On the Nevanlinna Order of Meromorphic Solutions to Linear Analytic Difference Equations

By Yik-Man Chiang and Simon N. M. Ruijsenaars

For various classes of linear ordinary analytic difference equations with meromorphic coefficients, we study Nevanlinna order properties of suitable meromorphic solutions. For a large class of first-order equations with coefficient of order $\rho \in [0, \infty)$, we explicitly construct meromorphic solutions of order $\leq \rho + 1$. For higher-order equations with coefficients of order $\rho \in [0, \infty)$, we show that meromorphic solutions with increase of order $\leq \rho + 1$ in a certain strip have order $\leq \rho + 1$. The assumptions made in the latter setting may seem quite restrictive, but they are satisfied for several classes of second-order difference equations that have been studied in recent years. The latter include Harper-type equations, "reflectionless" equations, Askey–Wilson-type equations, and equations of relativistic Calogero–Moser type.

1. Introduction

As is well known, there is a vast and comprehensive literature on (linear, ordinary) discrete difference and differential equations. It deals both with general features, associated to certain classes of equations, and with quite detailed characteristics of solutions to specific equations ("special functions"). By contrast, the literature on analytic difference equations is scarce and widely

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scattered. More precisely, the area was quite active in the nineteenth century, but during the twentieth century interest waned. Among the few twentieth century monographs focusing on analytic difference equations, we mention in particular Nörlund [1], Milne-Thompson [2], Meschkowski [3], and Immink [4], from which further references can be traced.

In more recent years, activity in the area has increased again, both from the perspective of general structural properties and from that of special analytic difference equations and suitably restricted solutions thereof. In the former area one may mention the work of the Groningen school (Braaksma, Faber, Immink, van der Put, ...) and that of Ramis and his school (A. Duval, Saulois, ...), whereas the special function activity is connected to the closely related fields of integrable systems and quantum groups.

The interest in *nonlinear* analytic difference equations has increased recently, too, especially in response to the program of finding a suitable analog of the Painlevé property (no movable singularities except poles) for second-order analytic difference equations. More specifically, scores of difference versions of the well-known six Painlevé equations have been found that share various integrability properties, and the problem is to construct a more systematic framework, enabling recognition and classification of such equations.

Among pioneering work in the latter area we mention in particular a recent paper by Ablowitz, Halburd, and Herbst [5]. Their starting point is the observation that all of the pertinent discrete difference equations admit obvious analytic versions, and hence can be studied in the setting of complex analysis. (As a spin-off, this yields novel perspectives on earlier Painlevé tests in the discrete setting, including singularity confinement [6,7] and the zero-step-size limit of Conte and Musette [8].) Although there are few results ensuring that a given nonlinear meromorphic difference equation admits meromorphic solutions, this generally appears to be the case, in stark contrast to meromorphic differential equations. To obviate the infinite-dimensional ambiguity due to meromorphic functions with the relevant period (cf. below for the linear case), and also for purposes of isolating Painlevé-type properties, it turns out that concepts from Nevanlinna theory are particularly useful [5].

Though the present paper deals solely with *linear* analytic difference equations (denoted AΔEs from now on), its outlook ties in with (and was partially inspired by) the work in the nonlinear Painlevé setting just mentioned. We recall in this connection that any linear meromorphic differential equation has the Painlevé property (because solutions can only be singular at the locations of poles of the coefficients). Passing to the meromorphic AΔE case, meromorphic solutions always exist (cf. below), but a natural question that arises from the viewpoint of [5] has only been answered for the case of first-order AΔEs. This question reads: Assuming the AΔE has coefficients with finite Nevanlinna order ρ, does it admit meromorphic solutions that have finite order, too? (If so, all of these equations are of Painlevé type in the sense of [5].)

At this point we should stress that we restrict the term AΔEs to linear equations of the form (1) below (and more generally (5)), which excludes in particular linear q-difference equations. The latter equation class is related to the former by a logarithmic change of the independent variable, which alters the meromorphy features of the solution. Indeed, by contrast to AΔEs, even quite simple linear meromorphic q-difference equations need not admit any meromorphic solutions (cf. the example (1.3) in [9]).

Returning to the above question, in the 1935 monograph by J. Whittaker [10] it has been answered in the affirmative for the first-order case. Stronger yet, Whittaker shows that there exist meromorphic solutions of order $\rho + 1$ at most (cf. Section 6 in [10]).

When the first-order restriction is dropped, the question seems not to have been answered in the literature. As already alluded to, meromorphic AΔEs always admit meromorphic solutions (as follows from the analytic theory of vector bundle sections [11]), but the relevant existence theorems are nonconstructive and yield no information on order properties.

Whittaker's first-order existence theorem is nonconstructive as well, in the sense that his solution of order $\leq \rho + 1$ is not explicitly expressed in terms of the given coefficient of order ρ. In the paper [12], one of the authors studied first-order AΔEs for which special ("minimal") solutions exist that can be written as an integral or series directly involving the given coefficient. A principal purpose of the present work is to reconsider these explicit first-order results from the perspective of Nevanlinna theory.

Our second aim is to study the higher-order case, our main result being that the order ρ restriction on the coefficients, together with certain assumptions on the existence of a special type of solution, implies that such a solution has order $\leq \rho + 1$. Admittedly, the assumptions at issue are strong, but they are satisfied for various second-order AΔEs. Accordingly, our results entail that the pertinent special function solutions have order $\leq \rho + 1$.

Having provided a general context and a rough sketch of our results, we turn to crucial preliminaries. To start with, only a good acquaintance with the beginnings of Nevanlinna theory is required to understand (Sections 3 and 4 of) the present paper. Indeed, we only need part of Chapter 1 (called "The elementary theory") in Hayman's monograph [13].

To be more specific, we do assume familiarity with the notions of counting, proximity, and Nevanlinna characteristic functions, and we have occasion to make explicit use of the Poisson–Jensen formula (Theorem 1.1 in [13]), which we recapitulate in the Appendix for ease of reference. On the other hand, we also derive some apparently new results (Lemmas 2 and 3) within the setting of elementary Nevanlinna theory. Lemma 2 plays a pivotal role in our study, and may also be useful in other contexts.

We proceed with some information on AΔEs and their minimal solutions (cf. also [14]). Our convention in the first-order case reads

$$F(z + ia/2) = \Phi(z)F(z - ia/2), \tag{1}$$

where Φ is a given coefficient in the space $\mathcal{M}^*$ of meromorphic functions that do not vanish identically, and where $a > 0$ is the step size. For the applied contexts we have in mind, this normalization is more convenient than the starting point $F(z + 1) = M(z)F(z)$, which can be found in much of the literature (in particular, in [1] and [10]). The first point to be emphasized is that solutions $F \in \mathcal{M}^*$ (to which we restrict attention) are highly nonunique: If $F(z)$ solves (1), then also $\mu(z)F(z)$ solves (1) for all multipliers μ in the space $\mathcal{P}_{ia}$, where

$$\mathcal{P}_\alpha \equiv \{\mu \in \mathcal{M}^* \mid \mu(z + \alpha) = \mu(z)\}, \quad \alpha \in \mathbb{C}^*. \tag{2}$$

For applications, therefore, it is crucial to single out solutions by further requirements. For example, we can require that a solution $F(a; z)$ be continuous in a and admit a limit as $a \downarrow 0$ (in keeping with [8]).

In this paper, however, a is assumed to be fixed. Our focus is on solutions that are *minimal*, as defined by the following two requirements. First, $F(z)$ should have no zeros and poles in the strip $|\mathrm{Im}\, z| \leq a/2$. Thus, $\log F(z)$ is well defined and analytic for $|\mathrm{Im}\, z| \leq a/2$, up to a multiple of $2\pi i$. The second requirement is that one have

$$\log F(z) = O(|z|^\xi), \quad |\mathrm{Im}\, z| \leq a/2, \quad |z| \to \infty, \tag{3}$$

for some $\xi \in \mathbb{R}$, with the bound uniform for $\mathrm{Im}\, z$ varying over $[-a/2, a/2]$. (We should add that the present definition is less restrictive than the one used in [12], where minimality involves a larger strip $|\mathrm{Im}\, z| < c + a/2$, $c > 0$.)

Of course, minimal solutions need not exist, a priori. For one thing, it is plain from (1) that Φ must be free of poles and zeros for real z for minimal solutions to exist. This necessary requirement on Φ is however not very restrictive, inasmuch as any meromorphic function has poles and zeros on at most countably many lines parallel to the real axis. If need be, one can therefore switch from $\Phi(z)$ to a function $\Phi(z + ib)$, $b \in \mathbb{R}^*$, without real poles and zeros. A much stronger restriction is also clear from (1): In view of (3), $|\log \Phi(z)|$ must be polynomially bounded for real z, and this need not be true for any shifted function $\Phi(z + ib)$, $b \in \mathbb{R}$. (Take for example shifts of $\Phi(z) = \exp(\exp(z))$.)

In Sections 2 and 3 we will obtain far more information about existence. At this introductory stage, however, we proceed to point out that when minimal solutions do exist, then they are essentially unique [12]. Indeed, the quotient F_1/F_2 of two minimal solutions is an ia-periodic entire function, and because its logarithm is polynomially bounded (due to (3)), it must be of the form

$$F_1(z)/F_2(z) = C \exp(2\pi kz/a), \quad C \in \mathbb{C}^*, \quad k \in \mathbb{Z}. \tag{4}$$

The upshot is that for solutions without zeros and poles in the strip $|\mathrm{Im}\, z| \leq a/2$, the minimal solutions $F(z)$ are singled out by “minimal increase” of $|\log F(z)|$ as $|z| \to \infty$ in the latter strip.

Even though a similar notion of minimal solution can be used for higher-order AΔEs, it is far less clear what restrictions on the coefficients this implies. Indeed, there are only certain special cases for which explicit solutions are known. To study general aspects, we actually find it convenient to consider the first-order $N \times N$ system

$$\mathcal{V}(z + ia/2) = \mathcal{M}(z)\mathcal{V}(z - ia/2), \tag{5}$$

with $\mathcal{M}(z)$ an $N \times N$ matrix with meromorphic elements satisfying further restrictions, and $\mathcal{V}(z)$ a vector solution with certain "minimal" features we assume. However, apart from some special cases, we do not know whether such solutions exist for suitable classes of coefficients.

We are now prepared to sketch the organization of this paper. Section 2 is concerned with minimal solutions to the AΔE (1), with $\Phi(z)$ free of zeros and poles in a strip around the real axis. It does not yet involve Nevanlinna theory. Rather, its results prepare the ground for a reappraisal of the first-order setting from the viewpoint of Nevanlinna theory, which we undertake in Section 3. Higher-order AΔEs are studied in Section 4, in the more general guise of the $N \times N$ system (5).

In more detail, in Section 2 we first derive a general explicit result (Theorem 1) on minimal solutions to (the logarithmic version of) (1). This result encompasses various previous results from [12] and [15]. We then show that the assumptions on the coefficients are satisfied for a large class of meromorphic functions obtained as ratios of entire functions in Weierstrass product form. (The zeros of the two entire functions are assumed to have features that are met in several concrete applications.) Hence, we deduce existence of minimal solutions for this class (Theorem 2).

The main result of Section 3 is Theorem 3. This concerns coefficients $\Phi(z)$ that are of order $\rho < \infty$ and for which a special type of minimal solution is assumed to exist. (Specifically, the uniform bound (3) should hold for all ξ greater than $\rho + 1$.) The conclusion is then that the order of such a solution is $\rho + 1$ at most. A crucial ingredient of the proof of Theorem 3 is Lemma 2, which concerns a bound on the proximity function of translates of an arbitrary meromorphic function that is uniform in the translation parameter. For expository reasons we have relegated the proof to the Appendix. With Theorem 3 in hand, we proceed to scrutinize its assumptions. In particular, using Lemma 3 (also proved in the Appendix), we show that order ρ coefficients that are free of poles and zeros in a strip around the real axis satisfy the assumptions (Theorem 4). Section 3 is then concluded with a reappraisal of various types of AΔEs admitting minimal solutions, in particular the ones obtained in Theorem 2.

Lemma 2 is also the main technical tool for our study of the higher-order case in Section 4. Even though our Theorem 5 seems a natural arbitrary-N generalization of Theorem 3, we should point out that we are not aware of general $N > 1$ results playing the role of Theorem 1. More precisely, we do not know whether our assumptions on solutions $\mathcal{V}(z)$ to (5) (which are once more of a certain minimal type) hold true for large classes of coefficient matrices $\mathcal{M}(z)$. On the other hand, for 2×2 systems arising from special second-order AΔEs that have been encountered in various contexts, we sketch results from which the validity of our assumptions follows. (Indeed, the latter were inspired by the features of the relevant special functions.)

2. First-order AΔEs: minimal solutions

In this section, we study the first-order AΔE (1) with $a > 0$ and $\Phi \in \mathcal{M}^*$. Introducing strips

$$S_d \equiv \{z \in \mathbb{C} \mid |\mathrm{Im}\, z| < d\}, \quad d \in (0, \infty), \tag{6}$$

we only consider functions Φ that have no poles and zeros in S_c for some $c > 0$. Therefore, $\log \Phi(z)$ is well defined in S_c up to a multiple of $2\pi i$. Setting

$$\phi(z) \equiv \log \Phi(z), \quad z \in S_c, \tag{7}$$

(with a specific branch choice understood), we now begin by obtaining special solutions to the logarithmic version

$$\lambda(z + ia/2) - \lambda(z - ia/2) = \phi(z), \quad z \in S_c, \tag{8}$$

of (1), assuming polynomial boundedness of $\phi(z)$. More precisely, we prove a somewhat more general theorem, whose specialization to (8) is immediate.

THEOREM 1. *Assume $\psi(z)$ is a function that is analytic in S_c for some $c > 0$ and that satisfies*

$$\psi(z) = O(|z|^{\nu}), \quad z \in S_c, \quad |z| \to \infty, \tag{9}$$

for some $\nu > -1$, uniformly on closed substrips of S_c. Then the function

$$h(z) \equiv \frac{\pi}{2ia^2} \int_{-\infty}^{\infty} (\psi(z - x)/ \cosh^2(\pi x/a))\, dx, \quad z \in S_c, \tag{10}$$

admits analytic continuation to $S_{c+a/2}$. Introducing

$$H(z) \equiv \int_0^z h(w)\, dw, \quad z \in S_{c+a/2}, \tag{11}$$

there exists $\gamma \in \mathbb{C}$ such that the function

$$\kappa(z) \equiv H(z) + \gamma z \tag{12}$$

satisfies the AΔE

$$\kappa(z + ia/2) - \kappa(z - ia/2) = \psi(z), \quad z \in S_c. \tag{13}$$

Moreover, we have the bound

$$\kappa(z) - \gamma z = O(|z|^{\nu+1}), \quad z \in S_{c+a/2}, \quad |z| \to \infty, \tag{14}$$

uniformly on closed substrips of $S_{c+a/2}$.

Evidently, we need only take $\psi \to \phi$ and $\kappa \to \lambda$ to apply this to (8), and then we obtain a solution to (1) by setting $F(z) \equiv \exp(\lambda(z))$. We emphasize that this solution is expressed explicitly in terms of $\phi(z)$. It is also clear that the solution is minimal, in the sense defined in the Introduction. We point out that the arbitrary choice of $2\pi i$-multiple in (7) gives rise to the integer k in (4). Note that the constant C in (4) equals 1 for the solutions derived from Theorem 1, because one clearly has $\kappa(0) = 0$. The above theorem contains as special cases various results on minimal solutions for coefficients with additional properties [12, 15]. Apart from a technical lemma that we state and prove first, its proof is adapted from the proof of theorem A.2 in [15].

LEMMA 1. *Assume* $A(z)$ *is a function that is analytic in* S_c *and satisfies a bound*

$$A(z) = O(|z|^{\xi}), \quad \xi \in \mathbb{R}, \quad z \in S_c, \quad |z| \to \infty, \tag{15}$$

uniformly on closed substrips of S_c. *Then the function*

$$B(z) \equiv \int_{-\infty}^{\infty} (A(z-x)/\cosh^2(\pi x/a))\, dx, \quad z \in S_c, \tag{16}$$

extends to an analytic function in $S_{c+a/2}$ *and satisfies a bound*

$$B(z) = O(|z|^{\xi}), \quad z \in S_{c+a/2}, \quad |z| \to \infty, \tag{17}$$

uniformly on closed substrips of $S_{c+a/2}$.

Proof of Lemma 1. Clearly, $B(z)$ is well defined and analytic in S_c. Shifting contours, we deduce it analytically continues to $S_{c+a/2}$. Specifically, choosing $z \in S_{c+a/2}$, we can find $v \in (-a/2, a/2)$ such that $z - iv \in S_c$. Then, the continuation reads

$$B(z) = \int_{\mathbb{R}+iv} (A(z-w)/\cosh^2(\pi w/a))\, dw. \tag{18}$$

We first assume $\xi \geq 0$. Then (15) entails we have a bound

$$|A(z-w)| \leq C(1 + |z-w|^{\xi}), \tag{19}$$

on the contour, where $C > 0$ can be chosen uniformly for $|\mathrm{Im}\,(z-w)| \leq d < c$. Also, letting $|z| > 1$, we get

$$|z - w| < |z|(1 + |w|). \tag{20}$$

Hence, we obtain

$$|B(z)| \leq C \int_{\mathbb{R}+iv} \frac{(1+|z|^{\xi}(1+|w|)^{\xi})}{|\cosh^2(\pi w/a)|} dw$$
$$\leq C(C_1 + C_2|z|^{\xi}), \qquad z \in S_{c+a/2}, \tag{21}$$

so that (17) results.

Turning to the case $\xi < 0$, we get from (15)

$$|A(z-w)| \leq C/\big(1+|z-w|^{|\xi|}\big), \tag{22}$$

uniformly for $|\mathrm{Im}\,(z-w)| \leq d < c$. Hence, choosing first $\mathrm{Re}\, z$ greater than $2c + a$ (say), we have from (18)

$$(\mathrm{Re}\, z)^{|\xi|}|B(z)| \leq C \int_{\mathbb{R}+iv} \frac{(\mathrm{Re}\, z)^{|\xi|}}{\big(1+\big|z-w\big|^{|\xi|}\big)} \frac{1}{|\cosh^2(\pi w/a)|} dw$$
$$\leq C \int_{-\infty+iv}^{\mathrm{Re}\,(z/2)+iv} \frac{(\mathrm{Re}\, z)^{|\xi|}}{\big(1+[\mathrm{Re}\,(z/2)]^{|\xi|}\big)} \frac{1}{|\cosh^2(\pi w/a)|} dw$$
$$+ C(\mathrm{Re}\, z)^{|\xi|} \int_{\mathrm{Re}\,(z/2)+iv}^{\infty+iv} \frac{dw}{|\cosh^2(\pi w/a)|}. \tag{23}$$

Obviously the first integral remains bounded as $\mathrm{Re}\, z \to \infty$. The second one is

$$O((\mathrm{Re}\, z)^{|\xi|} \exp(-\pi \mathrm{Re}\, z/a)) = O(1), \tag{24}$$

as $\mathrm{Re}\, z \to \infty$, too. Hence (17) follows for $\mathrm{Re}\, z \to \infty$, and likewise for $\mathrm{Re}\, z \to -\infty$. ■

Using this lemma, we now prove the theorem.

Proof of Theorem 1. Lemma 1 entails that $h(z)$ is analytic and $O(|z|^{\nu})$ in $S_{c+a/2}$, uniformly on closed substrips. We claim that $h(z)$ satisfies the AΔE

$$h(z+ia/2) - h(z-ia/2) = \psi'(z), \quad z \in S_c. \tag{25}$$

Taking this for granted, it is clear that the derivative of

$$H(z+ia/2) - H(z-ia/2) - \psi(z) \tag{26}$$

vanishes. Thus, there exists $\beta \in \mathbb{C}$ such that

$$H(z+ia/2) - H(z-ia/2) = \psi(z) + \beta. \tag{27}$$

Setting now $\gamma \equiv i\beta/a$, the assertions (13) and (14) readily follow.

It remains to prove the claim (25). To this end we fix z in the strip S_c and choose numbers t_-, t_+ in the interval $(-c, c)$ such that

$$t_+ - \mathrm{Im}\, z \in (0, a), \quad t_- - \mathrm{Im}\, z \in (-a, 0). \tag{28}$$

Then, we have

$$h(z \pm ia/2) = \frac{\pi}{2ia^2} \int_{-\infty}^{\infty} \left(\psi(x + it_\pm) \Big/ \cosh^2 \frac{\pi}{a} \left(z \pm \frac{ia}{2} - x - it_\pm \right) \right) dx, \tag{29}$$

so we may write

$$h(z + ia/2) - h(z - ia/2) = \frac{i\pi}{2a^2} \int_{-\infty}^{\infty} \frac{\psi(x + it_+)}{\sinh^2 \frac{\pi}{a}(z - (x + it_+))} dx - \frac{i\pi}{2a^2} \int_{-\infty}^{\infty} \frac{\psi(x + it_-)}{\sinh^2 \frac{\pi}{a}(z - (x + it_-))} dx. \tag{30}$$

The right-hand side may be viewed as a contour integral

$$\frac{i\pi}{2a^2} \int_\Gamma \left(\psi(w) \Big/ \sinh^2 \frac{\pi}{a}(z - w) \right) dw, \tag{31}$$

with Γ consisting of the lines $w = x + it_+$ and $w = x + it_-$, $x \in \mathbb{R}$, traversed from left to right and right to left, respectively. By a standard application of Cauchy's theorem, it equals $-2\pi i$ times the residue at the pole $w = z$. A routine calculation now shows that (31) equals $\psi'(z)$, and so our claim (25) follows. ■

We proceed to define a large class of coefficients $\Phi(z)$ that satisfy the above assumptions. To this end we introduce wedges

$$W_\chi \equiv \{z = re^{i\theta} \in \mathbb{C} \mid r > 0, \quad \theta \in (-\chi, \chi) \text{ or } \theta \in (\pi - \chi, \pi + \chi)\}, \quad \chi \in (0, \pi/2), \tag{32}$$

and assume that $a_k, b_l, k, l \in \mathbb{N}$, are complex numbers that belong to the set

$$\mathcal{S} \equiv \mathbb{C} \backslash (S_c \cup W_\chi), \tag{33}$$

and that also have the following properties. First, one has $a_k \neq b_l$ for all $k, l \in \mathbb{N}$, whereas $a_k = a_{k_0}$ and $b_l = b_{l_0}$ are allowed for finitely many $k \neq k_0$ and $l \neq l_0$. Second, there exists $\rho \in [0, \infty)$ such that

$$\sum_{k=0}^{\infty} \frac{1}{|a_k|^{\rho+\epsilon}} < \infty, \quad \sum_{l=0}^{\infty} \frac{1}{|b_l|^{\rho+\epsilon}} < \infty, \quad \forall \epsilon > 0, \tag{34}$$

and ρ is the smallest number so that (34) holds.

Next, we denote by M the integer part of ρ,

$$M \equiv [\rho], \tag{35}$$

and define the infinite products

$$\Phi_0(z) \equiv \prod_{k=0}^{\infty} E\left(\frac{z}{a_k}, M\right), \tag{36}$$

$$\Phi_\infty(z) \equiv \prod_{l=0}^{\infty} E\left(\frac{z}{b_l}, M\right), \tag{37}$$

where

$$E(w,0) \equiv 1 - w,$$
$$E(w,n) \equiv (1-w)\exp(w + w^2/2 + \cdots + w^n/n), \quad n \in \mathbb{N}^*. \tag{38}$$

(Here and below, $\mathbb{N}^*$ denotes $\mathbb{N}\backslash\{0\}$.) By virtue of our assumptions, these infinite products are convergent, yielding entire functions. Finally, let $Q_M(z)$ be a polynomial of degree $\leq M$. Then, the meromorphic function

$$\Phi(z) \equiv \exp(Q_M(z))\Phi_0(z)/\Phi_\infty(z) \tag{39}$$

has zeros a_k and poles b_l. We are now prepared for our next theorem, which concludes this section.

THEOREM 2. *For the above coefficients* $\Phi(z)$ *the* AΔE *(1) admits a meromorphic solution* $F(z)$ *that has no zeros and poles in* $S_{c+a/2}$; *it satisfies*

$$\frac{d^j}{dz^j}\log F(z) = O(|z|^{\rho+1-j+\epsilon}), \quad \forall j \in \mathbb{N}, \quad \forall \epsilon > 0,$$
$$z \in S_{c+a/2}, \quad |z| \to \infty, \tag{40}$$

with the bounds uniform on closed substrips of $S_{c+a/2}$.

Proof. We introduce the functions

$$\phi_j(z) \equiv \frac{d^j}{dz^j}\log \Phi(z), \quad j \in \mathbb{N}, \quad z \in S_c, \tag{41}$$

where the branch is fixed by requiring $\phi_0(0) = Q_M(0)$. Now we observe that we have

$$\phi_j(z) = (-1)^{j-1}(j-1)!\sum_{k=0}^{\infty}\left(\frac{1}{(z-a_k)^j} - \frac{1}{(z-b_k)^j}\right),$$
$$j \geq M+1, \quad z \in S_c. \tag{42}$$

Introducing the $\mathcal{S}$-subset

$$\mathcal{S}_r \equiv \{w \in \mathcal{S} \mid |w| > 2c/\sin\chi\}, \tag{43}$$

we claim that we have the estimates

$$|z - w| > |w|\sin(\chi)/2, \tag{44}$$

$$|z - w| > |z|\sin(\chi)/2, \tag{45}$$

for all $w \in \mathcal{S}_r$ and $z \in S_c$. Taking this claim for granted, we infer that for $j \geq M+1$, $z \in S_c$, $w \in \mathcal{S}_r$, $\epsilon \in (0, j-\rho]$, we have

$$\begin{aligned} |z-w|^j &= |z-w|^{j-\rho-\epsilon}|z-w|^{\rho+\epsilon} \\ &> |z|^{j-\rho-\epsilon}[\sin(\chi)/2]^{j-\rho-\epsilon}|w|^{\rho+\epsilon}[\sin(\chi)/2]^{\rho+\epsilon} \\ &> [\sin(\chi)/2]^j|w|^{\rho+\epsilon}|z|^{j-\rho-\epsilon}. \end{aligned} \tag{46}$$

Now because the numbers a_k, b_l belong to $\mathcal{S}$ and satisfy (34), only finitely many among them do not belong to $\mathcal{S}_r$. From (46) we therefore have

$$\left| \sum_k{}' \frac{1}{(z-a_k)^j} - \sum_l{}' \frac{1}{(z-b_l)^j} \right| \leq C \left(\sum_k{}' \frac{|z|^{\rho-j+\epsilon}}{|a_k|^{\rho+\epsilon}} + \sum_l{}' \frac{|z|^{\rho-j+\epsilon}}{|b_l|^{\rho+\epsilon}} \right), \quad j \geq M+1, \quad z \in S_c, \tag{47}$$

where the primes signify that we omit the finitely many terms where the bounds (44) and (45) are violated for $w = a_k$ and $w = b_l$. Using the convergence assumption (34), we deduce

$$\phi_j(z) = O(|z|^{\rho-j+\epsilon}), \quad j \geq M+1, \quad \epsilon > 0, \quad z \in S_c, \quad |z| \to \infty, \tag{48}$$

uniformly on S_c. It then follows by integrating $\phi_{M+1}(z)$ that we also have

$$\phi_j(z) = O(|z|^{\rho-j+\epsilon}), \quad j = 0, \ldots, M, \quad \epsilon > 0, \quad z \in S_c, \quad |z| \to \infty, \tag{49}$$

uniformly on S_c. (To see this, note that (35) implies $\rho - M - 1 + \epsilon > -1$ for all $\epsilon > 0$.)

In view of (49) with $j = 0$, we can invoke Theorem 1 with

$$\psi(z) = \phi_0(z). \tag{50}$$

This yields a function

$$\lambda(z) = \frac{\pi}{2ia^2} \int_0^z h(w)\, dw + \gamma z, \tag{51}$$

where

$$h(z) \equiv \frac{\pi}{2ia^2} \int_{-\infty}^{\infty} \left(\phi_0(z-x)\big/\cosh^2(\pi x/a)\right) dx, \quad z \in S_c, \tag{52}$$

satisfying the AΔE

$$\lambda(z + ia/2) - \lambda(z - ia/2) = \phi_0(z), \quad z \in S_c. \tag{53}$$

Setting

$$F(z) \equiv \exp(\lambda(z)), \tag{54}$$

we obtain a solution to (1), satisfying (40) for $j = 0$. Noting (51) and (52) imply

$$\frac{d}{dz}\lambda(z) = \frac{\pi}{2ia^2}\int_{-\infty}^{\infty}\big(\phi_0(z-x)/\cosh^2(\pi x/a)\big)\,dx + \gamma, \quad z \in S_c, \tag{55}$$

$$\frac{d^j}{dz^j}\lambda(z) = \frac{\pi}{2ia^2}\int_{-\infty}^{\infty}\big(\phi_{j-1}(z-x)/\cosh^2(\pi x/a)\big)\,dx, \quad j \geq 2, \quad z \in S_c, \tag{56}$$

we obtain (40) for $j > 0$ from the bounds (48) and (49) and Lemma 1.

It remains to prove the above claim. To this end we first observe that (43) entails that the distance between points in $\mathcal{S}_r$ and S_c is greater than c. More precisely, the minimal distance of the points $w \in \mathcal{S}_r$ on the circle $|w| = R$ to the closure of S_c equals $R\sin\chi - c$. Thus, we have $|w - z| > |w|\sin\chi - c$ for $w \in \mathcal{S}_r$ and $z \in S_c$. Because $c < |w|\sin(\chi)/2$ on $\mathcal{S}_r$, we deduce (44).

Turning to (45), consider first $z \in S_c$ with $|z| < 2c/\sin\chi$. Then, we get $|z - w| > c > |z|\sin(\chi)/2$ for all $w \in \mathcal{S}_r$, and so (45) results. Thus, we are left with points in S_c satisfying $|z| \geq 2c/\sin\chi$. By symmetry, it suffices to handle the case $\mathrm{Re}\, z > 0$, $\mathrm{Im}\, w > 0$.

Consider $z = Re^{i\psi}$ with $R \geq 2c/\sin\chi$ fixed. Then, we have $\psi < \psi_+$, where $\sin\psi_+ = c/R$, and hence

$$|z - w| > R\sin(\chi - \psi_+) = |z|\sin(\chi - \psi_+). \tag{57}$$

Now the largest ψ_+ arises for $R = 2c/\sin\chi$, so that

$$\sin\psi_+ = c/R \leq \sin(\chi)/2 \equiv \sin\psi_{\max}. \tag{58}$$

Consequently, we have

$$\begin{aligned}\sin(\chi - \psi_+) \geq\ & \sin(\chi - \psi_{\max}) = \sin\chi[\cos\psi_{\max} - \cos(\chi)/2] \\ =\ & 2^{-1}\sin(\chi)[(4 - \sin^2(\chi))^{1/2} - \cos\chi] > \sin(\chi)/2.\end{aligned} \tag{59}$$

Combining this with (57), we obtain (45). ■

3. First-order AΔEs: Nevanlinna-type results

In this section we reconsider the AΔE (1) from the viewpoint of Nevanlinna theory. From now on we denote the class of meromorphic functions with order $\rho \in [0, \infty)$ by $\mathcal{C}_\rho$. We first obtain a general result (Theorem 3) concerning certain minimal solutions for coefficients $\Phi(z)$ in a subclass $\mathcal{C}_{\rho,\min}$ of $\mathcal{C}_\rho$. It yields an upper bound $\rho + 1$ on the order of the solutions. Theorem 3 involves the technical Lemma 2 that is proved in the Appendix. With Theorem 3 available, we then proceed to show that the subclass $\mathcal{C}_{\rho,\min}$ contains all $\Phi \in \mathcal{C}_\rho$ that have no zeros and poles in a strip around the real axis (Theorem 4). A key ingredient of the proof is an illuminating corollary of the Poisson–Jensen formula, which we isolate in Lemma 3 (also proved in the Appendix). We conclude Section 3 with further observations on AΔEs admitting minimal solutions, including some special cases studied in [12] and [15].

By definition, $\mathcal{C}_{\rho,\min}$ consists of all $\Phi \in \mathcal{C}_\rho$ that admit solutions $F(z)$ to (1) with the following two properties: (a) $F(z)$ has no poles and zeros in the "period" strip

$$P_0 \equiv \{z \in \mathbb{C} \mid -a/2 \le \operatorname{Im} z \le a/2\}; \tag{60}$$

(b) $F(z)$ satisfies

$$\log F(z) = O(r^{\rho+1+\epsilon}), \quad \forall \epsilon > 0, \quad r = |z| \to \infty, \quad z \in P_0, \tag{61}$$

uniformly on P_0. We point out that these solutions are minimal, as defined in the Introduction.

THEOREM 3. *Assume $\Phi \in \mathcal{C}_{\rho,\min}$. Then, the order of a solution $F(z)$ with properties (a) and (b) is $\rho + 1$ at most.*

Our proof of this theorem involves quite a few technicalities. Even so, the underlying ideas are not hard to understand. To bring them out more clearly, we try and isolate them before embarking on the full proof.

To this end, we first recall that a meromorphic function $f(z)$ by definition has order σ when the characteristic function $T(r, f)$, $r = |z|$, has order σ for $r \to \infty$ [13]. From now on, we usually omit the argument α in the counting functions $n(r, \alpha, f)$, $N(r, \alpha, f)$ and proximity function $m(r, \alpha, f)$ when α equals ∞. With this convention, we continue to recall that we have [13]

$$T(r, f) = N(r, f) + m(r, f). \tag{62}$$

Therefore, to prove Theorem 3, it suffices to prove that both $N(r, f)$ and $m(r, f)$ have order $\le \rho + 1$.

Now it is not hard to see why the order ρ assumption on Φ entails that the pole counting function $N(r, F)$ has order $\le \rho + 1$, and we proceed to explain this in general terms. Defining shifted functions

$$\Phi_j(z) \equiv \Phi(z - (j - 1/2)ia), \quad j \in \mathbb{Z}, \tag{63}$$

we note (1) entails

$$F(z) = \prod_{j=1}^{N} \Phi_j(z) \cdot F(z - iNa), \tag{64}$$

and also

$$F(z) = \prod_{j=0}^{N-1} \frac{1}{\Phi_{-j}(z)} \cdot F(z + iNa). \tag{65}$$

If we now introduce the strips

$$P_j \equiv \{z \in \mathbb{C} \mid (j - 1/2)a < \operatorname{Im} z \leq (j + 1/2)a\}, \quad j = 1, 2, 3, \ldots, \tag{66}$$

and let z vary over P_N, $N > 0$, it is clear from property (a) and (64) that eventual poles of $F(z)$ arise from poles of $\Phi(z)$ in the strips $P_k - ia/2$, $k = 1, \ldots, N$. Likewise, setting

$$P_j \equiv \{z \in \mathbb{C} \mid (j - 1/2)a \leq \operatorname{Im} z < (j + 1/2)a\}, \quad j = -1, -2, -3, \ldots, \tag{67}$$

it follows from (65) that eventual poles of $F(z)$ for $z \in P_{-N}$ are due to zeros of $\Phi(z)$ in the strips $P_{-k} + ia/2$, $k = 1, \ldots, N$.

From this state of affairs it is easy to deduce that the pole counting function $n(r, F)$ (and hence $N(r, F)$ as well) has order $\leq \rho + 1$. The details can be found in the proof of Theorem 3 below. However, our main difficulty is to show that the proximity function $m(r, F)$ has order $\leq \rho + 1$, too.

To handle $m(r, F)$, we employ bounds on the proximity functions of the translates Φ_j (63) of Φ. The maximal translation that is needed to estimate $m(r, F)$ for large r is on the order of r. Thus, we can derive the pertinent bounds from the following lemma. (Recall $\mathcal{M}^*$ denotes the space of meromorphic functions that do not vanish identically.)

LEMMA 2. *Let $f \in \mathcal{M}^*$ and $\eta \in \mathbb{C}$, and set*

$$f_\eta(z) \equiv f(z - \eta). \tag{68}$$

Then, we have a uniform bound

$$m(r, f_\eta) \leq 5m(3r, f) + \log(4)n(3r, f), \quad |\eta| < r. \tag{69}$$

We relegate the proof of this lemma to the Appendix. It is clear from the proof that the numerical constants in (69) can be sharpened, but this is of no consequence. In fact, we will use the lemma in the slightly weaker form

$$m(r, f_\eta) \leq CT(4r, f), \quad |\eta| < r, \tag{70}$$

where C is an η-independent numerical constant. To see that (70) follows from (69), it suffices to observe that we have a bound

$$\log(4/3)n(3r, f) \leq \int_{3r}^{4r} n(s, f)\frac{ds}{s} = N(4r, f) - N(3r, f) \leq T(4r, f). \tag{71}$$

We would like to add that (70) with $m(r, f_\eta)$ replaced by $T(r, f_\eta)$ is true for entire f. This can be easily deduced from Theorem 1.6 in [13], in which maximum modulus and Nevanlinna characteristic are compared. However, the inequality

$$N(r, f_\eta) \le C_1 T(C_2 r, f) + C_3, \quad |\eta| < r, \tag{72}$$

does not hold *uniformly* in η whenever f has at least one pole for $|z| < r$. This is a consequence of the nonuniform weighting of an ϵ-neighborhood of the origin in the definition

$$N(r, f) = \int_0^r [n(s, f) - n(0, f)]\frac{ds}{s} + n(0, f)\log r. \tag{73}$$

To see this, assume a pole occurs at a location p with $|p| < r$ and consider a sequence $\eta_j = -p - \epsilon_j$ with $\epsilon_j \downarrow 0$ as $j \to \infty$. Then, there exists an $M \in \mathbb{N}$ such that

$$n(0, f_{\eta_j}) = 0, \quad n(\epsilon_j, f_{\eta_j}) \ge 1, \quad \forall j > M. \tag{74}$$

Hence we have

$$N(r, f_{\eta_j}) \ge \int_{\epsilon_j}^r \frac{ds}{s} = \log(r/\epsilon_j), \quad \forall j > M, \tag{75}$$

and because the right-hand side diverges as $j \to \infty$, it follows that no uniform bound of the form (72) can hold.

After this appraisal of the lemma, we turn to the proof of the theorem.

Proof of Theorem 3. We begin by estimating the pole counting function $n(r, F)$. Because $F(z)$ has by assumption no poles in P_0, we have $n(r, F) = 0$ for $r \le a/2$. Fixing $r > a/2$, we can find $M \in \mathbb{N}$ such that

$$(M - 1/2)a < r \le (M + 1/2)a. \tag{76}$$

Defining the intersections

$$I_j(r) \equiv P_j \cap \{|z| \le r\}, \quad j \in \mathbb{Z}, \tag{77}$$

we infer

$$\begin{aligned} n(r, F) &\equiv \{\text{no. of poles of } F(z) \text{ in } \{|z| \le r\}\} \\ &= \sum_{j=-M}^{M} \{\text{no. of poles of } F(z) \text{ in } I_j(r)\}. \end{aligned} \tag{78}$$

Using (64) and (65), this can be rewritten

$$
\begin{aligned}
n(r, F) &= \sum_{k=1}^{M}\left\{\text{no. of poles of } \prod_{j=1}^{k}\Phi_j(z) \text{ in } I_k(r)\right\}\\
&\quad + \sum_{k=1}^{M}\left\{\text{no. of zeros of } \prod_{j=0}^{k-1}\Phi_{-j}(z) \text{ in } I_{-k}(r)\right\}\\
&\le M \times \{\text{no. of poles of } \Phi(z) \text{ in } \{|z| \le r\} \cap \{\operatorname{Im} z > 0\}\}\\
&\quad + M \times \{\text{no. of zeros of } \Phi(z) \text{ in } \{|z| \le r\} \cap \{\operatorname{Im} z < 0\}\}\\
&\le M[n(r, \infty, \Phi) + n(r, 0, \Phi)]. \qquad (79)
\end{aligned}
$$

From this we deduce

$$
n(r, F) \le \frac{(r + a/2)}{a}[n(r, \infty, \Phi) + n(r, 0, \Phi)]. \qquad (80)
$$

We now invoke our assumption that Φ has order $\rho < \infty$. It entails

$$
n(r, \alpha, \Phi) = O(r^{\rho+\epsilon}), \quad \forall\epsilon > 0, \quad \forall\alpha \in \mathbb{C} \cup \{\infty\}. \qquad (81)
$$

Combining this with (80), we obtain

$$
n(r, F) = O(r^{\rho+1+\epsilon}), \quad \forall\epsilon > 0. \qquad (82)
$$

Hence, we get

$$
N(r, F) = \int_{a/2}^{r} n(s, F)\frac{ds}{s} = O(r^{\rho+1+\epsilon}), \quad \forall\epsilon > 0. \qquad (83)
$$

To complete the proof of the theorem, it remains to show that the proximity function $m(r, F)$ is also $O(r^{\rho+1+\epsilon})$ for any positive ϵ. (A priori, m might grow faster than this.) To this end we again begin by choosing r satisfying (76), and exploit the formulae (64) and (65) on the arcs $A_j(r)$, $j = -M, \dots, M$, where

$$
A_j(r) \equiv \{\theta \in [0, 2\pi) \mid re^{i\theta} \in P_j\}, \quad j \in \mathbb{Z}. \qquad (84)
$$

Specifically, we write (suppressing the dependence of the arcs on r to ease the notation)

$$
\begin{aligned}
2\pi m(r, F) &= \int_0^{2\pi} \log^+ |F(re^{i\theta})|\, d\theta \\
&= \sum_{k=1}^{M} \int_{A_k} \log^+ \left| \prod_{j=1}^{k} \Phi_j(re^{i\theta}) \cdot F(re^{i\theta} - ika) \right| d\theta \\
&\quad + \int_{A_0} \log^+ |F(re^{i\theta})|\, d\theta \\
&\quad + \sum_{k=1}^{M} \int_{A_{-k}} \log^+ \left| \prod_{j=0}^{k-1} \frac{1}{\Phi_{-j}(re^{i\theta})} \cdot F(re^{i\theta} + ika) \right| d\theta. \qquad (85)
\end{aligned}
$$

Clearly, this leads to a bound

$$
\begin{aligned}
2\pi m(r, F) &\le \sum_{k=1}^{M} \sum_{j=1}^{k} \left(\int_{A_k} \log^+ |\Phi_j(re^{i\theta})|\, d\theta + \int_{A_{-k}} \log^+ \left| \frac{1}{\Phi_{-j+1}(re^{i\theta})} \right| d\theta \right) \\
&\quad + \sum_{k=-M}^{M} \int_{A_k} \log^+ |F(re^{i\theta} - ika)|\, d\theta. \qquad (86)
\end{aligned}
$$

The functions $\log^+|\Phi_j(re^{i\theta})|$ and $\log^+|1/\Phi_{-j+1}(re^{i\theta})|$ on the right-hand side are integrated over the subsets $\cup_{k=j}^{M} A_k$ and $\cup_{k=j}^{M} A_{-k}$ of $[0, 2\pi]$, respectively. Also, the numbers $re^{i\theta} - ika$ with $\theta \in A_k$ belong to P_0 and have modulus $\le r$. Thus, we obtain a majorization

$$
\begin{aligned}
2\pi m(r, F) &\le \sum_{j=1}^{M} \int_0^{2\pi} \left(\log^+ \left|\Phi_j(re^{i\theta})\right| + \log^+ \left| \frac{1}{\Phi_{-j+1}(re^{i\theta})} \right| \right) d\theta \\
&\quad + 2\pi \max_{z \in P_0,\, |z| \le r} \log^+ |F(z)|. \qquad (87)
\end{aligned}
$$

Using the assumption (61), this implies

$$
m(r, F) = \sum_{j=1}^{M} [m(r, \Phi_j) + m(r, 1/\Phi_{-j+1})] + O(r^{\rho+1+\epsilon}). \qquad (88)
$$

We are now in the position to invoke the bound (70). Indeed, the argument shifts in $\Phi_k(z)$, $k = -M + 1, \ldots, M$, are smaller than r (cf., (63), (76)), so it yields

$$
m(r, \Phi_j) \le CT(4r, \Phi), \quad m(r, 1/\Phi_{-j+1}) \le CT(4r, 1/\Phi), \quad j = 1, \ldots, M. \qquad (89)
$$

Because $\Phi(z)$ and $1/\Phi(z)$ have order ρ, we finally obtain

$$
m(r, F) \le CM[T(4r, \Phi) + T(4r, 1/\Phi)] + O(r^{\rho+1+\epsilon}) = O(r^{\rho+1+\epsilon}), \quad \forall \epsilon > 0, \qquad (90)
$$

completing the proof. ∎

We proceed to study the class $\mathcal{C}_{\rho,\min}$ in more detail. Obviously, the existence of a solution $F(z)$ with the above properties is not compatible with arbitrary functions $\Phi(z)$ of order $\rho < \infty$. For one thing, as already noted in the Introduction, absence of zeros and poles of $F(z)$ in P_0 (60) implies that any $\Phi \in \mathcal{C}_{\rho,\min}$ has no poles and zeros for real z, cf. (1). Unfortunately, this necessary condition is very likely not sufficient to guarantee $\Phi \in \mathcal{C}_{\rho,\min}$. It seems not an easy matter to obtain illuminating, necessary, and sufficient conditions entailing $\Phi \in \mathcal{C}_{\rho,\min}$. (We elaborate on this after proving Theorem 4.)

On the other hand, we can exploit the Poisson–Jensen formula to derive a most useful sufficient condition for Φ to belong to $\mathcal{C}_{\rho,\min}$. The key point is encoded in the following lemma, whose proof is given in the Appendix.

LEMMA 3. *Assume $f \in \mathcal{M}^*$ has order $\sigma < \infty$ and has no poles and zeros in a strip S_d (6). Then, we have*

$$\log f(z) = O(r^{\sigma+\epsilon}), \quad \forall\epsilon > 0, \quad r = |z| \to \infty, \quad z \in S_d, \tag{91}$$

uniformly on closed substrips of S_d.

As a corollary of this lemma and several previous results, we can now show that the class $\mathcal{C}_{\rho,c}$ of functions $\Phi \in \mathcal{C}_\rho$ without zeros and poles in a strip S_c with $c > 0$ is a subclass of $\mathcal{C}_{\rho,\min}$. In more detail, we have the following theorem.

THEOREM 4. *Assume $\Phi \in \mathcal{C}_{\rho,c}$. Then (1) admits a solution $F(z)$ with the following properties: (i) $F(z)$ has no poles and zeros in $S_{c+a/2}$; (ii) $F(z)$ satisfies*

$$\log F(z) = O(r^{\rho+1+\epsilon}), \quad \forall\epsilon > 0, \quad r = |z| \to \infty, \quad z \in S_{c+a/2}, \tag{92}$$

uniformly on closed substrips of $S_{c+a/2}$; (iii) $F(z)$ has order $\leq \rho + 1$. Moreover, we have the inclusion

$$\mathcal{C}_{\rho,c} \subset \mathcal{C}_{\rho,\min}, \quad \forall c > 0. \tag{93}$$

Proof. Invoking Lemma 3 with $f = \Phi$, $\sigma = \rho$, and $d = c$, we see that $\psi \equiv \log\Phi$ satisfies the assumptions of Theorem 1 with $\nu = \rho + \epsilon$. Setting $F(z) = \exp(\kappa(z))$, where $\kappa(z)$ is given by (12) we therefore obtain a solution satisfying (i) and (ii). As a consequence, this solution satisfies the assumptions defining $\mathcal{C}_{\rho,\min}$, so that the inclusion (93) holds true. Therefore, (iii) follows from Theorem 3. ■

Now that we have established (93), we are in the position to shed more light on the question of necessary and sufficient conditions for Φ to belong to $\mathcal{C}_{\rho,\min}$. (Of course, we assume $\Phi \in \mathcal{C}_\rho$ to begin with.) As mentioned before, it is necessary that Φ have no poles and zeros for real z. In view of (93), it therefore remains to consider the case that $\Phi(z)$ has a sequence of poles or zeros converging to the real axis. Because the convergence can be arbitrarily fast as $r \to \infty$ (take for example poles at $n + i \exp(-\exp(n))$ with $n \in \mathbb{N}$), the function $\log \Phi(z)$ can diverge arbitrarily fast for a corresponding sequence z_n on the real axis. If we now suppose a solution $F(z)$ without poles and zeros in P_0 exists, it cannot satisfy a bound (61) for $|\mathrm{Im}\, z| = a/2$, cf. (1). On the other hand, if the sequence approaches the real axis sufficiently slowly, then a solution with the above two features (a) and (b) may well exist, in which case Φ does belong to $\mathcal{C}_{\rho,\min}$.

Returning to the class $\mathcal{C}_{\rho,c}$, we proceed to observe that all of the functions (39) belong to it. (Indeed, the assumptions on the zeros and poles entail that at least one of the entire functions Φ_0 and Φ_∞ has order ρ, so that Φ has order ρ. Moreover, by definition the strip S_c is free of zeros and poles.) Therefore, we can reobtain part of Theorem 2 as a corollary of Theorem 4.

To be specific, the properties (i) and (ii) of $F(z)$ obtained in Theorem 4 are also asserted to hold in Theorem 2. Moreover, the Poisson–Jensen formula reasoning yielding (91) gives rise to a quick proof of the key bound (49) with $j = 0$, because the wedge features of the zeros and poles need not be invoked. On the other hand, the bounds (48) and (49) with $j > 0$ (and hence (40) with $j > 0$) cannot be obtained in this way. Indeed, all logarithmic derivatives of functions of the form (39) have order ρ, so that one gets the same bound as for $j = 0$.

Consider next the larger class of order ρ coefficients $\Phi(z)$ (39) obtained by relaxing the restriction $a_k, b_l \in \mathcal{S}$ (cf. (33)) to $a_k,\ b_l \notin S_c$. These coefficients still belong to $\mathcal{C}_{\rho,c}$, so that Theorem 4 applies. However, now it can happen that $\Phi(z)$ has a real period. If so, minimal solutions may have a real period, too, entailing all of their logarithmic derivatives have a real period. Thus, they are $O(1)$ in S_c, but not $o(1)$. Hence, they do not satisfy the bounds (40) when j is larger than $\rho + 1$.

Concrete examples of the latter state of affairs are the trigonometric and elliptic gamma functions from [12], both of which can be viewed as minimal solutions with a real period. Their associated coefficients belong to $\mathcal{C}_{1,c}$ and $\mathcal{C}_{2,c}$ for a suitable $c > 0$, so Theorem 4 implies their orders are smaller than or equal to 2 and 3, respectively. (In the trigonometric case, one has $\Phi(z) = 1 - \exp[2ir(z + ic)], r > 0$, whereas the elliptic Φ involves a shifted Weierstrass σ-function.) Because the orders of their pole counting functions are 2 and 3 (as is clear by inspection), their orders are in fact equal to 2 and 3, respectively. The hyperbolic gamma function from [12] is a minimal solution for the coefficient $\Phi(z) = 2\cosh(\pi z/b)$ with $b > 0$, so that $\Phi \in \mathcal{C}_{1,b/2}$. In this case, both Theorems 2 and 4 apply. The order of the pole counting functions of the hyperbolic gamma function equals 2, so it follows from Theorem 4 that it has order 2. (Here the bounds (40) can be substantially improved, cf. equation (3.49) in [12] and its derivatives.)

Barnes' multiple gamma function $\Gamma_N(w)$ [16] may also be reinterpreted as a minimal solution to which Theorems 2 and 4 apply, cf. [15]. This implies in particular that $\Gamma_N(w)$ has order N. Moreover, for Γ_N the bounds (40) yield new information that does not follow from the generalized Stirling series given by equation (3.13) in [15].

In this connection it should be stressed that the assumption $f \in \mathcal{C}_{\sigma,d}$ does not exclude polynomial (or even exponential) decrease to 0 of $\log f(z)$ and its derivatives as $|z|$ tends to ∞ in S_d. Stronger yet, also infinite-order meromorphic functions can be at once free of zeros and poles in a strip S_d (6) and have logarithmic derivatives that quickly converge to 0 as $|z|$ tends to ∞ in S_d.

We conclude this section with an example that not only illustrates the previous paragraph, but also shows that minimal solutions can exist for infinite-order Φ. Specifically, let

$$\Phi(z) \equiv \exp(\exp(-z^2)). \tag{94}$$

Clearly, Φ is an infinite-order function without poles and zeros. Moreover, we have

$$\exp(-z^2) = O(\exp(-(\mathrm{Re}\, z)^2)), \quad |\mathrm{Re}\, z| \to \infty, \tag{95}$$

uniformly for $\mathrm{Im}\, z$ in $\mathbb{R}$-compacts. Therefore, Theorem 1 may be invoked, yielding minimal solutions to the AΔE (1) with coefficient (94).

4. Higher-order AΔEs

Second-order AΔEs with meromorphic coefficients may be written in the form

$$F(z + ia) + C_1(z)F(z) + C_2(z)F(z - ia) = 0, \tag{96}$$

where $C_1, C_2 \in \mathcal{M}^*$. We assume that the coefficients C_1, C_2 have order $\leq \rho$, and aim to show that certain minimality assumptions on a solution $F \in \mathcal{M}^*$ entail that this solution has order $\leq \rho + 1$. (These assumptions can be verified for various special cases, as detailed below Theorem 5.)

Specifically, we allow $F(z)$ to have a finite number of poles in the strip $\{\mathrm{Im}\, z \in [-a, a]\}$. We also assume

$$\log^+|F(z)| = O(r^{\rho+1+\epsilon}), \quad \forall \epsilon > 0, \quad r = |z| \to \infty, \quad |\mathrm{Im}\, z| \leq a, \tag{97}$$

uniformly for $|\mathrm{Im}\, z| \leq a$. (Recall that $\log^+ x$ is defined by (A.10).)

It will follow from the theorem below that under these assumptions the order of the solution is indeed $\leq \rho + 1$. However, the theorem has a more general bearing. To explain this, let us first note that we can rewrite the above AΔE as a first-order 2×2 system

$$\mathcal{V}(z + ia/2) = \mathcal{M}(z)\mathcal{V}(z - ia/2), \tag{98}$$

where the components of the vector-valued meromorphic function are given by

$$\mathcal{V}(z)_1 \equiv F(z + ia/2), \quad \mathcal{V}(z)_2 \equiv F(z - ia/2), \tag{99}$$

and the elements of the matrix-valued meromorphic function by

$$\mathcal{M}(z)_{11} \equiv -C_1(z), \quad \mathcal{M}(z)_{12} \equiv -C_2(z), \quad \mathcal{M}(z)_{21} \equiv 1, \quad \mathcal{M}(z)_{22} \equiv 0. \tag{100}$$

Then the assumed properties of the solution $F(z)$ can be easily rephrased for $\mathcal{V}(z)$.

More generally, the theorem pertains to first-order $\Lambda \times \Lambda$ systems of the form (98), where the matrix elements $\mathcal{M}(z)_{\alpha\beta}, \alpha, \beta = 1, \dots, \Lambda$, are meromorphic functions of order $\leq \rho < \infty$ and where the determinant of $\mathcal{M}(z)$ does not vanish identically. (Thus, the matrix elements of the inverse $\mathcal{M}(z)^{-1}$ are also meromorphic functions of order $\leq \rho$.) Moreover, we assume properties of a given meromorphic vector solution $\mathcal{V}(z)$, which generalize those of the previous special case. Specifically, we assume: (1) the components $\mathcal{V}(z)_\alpha$ have no poles for z in the strip P_0 (60) with $|z| \geq R_0 \geq 0$; (2) for all $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$\log^+|\mathcal{V}(z)_\alpha| \leq C_\epsilon |z|^{\rho+1+\epsilon}, \quad \alpha = 1, \dots, \Lambda, \quad z \in P_0, \quad |z| \geq R_0. \tag{101}$$

(For the first-order case $\Lambda = 1$ these conditions are less restrictive than the conditions in Section 3. We elaborate on this after proving Theorem 5.)

In the following theorem the role of the formulae (64) and (65) is played by the relations

$$\mathcal{V}(z)_\alpha = \sum_{\beta_1,\dots,\beta_N=1}^{\Lambda} \mathcal{M}_1(z)_{\alpha\beta_1}\mathcal{M}_2(z)_{\beta_1\beta_2}\cdots\mathcal{M}_N(z)_{\beta_{N-1}\beta_N}\mathcal{V}(z-iNa)_{\beta_N}, \tag{102}$$

$$\mathcal{V}(z)_\alpha = \sum_{\beta_1,\dots,\beta_N=1}^{\Lambda} \mathcal{M}_0(z)^{-1}_{\alpha\beta_1}\mathcal{M}_{-1}(z)^{-1}_{\beta_1\beta_2}\cdots\mathcal{M}_{-N+1}(z)^{-1}_{\beta_{N-1}\beta_N}\mathcal{V}(z+iNa)_{\beta_N}, \tag{103}$$

which readily follow from (98); here we have put

$$\mathcal{M}_j(z)_{\alpha\beta} \equiv \mathcal{M}(z-(j-1/2)ia)_{\alpha\beta}, \quad j = 1, 2, \dots, \tag{104}$$

$$\mathcal{M}_j(z)^{-1}_{\alpha\beta} \equiv \mathcal{M}(z-(j-1/2)ia)^{-1}_{\alpha\beta}, \quad j = 0, -1, -2, \dots \tag{105}$$

THEOREM 5. *With the above assumptions in force, the orders of* $\mathcal{V}(z)_1, \dots, \mathcal{V}(z)_\Lambda$ *are* $\rho + 1$ *at most.*

Proof. The proof proceeds along the same lines as that of Theorem 3. Thus, we fix r satisfying (76) and start from (78) with F replaced by $\mathcal{V}(z)_\alpha$. Using then (102) and (103), we arrive at the estimate

$$n(r, \mathcal{V}(z)_\alpha) \leq M \times \sum_{\beta,\gamma=1}^{\Lambda} \{\text{no. of poles of } \mathcal{M}(z)_{\beta\gamma} \text{ in } \{|z| \leq r\} \cap \{\operatorname{Im} z > 0\}\}$$
$$+ M \times \sum_{\beta,\gamma=1}^{\Lambda} \left\{\text{no. of zeros of } \mathcal{M}(z)^{-1}_{\beta\gamma} \text{ in } \{|z| \leq r\} \cap \{\operatorname{Im} z < 0\}\right\}$$
$$+ M \times \sum_{\beta=1}^{\Lambda} \{\text{no. of poles of } \mathcal{V}_\beta \text{ in } P_0\}. \tag{106}$$

As before, this implies

$$n(r, \mathcal{V}(z)_\alpha) \leq \frac{(r + a/2)}{a} \left(\sum_{\beta,\gamma} \left[n(r, \infty, \mathcal{M}_{\beta\gamma}) + n\big(r, 0, \mathcal{M}^{-1}_{\beta\gamma}\big) \right] + C \right). \tag{107}$$

On account of the order ρ assumption on the matrix elements, we now obtain

$$n(r, \mathcal{V}(z)_\alpha) = O(r^{\rho+1+\epsilon}), \quad \forall \epsilon > 0, \tag{108}$$

so that

$$N(r, \mathcal{V}(z)_\alpha) = O(r^{\rho+1+\epsilon}), \quad \forall \epsilon > 0. \tag{109}$$

It remains to show

$$m(r, \mathcal{V}(z)_\alpha) = O(r^{\rho+1+\epsilon}), \quad \forall \epsilon > 0, \quad \alpha = 1, \ldots, \Lambda. \tag{110}$$

Proceeding as before, we fix r satisfying (76) and use (102) and (103) on the arcs $A_M, \ldots, A_{-M}$, so as to obtain functions $\mathcal{V}(z)_\beta$ evaluated in the strip P_0. However, now we encounter a snag that did not arise in the first-order case. Specifically, though each term in the sum pertinent to the arc A_k can be estimated by using (70), there are $\Lambda^{|k|}$ terms present, and for $|k|$ on the order of r this number grows exponentially with r.

Therefore, we need additional arguments to arrive at (110). To supply these, we begin by fixing $k \in \{1, \ldots, M\}$ and studying the pertinent integrand. It consists of terms

$$T_{kl}(re^{i\theta}), \quad l = 1, \ldots, \Lambda^k, \tag{111}$$

each of which is of the form

$$\mathcal{M}_1(re^{i\theta})_{\alpha\beta_1} \mathcal{M}_2(re^{i\theta})_{\beta_1\beta_2} \cdots \mathcal{M}_k(re^{i\theta})_{\beta_{k-1}\beta_k} \mathcal{V}(re^{i\theta} - ika)_{\beta_k}. \tag{112}$$

(The ordering of the terms (111) is supposed to be fixed, but the precise ordering prescription is irrelevant.) Each term may have a finite number of poles on A_k. Deleting all such pole locations from A_k, we obtain a finite union A_k^u of intervals. Clearly, we may and will replace A_k by A_k^u in the integration.

Next, we recursively partition A_k^u into disjoint subsets A_{km}^u, as follows:

$$A_{k1}^u \equiv \left\{\theta \in A_k^u \,\middle|\, \left|T_{k1}(re^{i\theta})\right| = \max_{l=1,\dots,\Lambda^k} \left|T_{kl}(re^{i\theta})\right|\right\},$$
$$A_{k2}^u \equiv \left\{\theta \in A_k^u \setminus A_{k1}^u \,\middle|\, \left|T_{k2}(re^{i\theta})\right| = \max_{l=2,\dots,\Lambda^k} \left|T_{kl}(re^{i\theta})\right|\right\}, \tag{113}$$

etc. As a consequence, we now have bounds

$$\log^+\left|\sum_{l=1}^{\Lambda^k} T_{kl}(re^{i\theta})\right| \le \log^+\left(\Lambda^k \max_{l=1,\dots,\Lambda^k}\left|T_{kl}(re^{i\theta})\right|\right)$$
$$\le k\log\Lambda + \log^+\left|T_{km}(re^{i\theta})\right|, \quad \forall\theta \in A_{km}^u. \tag{114}$$

From this we deduce that we have

$$\int_{A_k} \log^+\left|\mathcal{V}(re^{i\theta})_\alpha\right| d\theta \le k|A_k|\log\Lambda + \sum_{m=1}^{\Lambda^k}\int_{A_{km}^u}\log^+\left|T_{km}(re^{i\theta})\right| d\theta. \tag{115}$$

Now each of the terms in the sum is majorized by the sum of an integral

$$\int_{A_{km}^u} \log^+\left|\mathcal{V}(re^{i\theta} - ika)_\beta\right| d\theta, \tag{116}$$

and k integrals of the form

$$\int_{A_{km}^u} \log^+\left|\mathcal{M}_j(re^{i\theta})_{\beta\gamma}\right| d\theta, \quad j = 1,\dots,k, \tag{117}$$

with β and γ depending on *k, m*, and *j*, cf. (111) and (112). As a consequence we obtain

$$\int_{A_k} \log^+\left|\mathcal{V}(re^{i\theta})_\alpha\right| d\theta \le k|A_k|\log\Lambda + \sum_{\beta=1}^{\Lambda}\int_{A_k^u}\log^+\left|\mathcal{V}(re^{i\theta} - ika)_\beta\right| d\theta$$
$$+ \sum_{\beta,\gamma=1}^{\Lambda}\sum_{j=1}^{k}\int_{A_k^u}\log^+\left|\mathcal{M}_j(re^{i\theta})_{\beta\gamma}\right| d\theta, \tag{118}$$

so that summing over k yields

$$\int_0^\pi \log^+\left|\mathcal{V}(re^{i\theta})_\alpha\right| d\theta \le M\pi\log\Lambda + \sum_{\beta=1}^{\Lambda} J_\beta(r) + \sum_{\beta,\gamma=1}^{\Lambda} J_{\beta\gamma}(r), \tag{119}$$

where we have set

$$J_\beta(r) \equiv \sum_{k=0}^{M}\int_{A_k}\log^+\left|\mathcal{V}(re^{i\theta} - ika)_\beta\right| d\theta, \tag{120}$$

$$J_{\beta\gamma}(r) \equiv \sum_{k=1}^{M}\sum_{j=1}^{k}\int_{A_k}\log^+\left|\mathcal{M}_j(re^{i\theta})_{\beta\gamma}\right| d\theta. \tag{121}$$

We now choose $r > R_0$ and use our assumptions on $\mathcal{V}(z)_\beta$, cf. the paragraph containing (101). A moment's thought suffices to see they entail that for any $\epsilon > 0$ there exists $D_\epsilon > 0$ such that

$$J_\beta(r) \le 2\pi C_\epsilon r^{\rho+1+\epsilon} + D_\epsilon, \quad \beta = 1, \ldots, \Lambda. \tag{122}$$

On combining this with (119) and the estimate

$$J_{\beta\gamma}(r) \le \sum_{j=1}^{M} m(r, \mathcal{M}_j(\cdot)_{\beta\gamma}) \le CMT(4r, \mathcal{M}(\cdot)_{\beta\gamma}), \tag{123}$$

(where we used (70)), we deduce as before

$$\int_0^\pi \log^+ \left| \mathcal{V}(re^{i\theta})_\alpha \right| d\theta = O(r^{\rho+1+\epsilon}), \quad \forall \epsilon > 0. \tag{124}$$

The above reasoning can be repeated with obvious changes to obtain the bound

$$\int_\pi^{2\pi} \log^+ \left| \mathcal{V}(re^{i\theta})_\alpha \right| d\theta = O(r^{\rho+1+\epsilon}), \quad \forall \epsilon > 0. \tag{125}$$

Combined with (124), this yields the estimate (110), concluding the proof. ■

As already mentioned, for $\Lambda = 1$ the assumptions (1) and (2) of Theorem 5 are weaker than the corresponding assumptions (a) and (b) of Theorem 3. Specifically, in (a) we do not allow poles and zeros in P_0, whereas in (1) we allow finitely many poles and any number of zeros; moreover, (3) imposes lower and upper bounds on $\log |F(z)|$, whereas (97) only requires an upper bound. We prefer the stronger assumptions (a) and (b) in the first-order case for reasons of uniqueness: They reduce the multiplier ambiguity to (4), singling out the most useful solutions in the process. The multiplier ambiguity associated with (1) and (2) is much larger: Any μ in the space $\mathcal{P}_{ia}$ (2) with a finite number of poles in P_0 and an arbitrary (possibly infinite) number of zeros in P_0 can be admitted, provided $\mu(z)\mathcal{V}(z)$ still satisfies (3).

The reason why the assumptions must be relaxed for $\Lambda > 1$ so as to capture all of the "useful" solutions is, briefly put, linearity. Indeed, one expects Λ solutions that are independent over the field of scalars $\mathcal{P}_{ia}$. Upon taking linear combinations, zeros can arise that do not satisfy any restrictions other than an order σ restriction on their counting functions if the summands are functions of order $\le \sigma$. In particular, even when the summands have no zeros in P_0, the sum may have an infinity of zeros.

We add a simple example to illustrate this. (In fact, this example is included as a special case in several of the applied contexts below.) Taking

$$C_1(z) = -2\cosh(ap), \quad p \in \mathbb{C}, \quad C_2(z) = 1, \tag{126}$$

in (96), it is evident that the plane waves

$$F_{\pm}(z, p) = \exp(\pm izp) \tag{127}$$

are solutions satisfying (1) and (2) with $\rho = 0$. They have no zeros at all and they still satisfy (97) with $\rho = 0$ when $\log^+|F_{\pm}(z, p)|$ is replaced by $\log F_{\pm}(z, p)$. However, this is not the case for the linear combinations sin (zp) and cos (zp): For $p \in \mathbb{R}^*$ the latter have infinitely many real zeros.

In this connection we would like to add that although at face value (1) and (2) may seem to yield no restrictions on zeros whatsoever, Theorem 5 actually implies that the zero counting functions $n(r, 0, \mathcal{V}(\cdot)_\alpha)$ must have order $\leq \rho + 1$. Thus, it follows a posteriori that there cannot be "too many" zeros in P_0 (as is also exemplified by the above functions sin (zp) and cos (zp)).

To conclude, we mention a number of far less trivial cases where the hypotheses of Theorem 5 can be checked. Because general methods yielding minimal solutions are presently not available for $\Lambda > 1$, we restrict attention to a number of $\Lambda = 2$ equations where solutions $\mathcal{V} \in \mathcal{M}^*$ are known in sufficient detail so that the minimality assumptions (1) and (2) on $\mathcal{V}(z)$ can be verified.

The first type of 2×2 first-order AΔEs where a comparison is feasible stems from a series of papers by Buslaev and Fedotov, culminating in [17]. Rephrasing their assumptions on $\mathcal{M}(z)$ in terms of our conventions, they study the case where the four matrix elements $\mathcal{M}(z)_{\alpha\beta}$ are Laurent polynomials in $\exp(z)$, hence entire functions of order 1; moreover, $\det\mathcal{M}(z)$ is assumed to be identically 1. This class contains the Harper equation, i.e., (96) with $C_2(z) = 1$ and $C_1(z) = 2\cosh(z) - 2E$, $E \in \mathbb{C}$ (cf. also (100)), whose study motivated their work.

The vector solutions constructed in [17] are entire and they satisfy

$$\log^+|\mathcal{V}(z)_\alpha| = O(|\mathrm{Re}\, z|^2), \quad \alpha = 1, 2, \quad |\mathrm{Re}\, z| \to \infty, \tag{128}$$

uniformly for $|\mathrm{Im}\, z|$ in $\mathbb{R}$-compacts. Thus the assumptions (1) and (2) are satisfied with $R_0 = 0$ and $\rho = 1$ in (101). As a consequence of Theorem 5, these solutions have order ≤ 2.

Our remaining examples are all of the scalar second-order form (96). They may be viewed as time-independent Schrödinger equations in a context of (one-dimensional) relativistic quantum mechanics [18]. Thus, we have in (96)

$$C_1(z) = V_1(z) - 2E, \quad E \in \mathbb{C}, \tag{129}$$

where E is viewed as the energy of a reduced 2-particle system. (In most of these cases, assumption (101) on the solutions can only be verified under certain reality restrictions, both on E and on certain parameters. It is beyond our scope to detail such restrictions.)

The first class of examples of form (96) is rather extensive, yet quite special from the viewpoint of Nevanlinna order. They are the "reflectionless" equations studied in various papers by one of us, which can be traced from [19].

Here the coefficients $V_1(z)$ and $C_2(z)$ are rational functions of exponentials $\exp(c_j z)$, $c_j \in \mathbb{C}^*$, $j = 1, \ldots, N$, hence of order 1. (Also, the step size a is normalized to 1.) Writing E in (129) as $\cosh p$, the special feature of the reflectionless solutions from the Nevanlinna perspective is that they have order 1, too. Specifically, they are of the form

$$F_\pm(z, p) = \exp(\pm izp) R_\pm(e^{c_1 z}, \ldots, e^{c_N z}, e^p), \tag{130}$$

where $R_\pm$ are rational functions of their arguments. Their asymptotic behavior reads

$$F_\pm(z, p) \sim \exp(\pm izp), \qquad \mathrm{Re}\, z \to -\infty, \tag{131}$$

$$F_\pm(z, p) \sim a(p) \exp(\pm izp), \quad \mathrm{Re}\, z \to \infty, \tag{132}$$

uniformly for $\mathrm{Im}\, z$ in $\mathbb{R}$-compacts, with $a(p) \neq 0$ except for N (bound state) energies $E_1, \ldots, E_N \in (-1, 1)$.

These features ensure that the hypotheses of Theorem 5 are satisfied with $\rho = 1$. (This can also be checked for the bound state solutions.) In this case, however, it is already obvious from (130) and rationality of $R_\pm$ that the order of $F_\pm$ equals 1, and not the upper bound 2 of Theorem 5. Thus, the reflectionless solutions have smaller order than "expected" from the coefficients in the AΔE.

The well-known Askey–Wilson polynomials $P_n(\cos z)$ [20, 21] can be viewed as solutions to AΔEs (96) with E in (129) depending on the degree n, and certain trigonometric coefficients $V_1(z)$, $C_2(z)$. (More precisely, V_1 and C_2 are rational functions of $\exp(iz/2)$.) Thus, the elements (100) have order 1 or 0, and it is plain that the properties (1) and (2) hold true. Of course, in this case it is immediate that the order of the special solutions equals 0 for $n = 0$ and 1 for $n \in \mathbb{N}^*$, so it is smaller than the upper bound 2 given by Theorem 5.

Solutions in terms of ${}_8\phi_7$ basic hypergeometric series that correspond to arbitrary E in the Askey–Wilson AΔE were found by Ismail and Rahman [22, 23]. For energy values E different from the polynomial energies E_n, it is quite likely that these solutions have not only properties (1) and (2), but also order 2. (For a special linear combination and special parameters, the order 2 feature follows from proposition 5.1 in Stokman's paper [24].)

For a hyperbolic version of the Askey–Wilson AΔE (i.e., V_1 and C_2 are now certain rational functions of $\exp(\pi z/b)$, $b > 0$), the assumptions of Theorem 5 can be seen to hold as well, the special minimal solution being the function $\mathcal{E}(-z)$ from [25]. For a dense set in the natural parameter space this function has order 1 (as follows from [26]), whereas on the complement of this set the order is generically 2 (it still equals 1 for a discrete set of E-values).

In fact, for the latter dense parameter set there exist two independent order 1 solutions with properties (1) and (2) [26]. There is a 1-parameter subfamily of the 4-parameter Askey–Wilson family for which the order 1 hyperbolic coefficients and order 1 solutions for a dense set of real parameters generalize to order 2 elliptic coefficients and order 2 solutions for the same dense set [27]. Again, the assumptions of Theorem 5 are met for the pertinent elliptic AΔEs, which are special instances of the (reduced 2-particle) elliptic relativistic Calogero–Moser system [18].

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Appendix: Proofs of Lemmas 2 and 3

The proofs of both lemmas are based on the Poisson–Jensen formula. Therefore, we begin by recapitulating this formula and a few bounds associated with it, cf. Theorem 1.1 in [13]. First, we fix $R > 0$ and define

$$Q_R(z;\alpha) \equiv \left| \frac{R^2 - \bar{\alpha} z}{R(z-\alpha)} \right|, \quad |z| < R, \quad |\alpha| \leq R. \tag{A.1}$$

Thus, we have

$$Q_R(z;\alpha) = 1, \quad |\alpha| = R, \tag{A.2}$$

$$Q_R(z;\alpha) \in (1, \infty], \quad |\alpha| < R. \tag{A.3}$$

Second, setting

$$z = re^{i\theta}, \quad r < R, \quad \theta \in [0, 2\pi), \tag{A.4}$$

we define

$$K_R(z;\phi) \equiv \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2}, \tag{A.5}$$

which entails

$$K_R(z;\phi) \in \left[\frac{R-r}{R+r}, \frac{R+r}{R-r} \right]. \tag{A.6}$$

Now let $f \in \mathcal{M}^*$ and denote the zeros and poles of $f(z)$ in the disc $|z| < R$ by $a_0, \ldots, a_K$ and $b_0, \ldots, b_L$, respectively. Fixing z satisfying (A.4) and such that $f(z) \neq 0, \infty$, the Poisson–Jensen formula is given by

$$\log|f(z)| = \frac{1}{2\pi}\int_0^{2\pi} K_R(z;\phi)\log|f(Re^{i\phi})|\,d\phi - \sum_{k=0}^{K}\log Q_R(z;a_k) + \sum_{l=0}^{L}\log Q_R(z;b_l). \tag{A.7}$$

We not only use (A.7), but also its special case $z = 0$, known as Jensen's formula

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi}\log|f(Re^{i\phi})|\,d\phi - \sum_{k=0}^{K}\log|R/a_k| + \sum_{l=0}^{L}\log|R/b_l|. \tag{A.8}$$

It holds true for $f(0) \neq 0, \infty$.

Next, we recall that the proximity function is defined by

$$m(R, f) \equiv \frac{1}{2\pi}\int_0^{2\pi}\log^+|f(Re^{i\phi})|\,d\phi, \tag{A.9}$$

with

$$\log^+ x \equiv \begin{cases}\log x, & x \geq 1,\\ 0, & 0 \leq x < 1.\end{cases} \tag{A.10}$$

Combining (A.7) with the bounds (A.3) and (A.6), one readily obtains upper and lower bounds

$$\log|f(z)| \leq \frac{R+|z|}{R-|z|}m(R, f) + \sum_{l=0}^{L}\log Q_R(z;b_l), \tag{A.11}$$

$$\log|f(z)| \geq -\frac{R+|z|}{R-|z|}m(R, 1/f) - \sum_{k=0}^{K}\log Q_R(z;a_k). \tag{A.12}$$

Clearly, (A.11) and (A.12) are still valid when $f(z)$ equals 0 or ∞. We are now prepared to embark on the proofs of the lemmas.

Proof of Lemma 2. Letting

$$z = re^{i\theta}, \quad |\eta| < r, \quad R > 2r, \tag{A.13}$$

we may invoke (A.11) with z replaced by $z - \eta$. Using $|z - \eta| < 2r$, this yields

$$\log|f(z-\eta)| < \frac{R+2r}{R-2r}m(R, f) + \sum_{l=0}^{L}\log Q_R(z-\eta;b_l). \tag{A.14}$$

Next, we observe that we have (cf. (A.9) and (A.10))

$$m(r, f_\eta) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\phi} - \eta)|\, d\phi$$
$$= \frac{1}{2\pi}\int_{P_\eta} \log |f(re^{i\phi} - \eta)|\, d\phi, \tag{A.15}$$

with

$$P_\eta \equiv \{\phi \in [0, 2\pi) \mid |f(re^{i\phi} - \eta)| \geq 1\}. \tag{A.16}$$

Using (A.14) and recalling $Q_R(re^{i\phi} - \eta; b_l) > 1$ (cf. (A.1), (A.3)), we deduce

$$m(r, f_\eta) < \frac{R + 2r}{R - 2r} m(R, f) + \sum_{l=0}^{L} I_l(r, \eta), \tag{A.17}$$

where

$$I_l(r, \eta) \equiv \frac{1}{2\pi}\int_0^{2\pi} \log Q_R\big(re^{i\phi} - \eta; b_l\big)\, d\phi. \tag{A.18}$$

We now exploit Jensen's formula (A.8), with R replaced by r, for the functions

$$J_l(z) \equiv \frac{R^2 - \bar{b}_l(z - \eta)}{R(z - \eta - b_l)}, \quad l = 0, \ldots, L. \tag{A.19}$$

The point of this is that the integral on the right hand side of (A.8) then amounts to $I_l(r, \eta)$ (A.18), cf. (A.1). Because we have $|\bar{b}_l(z - \eta)| < 2rR < R^2$ for $|z| < r$, we do not get a zero for $J_l(z)$ when $|z| < r$. Depending on $|\eta + b_l|$, however, we may get a pole. Hence, we distinguish three cases.

$$\text{(I)} \quad |\eta + b_l| \geq r : \tag{A.20}$$

In this case $J_l(z)$ has no pole for $|z| < r$, so we obtain

$$I_l(r, \eta) = \log |J_l(0)| = \log \left| \frac{R^2 + \bar{b}_l \eta}{R(\eta + b_l)} \right|$$
$$\leq \log \left| \frac{R^2 + Rr}{Rr} \right| = \log(1 + R/r). \tag{A.21}$$

$$\text{(II)} \quad |\eta + b_l| \in (0, r) : \tag{A.22}$$

Now we get a pole for $z = \eta + b_l$, so we deduce from (A.8)

$$I_l(r, \eta) = \log |J_l(0)| + \log \frac{|\eta + b_l|}{r} = \log \left| \frac{R^2 + \bar{b}_l \eta}{Rr} \right|$$
$$\leq \log \left| \frac{R^2 + Rr}{Rr} \right| = \log(1 + R/r). \tag{A.23}$$

$$\text{(III)} \quad |\eta + b_l| = 0 : \tag{A.24}$$

In this case $J_l(z)$ has a pole for $z = 0$, so that (A.8) does not apply. However, we need only substitute $b_l = -\eta$ in (A.18) to obtain

$$\begin{aligned} I_l(r, \eta) &= \frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{R^2 + \bar{\eta}(re^{i\phi} - \eta)}{Rre^{i\phi}} \right| d\phi \\ &\leq \log \left(\frac{R^2 + 2r^2}{Rr} \right) = \log \left(\frac{R}{r} + \frac{2r}{R} \right). \end{aligned} \tag{A.25}$$

Recalling $R > 2r$, we get

$$I_l(r, \eta) \leq \log(1 + R/r), \quad l = 0, \dots, L, \tag{A.26}$$

in all three cases. Using this in (A.17), we infer

$$m(r, f_\eta) < \frac{R + 2r}{R - 2r} m(R, f) + n(R, f) \log(1 + R/r). \tag{A.27}$$

Finally, choosing $R = 3r$, we obtain (69). ■

Proof of Lemma 3. It is not hard to see that we have a bound (recall (A.1))

$$Q_R(z; \alpha) \leq 2R/(d - |\mathrm{Im}\, z|), \quad |z| < R, \quad z \in S_d, \quad |\alpha| < R, \quad \alpha \notin S_d. \tag{A.28}$$

Hence, taking $R = 2r$ from now on, we deduce from (A.11) and (A.12) the upper and lower bounds

$$\log |f(z)| \leq 3m(2r, f) + n(2r, \infty, f)[\log r + C(\mathrm{Im}\, z)], \tag{A.29}$$

$$\log |f(z)| \geq -3m(2r, 1/f) - n(2r, 0, f)[\log r + C(\mathrm{Im}\, z)], \tag{A.30}$$

where

$$C(x) \equiv \log(4/(d - |x|)), \quad |x| < d. \tag{A.31}$$

Now $\log r$ is $O(r^\delta)$ for any $\delta > 0$ and $C(\mathrm{Im}\, z)$ is uniformly bounded on closed substrips of S_d. Thus, using also the order σ assumption on f, the lemma readily follows from (A.29) and (A.30). ■

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