems involving spherical



and circular measurements in that chapter. Based on the fact that the same value for $\pi$ was also used in the previous chapters (ten problems in chapter 1 and two problems in chapter 4), it may be assumed that Liu might not have had the opportunity of seeing a *Jia liang hu* before he set out to write his commentary on the *fang tian* chapter. This note about the *Jia liang hu* in the *fang tian* chapter could have been an addendum when Liu made a revision and updating of the whole commentary soon after the collapse of the Wei kingdom in 265. Therefore, as a mathematician whose life straddled two kingdoms, it was proper for him to mention the armory where the *Jia liang hu* was kept as the Jin armory. The revision would have spurred him to make a thorough investigation and calculation of the dimension of the *Jia liang hu*. He would have discovered that the vessel was not a perfect measure and that the value for $\pi$ obtained from the vessel was almost the same as that he had derived earlier on. This realization led him to improve the value for $\pi$ by continuing the process of calculation of areas of polygons up to 192 sides. He finally arrived at $\pi = \frac{3927}{1250}$ and was satisfied with it. It is no surprise that the great mathematician of the Tang dynasty Wang Xiaotong [bl] praised Liu as one who could stretch his "thought to the minutest detail" (*si ji hao mang* [bm]) [Qian 1963, 493].

Zu Chongzhi (429–500) came from a bureaucratic family of calendar experts [9]. The calendrical system of his time was the Yuanjia li [bn], compiled by Ho Chengtian [bo] (370–447). Zu found the system inaccurate and therefore unsuitable for civil use. He made two very bold attempts to rectify the system, by taking the precession of equinoxes (*sui cha* [bp]) into consideration, and by deviating from the traditional 19-year cycle (i.e., the Metonic Cycle) by putting 144 intercalary months in every 391 years. To gather the necessary data he needed precise mathematical techniques. It was for this reason that he had to reexamine the mathematical constants used in his computations.

Zu's expertise in astronomy and mathematics was unquestionable. Apart from successfully composing an astronomical system known as the *Daming li* [bq], he also wrote a mathematical text called *Zhui shu* [x], which was prescribed as a textbook for advanced students of mathematics in the official academies of the Tang dynasty. While the treatise on the *Daming li* is still available in the *Song shu* [ar] [192 ff.], the *Zhui shu* was purported to have been lost toward the end of the Northern Song dynasty (960–1127). The fact that students in the Tang academies had to devote four out of their seven years to the study of this text suggests that it must be an important and difficult treatise on mathematics. Ironically, as *Sui shu* [y] [388] says, "the official students were unable to understand the profoundity of the text so it was subsequently abandoned and neglected altogether."

Qian Baocong [1923, 56–57] and Sun [1955, 9] assigned Liu credit for $\pi = \frac{3927}{1250}$ and believed that Zu continued where Liu left off. The "very close" value of $\pi = \frac{355}{113}$ by Zu was a great advance in the history of mathematics (see [Yan 1936b, 518–519]). Mikami [1913, 50] even suggested that it should be named after Zu. As regards the familiar Archimedean ratio of $\frac{22}{7}$, the general consensus is that it was not Zu's discovery [10]. Prior to Zu it was thought that either Ho Chengtian [bo] or Pi Yancong [ac] (fl. 445) had already found a value of $\pi = 3.1428$ through an interest in astronomy. What is intriguing is the derivation of the "very close" value of $\pi = \frac{355}{113}$. Ho had earlier invented a "method of averaging days" (*tiao ri fa* [br]) in his astronomical system. He took two fractions, one known as the "strong ratio" and the other as the "weak ratio," to determine the fractional day part for the synodic period. It has been suggested by Qian [1923, 57–58] that Zu might have used a similar method, taking Liu's $\frac{157}{50}$ as the "weak ratio" and Ho's $\frac{22}{7}$ as the "strong ratio" such that $\frac{157}{50} < \pi < \frac{22}{7}$. By using Ho's method of averaging, he would have obtained the following ratio after the ninth iteration:

$$\frac{157 + (9 \times 22)}{50 + (9 \times 7)} = \frac{355}{113}.$$

Whether Zu obtained his "very close" ratio by the method suggested above is questionable. Apart from the extreme precision that was required for specific purposes, such as in calendrical calculations, mathematicians in general were quite happy with the approximate ratio of $\frac{22}{7}$. Li Chunfeng [w], for example, employed $\pi = \frac{22}{7}$ in almost all his mathematical commentaries. One can, in fact, say that after Zu Chongzhi the development of $\pi$ was in limbo. It was not until the turn of the 14th century that new scholars probed into the value of $\pi$. One such person was the Yuan mathematician Zhao Youqin [bs], who returned to the question of Zu's approximation for $\pi$. Following Liu's use of inscribed polygons, Zhao Youqin continued the process persistently and reached a regular polygon of 16,384 ($= 4 \times 2^{12}$) sides to derive $\pi = 3.1415926$, thus confirming Zu's accuracy [Ruan 1799, 333–345].

Some interest in the evaluation of $\pi$ was evidenced by the Ming mathematicians. Toward the end of the 16th and the beginning of the 17th century, Zhu Zaiyu [at] found $\pi = \sqrt{2}/0.45$ and $\pi = 3.1426968$, while Xing Yunlu [bt] adopted $\pi = 3.1126$ and $\pi = 3.12132034$. Then Chen Jinmo [bu] and Fang Yizhi [bv] used $\pi = 3.1525$ and $\pi = \frac{52}{17}$, respectively. Apart from these, there were other values such as $\frac{63}{20}$ and $\frac{25}{8}$. None of these was, however, as accurate as Zu's value [11].

There was another attempt to exhaust the value of $\pi$ by the method of "cutting the circle." This is found in chapter 15 of *Shu li jing yun* [bw] (*Collected basic principles of mathematics*), which was commissioned by Emperor Kang Xi [bx] and edited by Mei Gucheng [by] and He Guozong [bz]. Starting from an inscribed hexagon of radius $10^{12}$, the mathematicians here found that the length of one side of a regular $6 \times 2^{23}$-sided polygon was 121, with the sum of sides being 6283185307179. Again starting with a circumscribed regular hexagon and going up to the same number of sides, they found that the side and sum of sides remained the same. Hence, it was then fixed that $\pi = 3.14159265$ (a value correct to eight decimal places).

Following the arrival of the Jesuits, the traditional approach to the evaluation of $\pi$ came to a halt. Mathematicians, like Ming Antu [ca] (d. 1765), Xiang Mingda [cb] (1789–1850), Li Shanlan [cc] (1811–1882), and Zeng Jihong [cd] (1848–1877), began to evaluate $\pi$ by analytical methods.



## APPENDIX: Translation of Liu's Method on Circle Measurement [Qian 1963, 103–106]

### [*Section 1*]

If half the circumference [of a circle] is the length and half the diameter is the width, then the product of the length and width gives the area [of a circle]. Let the diameter of the circle be 2 *chi* [e]. The length of one side of a hexagon inscribed in the circle is equal to the radius. The ratio of the diameter to the perimeter [of the hexagon] is 1 to 3. Next, referring to the diagram [12], if the radius is multiplied by one side of the hexagon and then by 3, the product obtained is the area of an inscribed dodecagon. If the circle is cut (*ge* [ce]) again so that the radius is multiplied by one side of the dodecagon and then multiplied by 6, the product obtained is the area of an inscribed polygon of 24 sides. The more finely [the circle] is cut, the less loss there is [in area]. Cut it again and again until one is unable to cut further, that is, when the shape [of the polygon] coincides with that of the circle and there is no loss [in area]. Beyond one side of a polygon [from its mid-point to the circle], there is a remaining portion of the diameter. Multiply this remaining portion by a side of the polygon to give a rectangular area which extends beyond the arc [of the circle]. When a side of the inscribed polygon becomes so fine that it finally merges with the [circumference of the] circle, then there is no remaining portion of the diameter. When no remaining portion of the diameter is shown, this means that the area does not protrude out. When a side [of an inscribed polygon] is multiplied by the radius, [this implies that] in the process of extending the polygon, the number of sides is doubled. This is why half the circumference multiplied by the radius is the area of the circle. The relation of circumference and diameter considered in this manner gives the most satisfactory result.

The ratio of circumference to diameter is in fact not 3 to 1. The figure 3 attached to the circumference actually means that the perimeter of an inscribed hexagon is three times the diameter. Hence, if the ratio [i.e., 3] is used to compute the length of an arc, the result obtained is not the arc but the chord. The value of this ratio was, however, passed on from generation to generation indicating the reluctance of the people to strive for accuracy. The learners, too, just followed the ancients and simply learned the incorrect method. Nevertheless, without concrete examples, it is difficult to argue on [this inaccuracy]. Generally speaking, the forms of things around us are either square or round. If we understand the ratio of their forms at close range, we can also determine this ratio when viewed from afar. Considering the ratio in this manner, we can say that its use is indeed far and wide. For this reason, I have been using drawings for verifications in order to construct a more precise ratio. However, I fear that if only the method is given, the various numerical values involved may appear obscure and difficult to explain. Therefore, I do not consider it laborious here to show the derivation of the ratio in various stages with the necessary commentaries.

### [*Section 2*]

*Method of cutting an inscribed hexagon into an inscribed dodecagon.* Put down [on the counting board] the diameter of the circle, 2 *chi* [e], and halve it to obtain 1 *chi*, which is the length of one side of the hexagon. Let the radius of length 1 *chi* be the hypotenuse (*xian* [cf]); half of a side of the hexagon, 5 *cun* [v] [5], be the *gou* [cg] (the shorter orthogonal side of a right-angled triangle); and it is required to find the *gu* [ch] (the longer orthogonal side). When 25 *cun*, which is the square of the *gou*, is subtracted from the square of the hypotenuse, there is a remainder of 75 *cun*. Find the square root up to the place values of *miao* [t] and *hu* [u]. Again move the number in the *fa* [ci] row back by one jump [of two places] to find the other "minute numbers" (*wei shu* [cj]) [13]. These "minute numbers" have no place names, so convert them into a fraction with 10 as the denominator so that they are approximated to $\frac{2}{5}$ *hu* [u]. Hence the *gu* is 8 *cun* 6 *fen* [q] 6 *li* [r] 2 *miao* [t] 5$\frac{2}{5}$ *hu* [u]. Subtract this from the radius to leave a remainder of 1 *cun* 3 *fen* 3 *li* 9 *hao* [s] 7 *miao* 4$\frac{3}{5}$ *hu* which we shall call the smaller *gou* [that is, the shorter orthogonal side of a smaller right-angled triangle]. Then half the side of the hexagon will be called the smaller *gu* and it is required to find the hypotenuse. The square [of the hypotenuse] is 267949193445 [square] *hu*, after the remaining terms are discarded. The square root of this gives the length of one side of a dodecagon.

*Method of cutting an inscribed dodecagon into an inscribed polygon of 24 sides.* Again, let the radius be the hypotenuse and half the side of the dodecagon be the *gou*. From this, find the *gu*. Put down the square of the smaller hypotenuse of the preceding [triangle] and divide it by 4 to obtain 66987298361 [square] *hu* after the remaining terms are discarded, and this is the square of the *gou*. Subtract this from the square of the hypotenuse and find the square root of the difference to obtain the *gu*, which is 9 *cun* 6 *fen* 5 *li* 9 *hao* 2 *miao* $5\frac{4}{5}$ *hu*. Subtract this from the radius to give a remainder of 3 *fen* 4 *li* 7 *miao* $4\frac{1}{5}$ *hu*, which is called the smaller *gou*; half the side of the dodecagon is called the smaller *gu* and from these, the smaller hypotenuse is derived. Its square is 68148349466 [square] *hu* after the remaining terms are discarded. The square root of this is the length of a side of a polygon of 24 sides.

*Method of cutting an inscribed polygon of 24 sides into one of 48 sides.* Again, let the radius be the hypotenuse and half the side of the polygon of 24 sides be the *gou*. From this, find the *gu*. Put down the square of the smaller hypotenuse of the preceding [triangle] and divide it by 4 to obtain 17037087366 [square] *hu* after the remaining terms are discarded, and this is the square of the *gou*. Subtract this from the square of the hypotenuse and find the square root of the difference to obtain the *gu*, which is 9 *cun* 9 *fen* 1 *li* 4 *hao* 4 *miao* $4\frac{3}{5}$ *hu*. Subtract this from the radius to give a remainder of 8 *li* 5 *hao* 5 *miao* $5\frac{2}{5}$ *hu*, which is called the smaller *gou*; half the side of the 24-sided polygon is called the smaller *gu* and from these, the smaller hypotenuse is derived. Its square is 17110278813 [square] *hu* after the remaining terms are discarded. The square root of this is 1 *cun* 3 *fen* 8 *hao* 6 *hu* neglecting the lower terms and this is the length of the smaller hypotenuse, which is also the length of a side of an inscribed polygon of 48 sides. Multiply this by the radius 1 *chi* and also by 24 to obtain 3139344000000 [square] *hu*. Divide by 10000000000 to obtain $313\frac{584}{625}$ [square] *cun* which is the area of an inscribed polygon of 96 sides.

*Method of cutting an inscribed polygon of 48 sides into one of 96 sides.* Again, let the radius be the hypotenuse and half the side of the polygon of 48 sides be the *gou*. From this find the *gu*. Put down the square of the hypotenuse of the preceding [triangle] and divide it by 4 to obtain 4277569703 [square] *hu* after the remaining terms are discarded, and this is the square of the *gou*. Subtract this from the square of the hypotenuse and find the square root of the remainder to obtain the *gu* which is 9 *cun* 9 *fen* 7 *li* 8 *hao* 5 *miao* $8\frac{9}{10}$ *hu*. Subtract this from the radius to give a remainder of 2 *li* 1 *hao* 4 *miao* $1\frac{1}{10}$ *hu* which is called the smaller *gou*; half the side of the 48-sided polygon is called the smaller *gu* and from these, the smaller hypotenuse is derived. Its square is 4282154012 [square] *hu* after the remaining terms are discarded. The square root of this is 6 *fen* 5 *li* 4 *hao* 3 *miao* 8 *hu*, neglecting the lower terms, and this is the length of the smaller hypotenuse which is also the length of a side of an inscribed polygon of 96 sides. Multiply this by the radius 1 *chi* and also by 48 to obtain 3141024000000 [square] *hu*. Divide the area by 10000000000 yielding $314\frac{64}{625}$ [square] *cun*, which is the area of an inscribed polygon of 192 sides. Subtract the area of the 96-sided polygon from this to obtain a remainder of $\frac{105}{625}$ [square] *cun* and this is called a residual area (*cha mi* [ck]). Double this amount [so that the numerator of the fraction] becomes 210 and this amount represents the area outside the 96-sided polygon extending beyond the arcs [of the circle]. This can be said to be the sum of the products of the chord (*xian* [cf]) and the sagitta (*shi* [cl]). Add this amount to the area of the 96-sided polygon to obtain $314\frac{169}{625}$ square *cun* which gives an area covering the circle and protruding beyond it. This is the reason why 314 [square] *cun* which is the area of the 192-sided polygon to the nearest integer is taken as the standard area of the circle after the smaller terms are discarded. Divide the area of the circle by its radius, 1 *chi*, and then double this to obtain 6 *chi* 2 *cun* 8 *fen* which gives the circumference. Squaring the diameter gives a square area of 400 [square] *cun*. Compare the area of the circle with this, and obtain the ratio of the areas of the circle to the [circumscribed] square as 157 to 200. If a square has 200 [square units] then its inscribed circle has 157 [square units]. The rate for the circle actually contains some smaller units as well. Referring to the diagram of the arc of a circle, we see a circle inscribed in the square and a square inscribed in the circle. The area of the inner square is half the area of the outer square. If the area of the circle is 157 [square units] then the area of the inscribed square is 100 [square units]. Next, if the diameter, 2 *chi*, is compared with the circumference, 6 *chi* 2 *cun* 8 *fen*, the ratio of the circumference to the diameter is obtained as 157 to 50. Actually, the circumference still has other smaller units.

## [*Section 3*]

There was in the armory of the Jin dynasty a copper *hu* [cm] measure constructed by Wang Mang [ai] during the Han dynasty. The inscription on it reads: "The standardized *chia liang hu* [aj] measure has a square with each side 1 *chi* long, and outside it a circle. The distance from each corner of the



square to the circle is 9 *li* 5 *hao*. The area of the circle is 162 [square] *cun*, the depth is 1 *chi* and the volume is 1620 [cubic] *cun*. The measure has a capacity for 10 *dou* [cn] (bushels)." If we use the present ratio [i.e., $\frac{157}{50}$] to compute, we obtain the area [of the circle] as 161 *cun* and a fraction. The two values are very close indeed. But as the present ratio is not precise, we have to consider the area of the dodecagon as a basis to augment or diminish the area of $\frac{105}{625}$ [square] *cun*. This yields $\frac{36}{625}$ [square] *cun*. Add this to the area of the 192-sided polygon to obtain $314\frac{4}{25}$ [square] *cun* as the area of the circle. Put down the square of the diameter 400 [square] *cun* and compare this with the area of the circle. If the area of the circle is 3927 then that of the square is 5000. Taking this as the ratio [i.e., $\frac{3927}{5000}$] we can say that the area of the inscribed circle in the square of 5000 [square units] is 3927 [square units] and the area of the square inscribed in the circle of area 3927 [square units] is 2500 [square units]. Divide the area of the circle $314\frac{4}{25}$ [square] *cun* by its radius 1 *chi* and double the result to obtain 6 *chi* 2 *cun* $8\frac{8}{25}$ *fen* which gives the circumference. Take the diameter 2 *chi* and compare it with the circumference, getting 1250 for the diameter and 3927 for the circumference. The ratio [of the circumference to the diameter] thus obtained has been exhausted to very minute numbers. However, for practical purposes, the former ratio [i.e., $\frac{157}{50}$] is still a simpler one. The area of a 3072-sided polygon is computed from a side of a 1536-sided polygon. When the lower terms are discarded, the ratio obtained is the same as the previous one. In this way, the ratio is again verified.

## NOTES

1. The *Jiu zhang suanshu* is printed in [Qian 1963, 81–258]. The data assigned to this book is between 200 B.C. and A.D. 200; see [Needham 1959, 24–25].

2. There is a fairly large literature on Liu's method written in Chinese, of which one of the more up-to-date is [Bai Shangshu 1983, 35–53]. In English, Liu's method is found in [Mikami 1913, 47–49], which contains some misprints, and, more briefly, in [He Shaogeng 1983, 90–98]. In our translation, we have tried to give as close an interpretation of the original text [Qian 1963, 103–106] as possible.

3. However, the ancient Greeks were aware that the area of a circle to the square of its radius is a constant ratio [Euclid, Bk. XII, Prop. 2].

4. The phrase "counting board" means any flat surface suitable for the placement of rod numerals and should not be confused with the specially constructed counting boards of medieval Europe.

5. Note that 10 *cun* [v] = 1 *chi* [e] and 10 *chi* [e] = 1 *zhang* [ae].

6. We would like to thank Lam Chih Chao and Lam Chih Ming for the software program.

7. This is evident in the earliest mathematical texts of *Zhou bi suanjing* [co] (*The arithmetic classic of the gnomon and the circular paths of heaven*) and *Jiu zhang suanshu*, as also in the *Kao gong ji* [cp] (*Artificer's record*) of the *Zhou li* [cq] (*Record of the rites of the Zhou dynasty*).

8. For an objective review of the polemic of this problem, see [Li Di 1982, 35–44].

9. For a brief biography of Zu Chongzhi, see [Li Di 1962]. See also [Ruan Yuan 1799 353, 91–105].

10. See [Qian 1923, 59; Yan 1936b, 43; Sun 1955, 9].

11. For a brief survey of this period, see [Xu 1957, 48].

12. The diagram is missing, but it should be similar to Fig. 1.

13. For the square-root extraction procedure, see [Wang & Needham 1955, 350–356; Lam 1969, 93–97].

## REFERENCES

Ang Tian Se. 1977. Chinese computation with the counting-rods. *Papers on Chinese Studies*, University of Malaya 1, 97–109.

Bai Shangshu [ap]. 1982. Cong Wang Mang liang qi dao Liu Xin yuan lu [cr] [From Wang Mang's measuring vessel to Liu Xin's ratio of circumference to diameter]. *Beijing daxue xuebao*, No. 2, 75–79.

——— 1983. *Jiu zhang suanshu zhushi* [cs] [Annotations on *Nine chapters of the mathematical art*]. Beijing: Kexue chubanshe.

Beckmann, P. 1970. *A history of π (pi)*. All page references are to the second edition, Boulder, Colo.: Golem, 1971.

Gupta, R. C. 1975. Some ancient values of pi and their use in India. *Mathematics Education* **9,** 1–5.

He Shaogeng [bb]. 1983. Method for determining segment areas and evaluation of π. In *Ancient China's technology and science*. Compiled by the Institute of the History of Natural Sciences, Chinese Academy of Sciences. Beijing: Foreign Language Press.

Heath, T. L. (ed.). 1897. *The works of Archimedes*. All page references are to the reissued edition, New York: Dover, 1953.

——— 1921. *A history of Greek mathematics*. 2 vols. London/New York: Oxford Univ. Press (Clarendon).

Lam Lay-Yong. 1969. The geometrical basis of the ancient Chinese square-root method. *Isis* **61,** 92–101.

Lam Lay-Yong and Shen Kangsheng. 1985. The Chinese concept of Cavalieri's principle and its applications. *Historia Mathematica* **12,** 219–228.

Li Di [bg]. 1962. Da kexuejia Zu Chongzhi [ct] [The great scientist Zu Chongzhi]. Shanghai: Renmin chubanshe.

——— [bg]. 1982. Jiu zhang suanshu zhengming wentide gai shu [cu] [A summary of the various views on the Jiu zhang suanshu]. In *Jiu zhang suanshu yu Liu Hui* [cw] [The *Jiu zhang suanshu* and Liu Hui], Wu Wenjun [cv], ed., pp. 28–50. Beijing: Shifan daxue chubanshe.

Mikami, Y. 1913. *The development of mathematics in China and Japan*. All page references are to the second edition, New York: Chelsea, 1974.

Needham, J. 1959. *Science and civilisation in China*. Vol. 3. Cambridge: Cambridge Univ. Press.

Neugebauer, O. 1952. *The exact sciences in antiquity*. All page references are to the second edition, Providence, R.I.: Brown Univ. Press, 1957.

Qian Baocong [al]. 1923. Zhongguo suan shu zhong zhi zhoulu yanjiu [cx] [A study of π in Chinese mathematical texts]. In *Qian Baocong kexueshi lunwen xuanji* [cy] [Selected essays of Qian Baocong on the history of Chinese science], pp. 50–74. All page references are to the 1983 book edition, Beijing: Kexue chubanshe 1983.

——— (ed.). 1963. *Suanjing shi shu* [cz] [Ten mathematical manuals]. Shanghai: Zhonghua shuju.

Qudan Xida [da]. 729. *Kaiyuan zhan jing* [aq] [Kaiyuan treatise on astrology].

Ruan Yuan [db]. 1799. *Chou ren zhuan* [dc] [Biographies of mathematicians and scientists]. In *Guoxue jiben congshu*. Taipei: Shangwu yinshuguan, 1968.

*Song shu* [ar]. [Standard history of the Liu Song dynasty]. 1973 edition. Beijing: Zhonghua shuju.

*Sui shu* [y] [Standard history of the Sui dynasty]. 1973 edition. Beijing: Zhonghua shuju.

Sun Zhifu [an]. 1955. Zhongguo gudai shuxue jia guanyu yuanzhoulu yanjiu de chengjiu [dd] [Achievements in the research of π by ancient Chinese mathematicians]. *Shuxue Tongbao*, No. 5, 5–12.

Wagner, D. 1978. Doubts concerning the attribution of Liu Hui's commentary on the *Chiu-chang suan-shu*. *Acta Orientalia* **39,** 199–212.

Wang, L., and Needham, J. 1955. Horner's method in Chinese mathematics: Its origins in the root-extraction procedures of the Han dynasty. *T'oung Pao* **43,** 345–401.

Xu Chunfang [ao]. 1957. *Zhong suan jiade jihexue yanjiu* [de] [Research on geometry by Chinese mathematicians]. Hong Kong.

Yan Dunjie [as]. 1936a. Sui shu luli zhi Zu Chongzhi yuan lu ji shi shi [df] [An analysis of records for π by Zu Chongzhi as found in the harmonics and calendar chapters of *Sui shu*]. *Xue yi zazhi* [dg] (Shanghai) **15,** No. 10, 27–57.

——— 1936b. Zhongguo suanxuejia Zu Chongzhi ji qi yuanzhou lu zhi yanjiu [dh] [Chinese mathematician Zu Chongzhi and his work on π]. *Xue yi zazhi* [dg] (Shanghai) **15,** No. 5, 37–50.

Youschkevitch, A. P., and Rosenfeld, B. A. 1973. Al-Kashi (or Al-Kashani), Ghiyath al-din Jamshid Mas'ud. In *Dictionary of scientific biography*, Vol. 7, pp. 255–262. New York: Scribner's.



## GLOSSARY

a 刘徽 b 祖冲之 c 方田 d 九章算术
e 尺 f 十 g 百 h 千
i 万 j 十万 k 百万 l 千万
m 亿 n 十亿 o 百亿 p 千亿
q 分 r 厘 s 毫 t 秒
u 忽 v 寸 w 李淳风 x 缀术
y 隋书 z 刘歆 aa 张衡 ab 王蕃
ac 皮延宗 ad 南徐 ae 丈 af 密率
ag 约率 ah 祖暅 ai 王莽 aj 嘉量斛
ak 李俨 al 钱宝琮 am 励乃骥 an 孙炽甫
ao 许莼舫 ap 白尚恕 aq 开元占经 ar 宋书
as 严敦杰 at 朱载堉 au 踵古 av 李潢
aw 九章算术细草图说 ax 杜石然 ay 何洛
az 王守义 ba 华罗庚 bb 何绍庚 bc 梅荣照
bd 沈康身 be 余宁生 bf 余介石 bg 李迪
bh 魏 bi 司马炎 bj 晋 bk 商功
bl 王孝通 bm 思极毫芒 bn 元嘉历 bo 何承天
bp 岁差 bq 大明历 br 调日法 bs 赵友钦
bt 邢云路 bu 陈荩谟 bv 方以智 bw 数理精蕴
bx 康熙 by 梅谷成 bz 何国宗 ca 明安图
cb 项名达 cc 李善兰 cd 曾纪鸿 ce 割
cf 弦 cg 勾 ch 股 ci 法
cj 微数 ck 差幂 cl 矢 cm 斛
cn 斗 co 周髀算经 cp 考工记 cq 周礼
cr 从王莽量器到刘歆圆率 cs 九章算术注释 ct 大科学家祖冲之
cu 九章算术争鸣问题的概述 cv 吴文俊 cw 九章算术与刘徽
cx 中国算书中之周率研究 cy 钱宝琮科学史论文选集
cz 算经十书 da 瞿昙悉达 db 阮元 dc 畴人传
dd 中国古代数学家关于圆周率研究的成就 de 中算家的几何学研究
df 隋书律历志祖冲之圆率记事释 dg 学艺杂志
dh 中国算学家祖冲之及其圆周率之研究