When does a formal finite-difference expansion become "real"?[1]

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Dedicated to the fond memory of J. Milne Anderson

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Outline

Notations

- We denote

$$\begin{aligned}\Delta f(x) &= f(x+1) - f(x),\\ \Delta^2 f(x) &= \Delta[f(x+1) - f(x)] = f(x+2) - 2f(x+1) + f(x)\\ \Delta^3 f(x) &= f(x+3) - 3f(x+2) + 3f(x+1) - f(x)\\ &\cdots\cdots.\end{aligned}$$

- Also

$$Ef(x) = f(x+1)$$

so that

$$E = 1 + \Delta, \quad \Delta = E - 1.$$

Formal calculus

- We develop the formal Taylor series

$$Ef(x) = f(x+1)$$
$$= f(x) + \frac{f'(x)}{1!}\cdot 1 + \frac{f''(x)}{2!}\cdot 1^2 + \frac{f^{(3)}(x)}{3!}\cdot 1^3 + \cdots$$

- But then

$$Ef(x) = \Big(I + \frac{d}{dx} + \frac{1}{2!}\frac{d^2}{dx^2} + \frac{1}{3!}\frac{d^3}{dx^3} + \cdots\Big)f(x)$$
$$= (I + D + \frac{1}{2!}D^2 + \frac{1}{3!}D^3 + \cdots)f(x)$$
$$= e^D f.$$

- Hence

$$\Delta f(x) = (E-1)f(x) = (e^D - 1)f(x).$$

and

$$\Delta^n f(x) = (e^D - 1)^n f(x).$$

Formal calculus

- Conversely, we have

$$e^D = 1 + \Delta,$$

so that

$$D = \log(1 + \Delta)$$

and

$$\Big(\frac{d}{dx}\Big)^n = \Big\{\log(1+\Delta)\Big\}^n = \Big\{\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \cdots\Big\}^n$$

- We need to introduce Stirling number of the first kind in order to describe the expansion.

Stirling numbers of the first kind

- Recall that the Stirling no. of the first kind $(-1)^{n-m}S_n^{(m)}$ counts the number of permutations of n symbols which have exactly m cycles. (Cycles are those related to permutation groups).
- Generating function:

$$x(x-1)\cdots(x-n+1)=\sum_{m=0}^{n}S_n^{(m)}x^m$$

- Recursions:

$$\binom{m}{r}S_{n+1}^{(m)}=S_n^{(m-1)}-nS_n^{(m)},\quad n\geq m\geq r$$

$$\binom{m}{r}S_n^{(m)}=\sum_{k=m-r}^{n-r}\binom{n}{k}S_{n-k}^{(r)}S_k^{(m-r)},\quad n\geq m\geq r.$$

A formal expansion

- $$\frac{d^m}{dx^m} f(x) = m! \sum_{n=m}^{\infty} \frac{S_n^{(m)}}{n!} \Delta^n f(x)$$

Another formal expansion

- The formula
$$\Delta^n f(x) = (e^D - 1)^n f(x).$$
can be formally expanded as
$$\begin{aligned}\Delta^n f(x) &= (e^D - 1)^n f(x)\\ &= n! \sum_{k=n}^{\infty} \frac{\mathfrak{S}_k^{(n)}}{k!} f^{(k)}(x)\\ &= \left(\eta^n D^n + \frac{n}{2!}\eta^{n+1} D^{n+1} + \cdots\right) f(x).\end{aligned}$$

- Here $\mathfrak{S}_k^{(n)}$ is the Stirling numbers of the second kind.

Stirling numbers of the second kind

- Generating function:

$$x^n = \sum_{m=0}^{n} \mathfrak{S}_n^{(m)} x(x-1)\cdots(x-m+1).$$

- $\mathfrak{S}_n^{(m)}$ counts the number of different ways to partition a set of n objects into m non-empty subsets.
- Explicit form:

$$\mathfrak{S}_n^{(m)} = \frac{1}{m!}\sum_{k=0}^{m}(-1)^{(m-k)}\binom{m}{k}k^n.$$

- Recursions:

$$\mathfrak{S}_n^{(m)} = m\mathfrak{S}_n^{(m)} + \mathfrak{S}_n^{(m-1)}, \quad n \geq m \geq 1,$$

$$\binom{m}{r}\mathfrak{S}_n^{(m)} = \sum_{k=m-r}^{n-r}\binom{n}{k}\mathfrak{S}_{n-k}^{(r)}\mathfrak{S}_k^{(m-r)}, \quad n \geq m \geq r.$$

Some Stirling numbers of the Second Kind

n	k = 0	1	2	3	4	5	6	7	8	9	10
0	1										
1	0	1									
2	0	1	1								
3	0	1	3	1							
4	0	1	7	6	1						
5	0	1	15	25	10	1					
6	0	1	31	90	65	15	1				
7	0	1	63	301	350	140	21	1			
8	0	1	127	966	1701	1050	266	28	1		
9	0	1	255	3025	7770	6951	2646	462	36	1	
10	0	1	511	9330	34105	42525	22827	5880	750	45	1

Figure: $\mathfrak{S}_n^{(k)}$: Digital Library of Mathematical Functions, National Institute of Standard and Technology

Logarithmic differences

- Theorem (C. & Feng (2009))

  Let f be meromorphic $\sigma = \sigma(f) < \infty$. Then

$$\frac{f(z+\eta)}{f(z)} = e^{\eta \frac{f'(z)}{f(z)} + O(r^{\beta+\varepsilon})},$$

  holds for $r \notin E \cup [0,1]$, where $\beta = \begin{cases} \max\{\sigma-2,\, 2\lambda-2\}, & \lambda < 1 \\ \max\{\sigma-2,\, \lambda-1\}, & \lambda \geq 1 \end{cases}$

  where λ is the maximum of the exponent convergence of zeros and poles of f.

  - No such comparison is possibly if $\sigma(f) = \infty$ in general. Consider e.g. $f(z) = e^{e^z}$. Then $f(z+1)/f(z) = \exp[(e-1)e^z]$ grows faster than f.

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Main result I

- Theorem (C. & Feng (2016))

  Let f be a meromorphic function with order $\sigma = \sigma(f) < 1$. Then for any positive integers n, N such that $N \geq n$, and for each $\varepsilon > 0$, there is a set $E \subset [1, +\infty)$ of finite logarithmic measure so that

$$\frac{\Delta^n f(z)}{f(z)} = n! \left(\sum_{k=n}^{N} \frac{\mathfrak{S}_k^{(n)}}{k!} \frac{f^{(k)}(z)}{f(z)} \right) + O\left(r^{(n+N+1)(\sigma-1)+\varepsilon}\right) \qquad (1)$$

  for $|z| = r \notin E \cup [0, 1]$

  - Here the exceptional set E are intervals that we remove arising from the zeros and poles of f.

Logarithmic measures

- A subset E of $\mathbb{R}$ has *finite logarithmic measure* if

$$\mathrm{lm}(E) = \int_{E\cap(1,\infty)} \frac{dr}{r}$$

  is finite. Otherwise, the set E is said to have an infinite logarithmic measure.
- E.g. If $E_n = [e^n, (e+1)^n]$, then $\mathrm{lm}(E) = \infty$
- E.g. if $E_n = [e^n, e^n(1 + 1/n^e)]$, then $\mathrm{lm}(E) < \infty$
- An ingenious combinatorial type estimate due to H. Cartan.

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Main result II

- Theorem (C. & Feng (2016))

Let f be a meromorphic function with order $\sigma = \sigma(f) < 1$. Then for each positive integer k, and for each $\varepsilon > 0$, there exists an exceptional set $E^{(\eta)}$ in $\mathbb{C}$ consisting of a union of disks centred at the zeros and poles of $f(z)$ such that when z lies outside of the $E^{(\eta)}$,

$$\begin{aligned}\frac{\Delta f}{f} &:= \frac{f(z+\eta)-f(z)}{f(z)}\\ &= \eta\frac{f'(z)}{f(z)} + \frac{\eta^2}{2!}\frac{f''(z)}{f(z)} + \cdots + \frac{\eta^k}{k!}\frac{f^{(k)}(z)}{f(z)} + O\big(\eta^{k+1} r^{(k+1)(\sigma-1)+\varepsilon}\big).\end{aligned} \tag{2}$$

Moreover, the set $\pi E^{(\eta)} \cap [1, +\infty)$, where $\pi E^{(\eta)}$ is obtained from rotating the exceptional disks of $E^{(\eta)}$ so that their centres all lie on the positive real axis, has finite logarithmic measure.

Taylor expansion

- Applying Taylor expansion

$$\frac{f(z+\eta)-f(z)}{f(z)}=\eta\frac{f'(z)}{f(z)}+\frac{\eta^2}{2!}\frac{f''(z)}{f(z)}+\cdots+\frac{\eta^n}{n!}\frac{f^{(n)}(z)}{f(z)}+\frac{R_n(z+\eta)}{f(z)}$$

where

$$\begin{aligned}\frac{R_n(z+\eta)}{f(z)}&=\frac{1}{n!}\int_z^{z+\eta}(z+\eta-t)^n\frac{f^{(n+1)}(t)}{f(z)}\,dt\\&=\frac{\eta^{n+1}}{n!}\int_0^1(1-T)^n\frac{f^{(n+1)}(z+T\eta)}{f(z)}\,dT.\end{aligned}\tag{3}$$

Then we rewrite:

$$\left|\frac{R_n(z+\eta)}{f(z)}\right|=\left|\frac{\eta^{n+1}}{n!}\int_0^1(1-T)^n\frac{f^{(n+1)}(z+T\eta)}{f(z+T\eta)}\,\frac{f(z+T\eta)}{f(z)}\,dT\right|$$

Taylor expansion

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Some estimates

Lemma (E. Hille, G. Gundersen (1988))

Let f be a meromorphic function of finite order $\sigma(f) = \sigma$. Then for any $\varepsilon > 0$, there exists a set $E \subset (1, \infty)$ that depends on f and it has finite logarithmic measure, such that for all z satisfying $|z| = r \notin E \cup [0, 1]$, we have

$$\left|\frac{f'(z)}{f(z)}\right| \leq |z|^{\sigma-1+\varepsilon}. \tag{4}$$

Lemma (C. & Feng (2008))

Let $f(z)$ be a finite order meromorphic function of order σ, then for each $\varepsilon > 0$,

$$\left|\frac{f(z+1)}{f(z)}\right| \leq \exp(|z|^{\sigma-1+\varepsilon}) \tag{5}$$

holds for all $|z|$ outside a set of finite logarithmic measure.

Linear difference equations

- We considered linear difference equations of the form

$$a_n(z)\Delta^n f(z) + \cdots + a_1(z)\Delta f(z) + a_0(z)f(z) = 0, \qquad (6)$$

where $a_0(z), \cdots, a_n(z)$ are polynomials.

- Theorem (C. & Feng (2016))

Let f be an entire solution of the difference equation (6) above with order $\sigma(f) = \chi < 1$. Then χ is a rational number which can be determined from a gradient of the corresponding Newton-Puisseux diagram for equation (6). In particular,

$$\log M(r, f) = Lr^{\chi}(1 + o(1))$$

where $L > 0$, $\chi > 0$ and $M(r, f) = \max_{|z|=r} |f(z)|$. That is, the solution has completely regular growth.

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An example

- The equation

$$z(z-1)(z-2)\Delta^3 f(z-3)+z(z-1)\Delta^2 f(z-2)+ \\ z\Delta f(z-1)+(z+1)f(z)=0$$

admits an entire solution of order $1/3$. This e.g. is due to Ishizaki & Yanagihara (2004). Our theory allows to conclude one entire solutions has growth

$$\log M(r,\ f)=Lr^{1/3}((1+o(1)).$$

Wiman-Valiron theory I

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- Let
$$f(z)=\sum_{n=0}^{\infty}a_nz^n$$
be an entire function in $\mathbb{C}$, $M(r,f)=\max_{|z|=r}|f(z)|$ denotes the maximum modulus of f on $|z|=r>0$.
- $a_nz^n\to 0$ as $n\to\infty$.
- $\mu(r,f)=\max_{n\geq 0}|a_n|r^n$ maximal term $\to 0$
- central index $\nu(r,f)$ is the greatest exponent m such that
$$|a_m|r^m=\mu(r,f),$$
- $\nu(r,f)$ is a real, non-decreasing function of r.
- $$\limsup_{r\to\infty}\frac{\log\nu(r,f)}{\log r}=\sigma=\limsup_{r\to\infty}\frac{\log\log M(r,\ f)}{\log r}$$

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Wiman-Valiron theory II

Theorem (C. & Feng (2016))

Let f be a transcendental entire function of order $\sigma(f) = \sigma < 1$, $0 < \varepsilon < \min\{\frac{1}{8}, 1 - \sigma\}$ and z is "close to" where $f(z)$ is maximal. Then for each positive integer k, there exists a set $E \subset (1, \infty)$ that has finite logarithmic measure, such that for all $r \notin E \cup [0, 1]$,

$$\frac{\Delta^k f(z)}{f(z)} = \left(\frac{\nu(r, f)}{z}\right)^k \big(1 + \mathcal{R}_k(z)\big) \tag{7}$$

where $\mathcal{R}_k(z) = O\big(\nu(r, f)^{-\kappa+\varepsilon}\big)$ and $\kappa = \min\{\frac{1}{8}, 1 - \sigma\}$.

Connection to little Picard's theorem

- Integrability of discrete Painlevé equations: 2nd-order non-linear difference equations. Ablowitz, Herbst, Hablurd, Korhonen (2000, 2007): *a finite order meromorphic solution*.
- This is an analogue for Painlevé's test for Painlevé's equations.
- Crucial estimate are *average estimates* of $f(z+1)/f(z)$ (Halburd-Korhonen (2006), C. & Feng (2008)) that are analogue for Nevanlinna's average estimate for $f'(z)/f(z)$.
- Picard type theorems (Nevanlinna theory) for difference operators. Chiang, Feng, Halburd, Korhonen (2006, 2016).

Difference-type Picard's theorem

Theorem (Halburd-Korhonen (2006))

If f is a finite-order meromorphic function that possesses three paired-values with separation η, then $f \in \ker(\Delta f)$, i.e., $0 \equiv \Delta_\eta f(z) := f(z+\eta) - f(z)$. i.e., f is a periodic with period η.

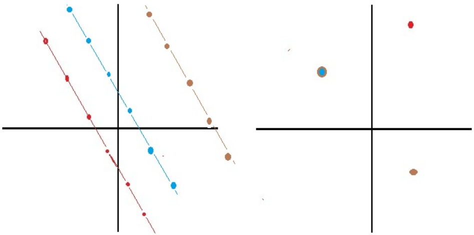

Figure: The left-side are the pre-images (two points differ by η) of the right-side.

- Original Picard's theorem can be thought of this way: The preimages of three points are empty sets. So $f \in \ker(\frac{d}{dz})$.

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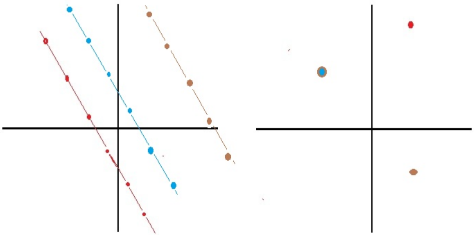

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Askey-Wilson type Picard's theorem

Theorem (C. & Feng (2015))

Let f be a meromorphic function with finite logarithmic order, and that f has three distinct $\mathrm{AW}-$Picard exceptional values. Then $f \in \ker(\mathcal{D}_q)$, i.e., f is an $\mathrm{AW}-$constant.

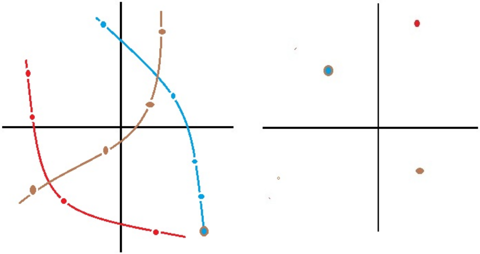

Figure: The left-side are the pre-images of the right-side

- Cheng & C. (2017) also establishes an analogue for the Wilson operator.

Summary and problems

- We have reviewed a classical finite-difference differential (classical-quantum) relationship:

$$\Delta f^n = (e^D - 1)^n f = n! \sum_{k=n}^{N} \frac{\mathfrak{S}_k^{(n)}}{k!} f^{(k)}(z) + o(1)$$

- Problem 1: Can we REALLY have

$$\Delta f^n(z) = (e^D - 1)^n f = n! \sum_{k=n}^{\infty} \frac{\mathfrak{S}_k^{(n)}}{k!} f^{(k)}(z) \ ?$$

- Problem 2: How about:

$$\left(\frac{d}{dx}\right)^n f = \left\{\log(1+\Delta)\right\}^n f = n! \sum_{k=n}^{\infty} \frac{S_k^{(n)}}{k!} \Delta^k f(x) \ ?$$

- Problem 3: Others, e.g.

$$\exp(2xt - ytD_x)\, f(x) = \exp(2xt - yt^2) \cdot f(x - yt)?$$

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Figure: Clear Water Bay, Hong Kong Thank you for your attention !!