Semifinite-gap problems of Whittaker-Hill equation and complex oscillation theory[1]

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Whittaker-Hill equation

- Schrödinger equations with potentials with period π.
- Mathieu equation (1868)

$$f''(z) + (A + B\cos 2z)f(z) = 0$$

(*Separation of variables* of 2D-Wave equation by *elliptic cylindiical coordinates*)

- Whittaker-Hill equation (1907/1915)

$$f''(z) + (A + B\cos 2z + C\cos 4z)f(z) = 0. \tag{1}$$

(*Separation of variables* of 3D-Helmholz equation by *paraboloidal coordinates*)

- Celestial machines, Quantum theory, Quantum chemistry, Integration of KdV with periodic BVP (Novikov), Quantum field theory, etc

Hill's equations

- Consider Hill's equation (1877)

$$\frac{d^2y}{dx^2} + Q(x)y(x) = 0, \quad . \tag{2}$$

which is a Schrödinger equation with periodic (even) potential

$$Q(x+\pi) = Q(x).$$

- Hill's original treatment was to assume

$$Q(x) = \lambda + 2\sum_{k=1}^{\infty} \theta_k \cos 2kx$$

to converge on $\mathbb{R}$.

- How much do we know about the eigenvalues λ ?
- Do there exist any periodic solutions?
- Coexistence: Do there exist two linearly independent (LI) periodic solutions?

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Floquet (Bloch) Theory

- G. H. Hill (1886), G. Floquet (1883), A. M. Lyapunov (1907)
- L. Brillouin (1953) Wave propagation in periodic structures (Dover)
- Magnus & Winkler (1966): Hill's Equations (Dover)
- Arscott (1964): Periodic Differential Equations (Pergamon press)
- Eastham (1973): Spectral Theory of Periodic Differential Equations (Scottish Academic Press)
- Floquet theory: $\exists \rho \neq 0$ and non-trivial soln $\psi(x)$ of Eqn (2) such that

$$\psi(x+\pi) = \rho\psi(x).$$

- Similar to *monodromy* at a *regular singular point* ($\mathbb{C}$).
  - $\rho = 1$ periodic soln;
  - $\rho = -1$ semi-periodic soln;
  - Since $Q(x)$ is even, so the solutions of (2) could be even/odd solns.

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Floquet (Bloch) Theory (II)

- Suppose $\phi_1(x)$ and $\phi_2(x)$ are two *linearly independent* solutions with initial conditions

$$\phi_1(0)=1, \quad \phi_1'(0)=0, \quad \phi_2(0)=0, \quad \phi_2'(0)=1$$

- We observe $\exists$ 2×2 matrix such that

$$\begin{pmatrix}\phi_1(x+\pi)\\ \phi_2(x+\pi)\end{pmatrix}=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}\phi_1(x)\\ \phi_2(x)\end{pmatrix}$$

- It follows from the initial conditions on ϕ_1, ϕ_2 above that

$$A=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}=\begin{pmatrix}\phi_1(\pi) & \phi_1'(\pi)\\ \phi_2(\pi) & \phi_2'(\pi)\end{pmatrix}$$

$$\mathrm{tr}\,(A)=\phi_1(\pi)+\phi_2'(\pi),\quad \det(A)=\phi_1(\pi)\phi_2'(\pi)-\phi_1'(\pi)\phi_2(\pi).$$

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- Suppose $\psi(x)$ is a solution such that $\psi(x+\pi)=\rho\psi(x)$ for certain ρ. Then $\psi(x)=c_1\phi_1(x)+c_2\phi_2(x)$ for some $c_1,\ c_2$
Then

$$(A^T-\rho\,\mathrm{I}_2)\begin{pmatrix}c_1\\c_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}=\left[\begin{pmatrix}a_{11}&a_{21}\\a_{12}&a_{22}\end{pmatrix}-\rho\,\mathrm{I}_2\right]\begin{pmatrix}c_1\\c_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$

i.e.,

$$\begin{vmatrix}a_{11}-\rho & a_{21}\\ a_{12} & a_{22}-\rho\end{vmatrix}=0$$

or just

$$\rho^2-\mathrm{tr}(A)\,\rho+\det(A)=0.$$

where $\det(A)\neq 0$

- $\rho=\rho_1,\ \rho_2$ are called the characteristic (Floquet) multipliers and

$$D=\Delta(\lambda)=\mathrm{tr}(A)=\phi_1(\pi)+\phi_2'(\pi)$$

is called Hill's discriminant (Lyapunov fn) of the DE (1).

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Stability intervals vs Gaps

Theorem

- *If $D = |\mathrm{tr}(A)| > 2$, then all non-trivial solutions of Eqn (2) are unbounded on $\mathbb{R}$ (unstable),*
- *If $D = |\mathrm{tr}(A)| < 2$, then all solutions of Eqn (2) are bounded on $\mathbb{R}$ (stable)*
- *If $D = |\mathrm{tr}(A)| = 2$, then there exists a non-trivial solution of Eqn (2) which is bounded on $\mathbb{R}$ (conditionally stable)*

Theorem

- *The Eqn (2) has a non-trivial soln with period $\pi \iff D = 2$.*
- *The Eqn (2) has a non-trivial soln with semi-period $\pi \iff D = -2$ (i.e., $f(x+\pi) = -f(x)$)*

Periodic boundary (eigenvalue) problem

- Periodic boundary value problem on $[0, \pi]$ ($D = 2$)

$$f(0) = f(\pi), \quad f'(0) = f'(\pi)$$

This is a self-adjoint problem and standard method of constructing Green's functions and defining compact symmetric linear opeartor with a suitable inner-product space on $[0, \pi]$ guarantees

  - the existence of countably many orthogonal eigen-functions ψ_n and eigenvalues λ_n such that

$$\lambda_0 \le \lambda_1 \le \lambda_2, \cdots, \quad \lambda_n \to \infty$$

- Semi-periodic boundary value problem on $[0, \pi]$ ($D = -2$)

$$f(0) = -f(\pi), \quad f'(0) = -f'(\pi)$$

  - corresponding eigenfunctions ϕ_n and eigenvalues λ'_n such that

$$\lambda'_1 \le \lambda'_2, \cdots, \quad \lambda'_n \to \infty$$

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Bands and Gaps

Theorem

To every differential equation

$$y'' + [\lambda + Q(x)]y = 0, \quad Q(x+\pi) = Q(x), \quad \lambda \in \mathbb{R} \qquad (3)$$

there exists two monotone sequences $(\lambda_n)_0^\infty$, $(\lambda'_n)_1^\infty$ *such that*

$$-\infty < \lambda_0 < \lambda'_1 \le \lambda'_2 < \lambda_1 \le \lambda_2 < \lambda'_3 \le \lambda'_4 < \lambda_3 \le \lambda_4 < \cdots,$$

for which

- *stability intervals (bands):* (λ_0, λ'_1), (λ'_2, λ_1), $(\lambda_2, \lambda'_3) \cdots$,
- *instability intervals (gaps):* $(\lambda_{2n+1}, \lambda_{2n+2})$, $(\lambda'_{2n+1}, \lambda'_{2n+2}), \cdots$, *(simple point).*
- *Instability interval disappears (shrink to a point) so that* $\lambda_{2n+1} = \lambda_{2n+2}$, *or* $\lambda'_{2n+1} = \lambda'_{2n+2}, \cdots$. *(stable double point).*

Mathieu equation
Ince (1922) infinitely many gaps

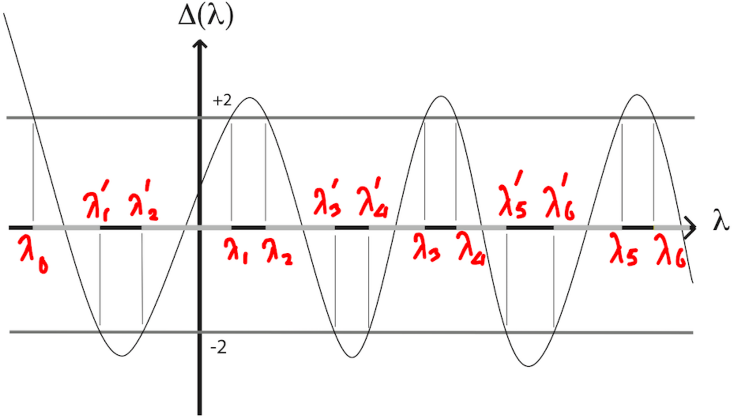
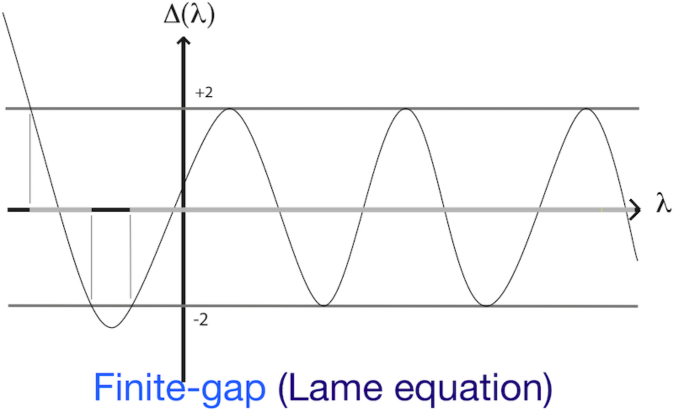


Figure: (Hemery & Veselov (2010))

Finite-gaps or semi-finite gap potentials



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Semi-finite gap (odd solutions)

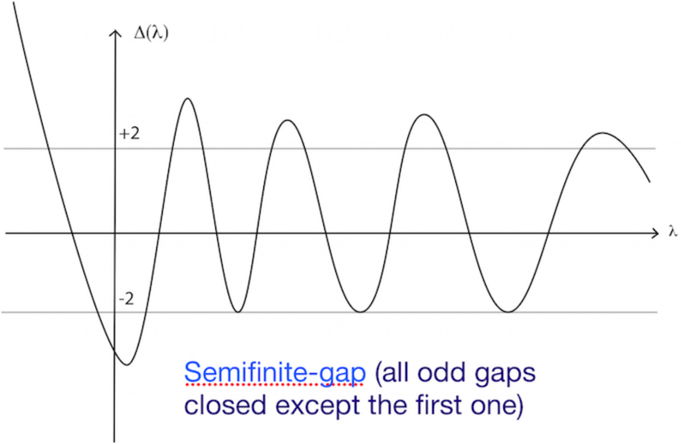
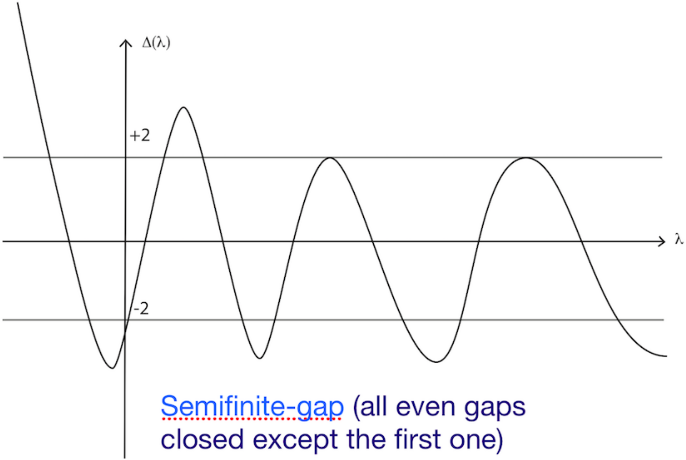


Figure: (Hemery & Veselov (2010))

Semi-finite gap (even solutions)



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Semifinite-gap for WH-operator

Theorem (Djakov & Mityagin (2005))

Let

$$\nu(x) = -4\alpha t \cos 2x - 2\alpha^2 \cos 4x$$

be a potential of the Hill operator

$$\lambda f = Lf = -f'' + \nu(x)f, \quad 0 \le x \le \pi,$$

where both $(0 \neq)\, \alpha$ *and* t *are real.*

1. *If* $t = 2p - 1$, $p \in \mathbb{N}$ *with periodic boundary conditions, then the first* $2p - 1$ *eigenvalues are simple, and others are double.*
2. *If* $t = 2m$, $m \in \mathbb{N}$ *with semi-periodic boundary conditions, then the first* $2m$ *eigenvalues are simple and others are double.*

Comments

- Compared with the Mathieu operator, the eigenvalues of Whittaker-Hill potential

$$\nu(x) = -(B\cos 2x + C\cos 4x)$$

may be *simple or double* both for *periodic* and *semi-periodic boundary conditions*.

- Djakov and Mityagin assumed that if $B = 4\alpha t$ and $C = 2\alpha^2$ for any real α and natural number t, i.e.,

$$B/(2\sqrt{2C}) = t \in \mathbb{Z},$$

then they conclude that

1. all finitely many gaps exist
2. Semi-periodic BVP: $t = 2m + 1$, all the even gaps except the first m are closed,
3. Periodic BVP: $t = 2m$, the first m are closed.

- Djakov and Mityagin's argument is very elaborate and heavy in spectra analysis (and also long).

Riemann-Hilbert (21st) Problem

- Prove that there always exists a Fuchsian system (equation) with given singularities in $\mathbb{CP}^1$ and a given monodromy (representation).
- "Fuchsian" means the "coefficients" have at most poles at the given singularities.
- Solved by Plemelj (1908), Arnold (1988), Bolibruch (1989)
- Closely related to Riemann-Hilbert methods in integrable systems and Random matrices theory.
- Two singularities in $\mathbb{CP}^1$:
  - E.g., Euler equation: $\{0, \infty\}$

$$x^2y'' + (1 - a - b)xy' + ab\, y = 0$$

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$$y = Ax^a + Bx^b$$

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2-3-4 Regular Singularties (I)

- Three singularities in $\mathbb{CP}^1$:
    - E.g., Bessel equation: $\{0, \infty^2\}$

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0.$$

$$y(x) = J_\nu(x) = \sum_{k=0}^{\infty} \frac{2^\nu}{2^k k!\Gamma(\nu + k)} x^{2k+\nu}$$

- E.g., Confluent hypergeometric equation: $\{0, \infty^2\}$

$$xy''(z) + (c - x)y'(x) - ay(x) = 0.$$

$${}_1F_1(a; c; x) = {}_1F_1\left(\begin{matrix} a \\ c \end{matrix}; x\right) := \sum_{n=0}^{\infty} \frac{(a)_n}{n!(c)_n} x^n, \quad c \neq 0, -1, -2, \cdots,$$

where $(a)_k = a(a+1)\cdots(a+k-1)$ In particular, when $a = -n = 0, -1, -2, \cdots$, ${}_1F_1(-n; c; x)$ is (the Laguerre) polynomial

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2-3-4 Regular Singularties (II)

- Four singularities in $\mathbb{CP}^1$:
  - Rational form of the Mathieu Eqn., ($x = e^{2iz}$)
  - Rational form of the Whittaker-Hill equation ($x = e^{2iz}$) ,
  - Rational form of Lamé equation ($x = \mathcal{P}(z)$)
  - Rational form of Darboux-Trebibich-Verdier equation
- Painlevé equations (II-VI) arise as *compatibility condition* (Lax pairs) of Isomonodromy deformation of some of the DEs above (Fuchs (1905/07), Garnier (1912), Schlesinger (1912), Jimbo-Miwa-Ueno (1981)).

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WH-Eqn: General soln I

Theorem (C. & Luo) Suppose the coefficients A, B and $C>0$ of the WH-eqn

$$f''(z)+(A+B\cos 2z+C\cos 4z)f(z)=0$$

are complex parameters, where $BC\neq 0$, then we have two linearly independent solutions

$$\begin{aligned} f_1(z) =&(e^{2iz})^{\mp\frac{B}{4\sqrt{2C}}+\frac{1}{2}}\cdot e^{\pm\frac{\sqrt{2C}}{4}e^{-2iz}}\cdot e^{\mp\frac{\sqrt{2C}}{4}e^{2iz}}\\ &\cdot\sum_{k=0}^{\infty}\frac{B_k^{\mp}}{\Gamma(k+2\mp\frac{B}{2\sqrt{2C}})}\cdot {}_1F_1\left(\begin{matrix}k+1\\ k+2\mp\frac{B}{2\sqrt{2C}}\end{matrix};\pm\frac{\sqrt{2C}}{2}e^{2iz}\right)\end{aligned}$$

and

$$\begin{aligned} f_2(z) =&(e^{2iz})^{\pm\frac{B}{4\sqrt{2C}}+\frac{1}{2}}\cdot e^{\mp\frac{\sqrt{2C}}{4}e^{-2iz}}\cdot e^{\mp\frac{\sqrt{2C}}{4}e^{2iz}}\\ &\cdot\sum_{k=0}^{\infty}\frac{\widehat{B}_k^{\pm}}{\Gamma(k+2\pm\frac{B}{2\sqrt{2C}})}\cdot {}_1F_1\left(\begin{matrix}k+1\pm\frac{B}{2\sqrt{2C}}\\ k+2\pm\frac{B}{2\sqrt{2C}}\end{matrix};\pm\frac{\sqrt{2C}}{2}e^{2iz}\right).\end{aligned}$$

WH-Eqn: General soln II

where the coefficients $B_k^{\mp}$ *and* $\widehat{B}_k^{\pm}$ *satisfy the following three-term recurrence relations (infinite determinants)*

$$-\frac{C}{2}(k+1)B_k^{\mp}+\Big[(k+1)\Big(k+2\mp\frac{B}{2\sqrt{2C}}\Big)+\frac{C-A+1}{4}\frac{B^2}{32C}\mp\frac{B}{4\sqrt{2C}}\Big]B_{k+1}^{\mp}-(k+2)B_{k+2}^{\mp}=0$$

and

$$\frac{C}{2}\Big(k+1\pm\frac{B}{2\sqrt{2C}}\Big)\cdot\widehat{B}_k^{\pm}+\Big[(k+1)\Big(k+2\pm\frac{B}{2\sqrt{2C}}\Big)\pm\frac{B}{4\sqrt{2C}}+\frac{B^2}{32C}+\frac{-A-C+1}{4}\Big]\cdot\widehat{B}_{k+1}^{\pm}-(k+2)\cdot\widehat{B}_{k+2}^{\pm}=0$$

respectively.

WH-Eqn: General soln III

Theorem (C. & Luo) *In particular, if* $\pm\frac{B}{2\sqrt{2C}} = -n-1 \in \mathbb{Z}_{<0}$ *and the tri-diagonal determinate* $|\widehat{D}^{\pm}_{n+1}| = 0$, *where* $\widehat{D}^{\pm}_{n+1} = \{\widehat{b}_{kj}\}_{1\le k,j\le n+1}$ *is defined as*

$$\widehat{b}_{k,k-1} = \frac{C}{2}\Big(k-1\pm\frac{B}{2\sqrt{2C}}\Big),$$

$$\widehat{b}_{kk} = (k-1)\Big(k\pm\frac{B}{2\sqrt{2C}}\Big)\pm\frac{B}{4\sqrt{2C}}+\frac{B^2}{32C}+\frac{-A-C+1}{4},$$

$$\widehat{b}_{k,k+1} = -k$$

and $\widehat{b}_{kj} = 0$ *for other* j, *then*

$$f_2(z) = (e^{2iz})^{-\frac{n}{2}}\cdot e^{\mp\frac{\sqrt{2C}}{4}e^{-2iz}}\cdot e^{\mp\frac{\sqrt{2C}}{4}e^{2iz}} \cdot\sum_{k=0}^{n}\widehat{B}^{\pm}_k\cdot\frac{(-n-k)_{n-k}}{(n-k)!}\cdot\Big(\pm\frac{\sqrt{2C}}{2}e^{2iz}\Big)^{n-k}.$$

Idea of proof I

- The WH-eqn in a rational form ($x = e^{2iz}$) looks like:

$$x^2y''(x) + \Big[-\frac{\sqrt{2C}}{2}x^2 + \Big(2 - \frac{B}{2\sqrt{2C}}\Big)x - \frac{\sqrt{2C}}{2}\Big]y'(x)$$
$$+ \Big(-\frac{\sqrt{2C}}{2}x + \frac{C - A + 1}{4} + \frac{B^2}{32C} - \frac{B}{4\sqrt{2C}}\Big)y(x) = 0. \tag{4}$$

when divide x on both sides yields

$$xy''(x) + \Big[-\frac{\sqrt{2C}}{2}x + \Big(2 - \frac{B}{2\sqrt{2C}}\Big) + O(1/x)\Big]y'(x)$$
$$+ \Big(-\frac{\sqrt{2C}}{2} + O(1/x)\Big)y(x) = 0. \tag{5}$$

which is asymptotically like the *confluent hypergeometric Eqn*.

- Recursion and asymptotic fomrulae of ${}_1F_1$
- Three-term recursion: *Poincaré's theorem* and *Perron's* theorems.

Complex Oscillation: Philosophy

- Liouville theorem (1840's): If f is a bounded entire function, then $f \equiv$ const. (i.e., f misses most of $\mathbb{C}$)
- Little Picard theorem (1879-80) states that if an entire function f on $\mathbb{C}$ misses *two* values in $\mathbb{C}$, then $f \equiv$ const.
- The number "2" is best possible: (e.g. $f(x) = e^x \neq 0$)
- Little Picard theorem says that entire functions are quite rigid.

- Consider

$$y''(x) + A(x)y(x) = 0$$

where $A(x)$ is an entire potential.
  - Suppose a solution $f(x)$ omits $x = 0$. Then what can we say about the solution f ?

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Measuring zeros of Entire Functions

We first review some complex analytic theory.

- Suppose $f(z)$ has zeros. We define

$$n(r) := \textbf{numbers of zeros of } f \textbf{ in } \{|z| < r\},$$

and define the exponent of convergence of the zeros of $f(z)$:

$$\lambda(f) = \limsup_{r\to+\infty} \frac{\log n(r)}{\log r}$$

- Relation between order and exponent of convergence of $f(z)$:

$$\limsup_{r\to+\infty} \frac{\log n(r)}{\log r} = \lambda(f) \le \limsup_{r\to+\infty} \frac{\log \log M(r,\ f)}{\log r} = \sigma(f).$$

- Essentially just Poisson-Jensen formula.

Bank-Laine's Complex Oscillation (Nevanlinna) Theory

Theorem (Bank & Laine (1983))

Let $f \not\equiv 0$ be a solution of

$$f''(z) + A(z)\,f(z) = 0$$

where

$$A(z) = B(e^z) = \sum_{j=-k}^{\ell} K_j e^{jz},$$

such that $\lambda(f) < +\infty\,(=\sigma(f))$. Moreover, we have

$$f(z) = \begin{cases} \psi(e^{z/2})\exp\left(\sum_{j=0}^{\ell} d_j e^{jz/2} + dz\right), & \ell \textit{ is odd and } k = 0;\\ \psi(e^{z})\exp\left(\sum_{j=-k/2}^{\ell/2} d_j e^{jz} + dz\right), & \ell \textit{ is even.} \end{cases}$$

$\frac{1}{16}$−theorem

Theorem

(Bank, Laine, Langley (1986), C. & Ismail (2006)) Let $K \in \mathbb{C}$. *Then the equation*

$$f'' + (e^z - K)f = 0. \tag{6}$$

admits linearly independent solutions

$$y_\pm(z) = A_\pm J_{2\sqrt{K}}\big(\pm 2e^{z/2}\big) + B_\pm Y_{2\sqrt{K}}\big(\pm 2e^{z/2}\big). \tag{7}$$

Each of the solutions of (6) *has* $\lambda(f) < \infty$ *if and only if*

$$K = (n+1)^2/16, \quad n \in \mathbb{N} \cup \{0\} \tag{8}$$

$$y_\pm(z) = \theta_n\big(\pm 2ie^{z/2}\big) \exp\big(\mp 2ie^{z/2} + dz\big), \tag{9}$$

where $\theta_n(x)$ *is the reversed Bessel polynomial of degree* n.

Complex oscillation of WH-Eqn

Theorem (C & Luo) *Suppose the Whittaker-Hill Eqn admits a non-trivial solution f with $\lambda(f) < \infty$, where $BC \neq 0$. Then,*

$$\pm B/(2\sqrt{2C}) = -n-1.$$

Moreover, if the tri-diagonal determinant $|\widehat{D}^{\pm}_{n+1}| = 0$, where the tri-diagonal matrix $\widehat{D}^{\pm}_{n+1} = \{\widehat{b}_{kj}\}_{1\le k,j\le n+1}$ is defined as

$$\widehat{b}_{k,k-1} = \frac{C}{2}\Big(k-n-2\Big), \quad \widehat{b}_{k,k+1} = -k$$

$$\widehat{b}_{kk} = (k-1)\Big(k-n-1\Big) + n^2 - A - C/4,$$

and $\widehat{b}_{kj} = 0$ for other j, then this solution can be represented by

$$f(z) = (e^{2iz})^{-\frac{n}{2}} \cdot e^{\mp\frac{\sqrt{2C}}{4}e^{-2iz}} \cdot e^{\mp\frac{\sqrt{2C}}{4}e^{2iz}} \cdot \sum_{k=0}^{n} \frac{\widehat{B}^{\pm}_k}{\Gamma(-n+k+1)} \cdot {}_1F_1\left(\begin{matrix} -n+k \\ -n+k+1 \end{matrix}; \pm\frac{\sqrt{2C}}{2}e^{2iz}\right)$$

Complex oscillation of WH-Eqn

where the coefficients $\widehat{B}_k$ satisfy the following three-term recurrence relation

$$\frac{C}{2}(k-n)\widehat{B}_k^{\pm}+\left[(k+1)(k-n+1)+\frac{n^2-A-C}{4}\right]\widehat{B}_{k+1}^{\pm}-(k+2)\widehat{B}_{k+2}^{\pm}=0.$$

(1) New explicit solutions

- Theorem

  Suppose the coefficients B and C of the Whittaker-Hill equation

$$f''(x) + (A + B\cos 2x + C\cos 4x)f(x) = 0$$

  are real, and $BC \neq 0$, $0 \leq x \leq \pi$. Then it admits two linearly independent solutions of periodic or semi-periodic π if and only if $\frac{B}{2\sqrt{2C}} \in \mathbb{Z} \ldots$

  - (1) If $\frac{B}{4\sqrt{2C}} = -n - \frac{1}{2}$ $n \geq 0$, and the solutions satisfy the periodic boundary conditions, then the first $2n+1$ eigenvalues are *simple* ($\lambda(f) < \infty$), and others are *double* ($\lambda(f) = \infty$):

$$A_0^+ < A_2^- < A_2^+ < A_4^- < A_4^+ < \cdots < A_{2n}^- < A_{2n}^+$$
$$< A_{2n+2}^- = A_{2n+2}^+ < A_{2n+4}^- = A_{2n+4}^+ < \cdots,$$

  where A_{2j}^+ and A_{2j}^- are the eigenvalues corresponding to non-trivial odd and even solutions with period π respectively.

(1) Even, Odd (stable) solutions

- Moreover, (i) when $A = A_{2k}^{-}$, $1 \le k \le n$,

$$f_{odd}^{\pi}(x) = e^{-\sqrt{\frac{C}{2}}\cdot\cos 2x}\cdot\sin 2x$$
$$\cdot\sum_{k=0}^{n-1} A_k^{-(1)}\cdot {}_1F_1\left(\begin{matrix}-n+1+k\\ k+3\end{matrix}; \sqrt{2C}(\cos 2x - 1)\right);$$

when $A = A_{2k}^{+}$, $0 \le k \le n$,

$$f_{even}^{\pi}(x) = e^{-\sqrt{\frac{C}{2}}\cdot\cos 2x}$$
$$\cdot\sum_{k=0}^{n} A_k^{-(2)}\cdot {}_1F_1\left(\begin{matrix}-n+k\\ k+1\end{matrix}; \sqrt{2C}(\cos 2x - 1)\right);$$

(1) Coexistence solutions

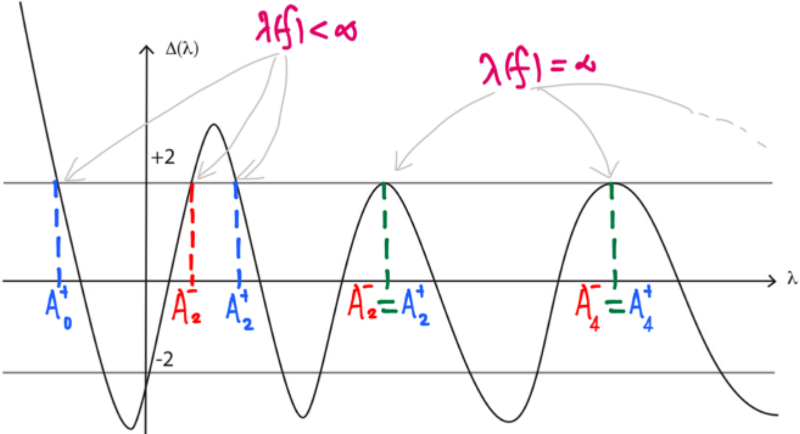
- (ii) when $A = A_{2k}^- = A_{2k}^+$, $k > n$, then the two linearly independent solutions of the Whittaker-Hill differential equation are given by

$$f_{odd}^{\pi}(x) = e^{-\sqrt{\frac{C}{2}}\cdot\cos 2x} \cdot \sin 2x \cdot \sum_{k=0}^{\infty} A_k^{-(1)} \cdot {}_1F_1\left(\begin{matrix} -n+1+k \\ k+3 \end{matrix}; \sqrt{2C}(\cos 2x - 1)\right)$$

and

$$f_{even}^{\pi}(x) = e^{-\sqrt{\frac{C}{2}}\cdot\cos 2x} \cdot \sum_{k=0}^{\infty} A_k^{-(2)} \cdot {}_1F_1\left(\begin{matrix} -n+k \\ k+1 \end{matrix}; \sqrt{2C}(\cos 2x - 1)\right).$$

Even gaps closed



Semifinite-gap (all even gaps closed except the first one)

(2) New explicit solutions

- (2) If $\frac{B}{4\sqrt{2C}} = -n-1,\ n \in \mathbb{N}^+ \cup \{0\}$, and the solutions satisfy the semi-periodic boundary conditions, then the first $2n+2$ eigenvalues are *simple*, and others are *double*, i.e.,

$$A_1^+ < A_1^- < A_3^+ < A_3^- < \cdots < A_{2n+1}^+ < A_{2n+1}^-$$
$$< A_{2n+3}^+ = A_{2n+3}^- < A_{2n+5}^+ = A_{2n+5}^- < \cdots ,$$

where A_{2j-1}^+ and A_{2j-1}^- are the eigenvalues corresponding to non-trivial odd and even solutions with semi-period π respectively.

Remark This appears to be the case that Djakov & Mityagin (2005) missed.

(2) Even, Odd (stable) solutions

- Moreover, (i) when $A = A^{-}_{2k-1}$, $1 \le k \le n+1$,

$$f^{2\pi}_{odd}(x) = \sin x \cdot e^{-\sqrt{\frac{C}{2}}\cdot\cos 2x} \cdot \sum_{k=0}^{n} A_k^{-(3)} \cdot {}_1F_1\left(\begin{matrix}-n+k\\ k+2\end{matrix};\sqrt{2C}(\cos 2x-1)\right);$$

when $A = A^{+}_{2k-1}$, $1 \le k \le n+1$,

$$f^{2\pi}_{even}(x) = \cos x \cdot e^{-\sqrt{\frac{C}{2}}\cdot\cos 2x} \cdot \sum_{k=0}^{n} A_k^{-(4)} \cdot {}_1F_1\left(\begin{matrix}-n+k\\ k+2\end{matrix};\sqrt{2C}(\cos 2x-1)\right);$$

(2) Coexistence solutions

- (ii) when $A = A^{-}_{2k-1} = A^{+}_{2k-1}$, $k > n+1$, then the two linearly independent solutions of the Whittaker-Hill differential equation are

$$f^{2\pi}_{odd}(x) = \sin x \cdot e^{-\sqrt{\frac{C}{2}}\cdot\cos 2x} \cdot \sum_{k=0}^{\infty} A_k^{-(3)} \cdot {}_1F_1\left(\begin{matrix} -n+k \\ k+2 \end{matrix}; \sqrt{2C}(\cos 2x - 1)\right)$$

and

$$f^{2\pi}_{even}(x) = \cos x \cdot e^{-\sqrt{\frac{C}{2}}\cdot\cos 2x} \cdot \sum_{k=0}^{\infty} A_k^{-(4)} \cdot {}_1F_1\left(\begin{matrix} -n+k \\ k+2 \end{matrix}; \sqrt{2C}(\cos 2x - 1)\right).$$

The remaining two cases

- (3) If $-\frac{B}{4\sqrt{2C}} = -n - \frac{1}{2}$, $n \in \mathbb{N}^+ \cup \{0\}$,
- (4) If $-\frac{B}{4\sqrt{2C}} = -n - 1$, $n \in \mathbb{N}^+ \cup \{0\}$,
- We skip the details.

Complex oscillation and semifinite-gaps

Theorem (C. & Luo) *Suppose the WH-Eqn admits a solution f with $\lambda(f) < \infty$. Then $B/(2\sqrt{2C}) \in \mathbb{Z}$ holds. Moreover, if f satisfies the normalised initial condition, then we can express f in explicit non-oscillatory-soln form:*

1. *the odd and even solutions $f^{\pi}_{\mathrm{odd}}(x)$ and $f^{\pi}_{\mathrm{even}}(x)$ in cases* (I) *and* (III) *of the last Theorem corresponding to the periodic boundary condition and for the eigenvalues $A = A^{-}_{2k}$ $(1 \le k \le n)$ and $A = A^{+}_{2k}$ $(1 \le k \le n)$;*
2. *the odd and even solutions $f^{2\pi}_{\mathrm{odd}}(x)$ and $f^{2\pi}_{\mathrm{even}}(x)$ in cases* (II) *and* (IV) *of the last Theorem corresponding to the semi-periodic boundary condition and for the eigenvalues $A = A^{-}_{2k-1}$ $(1 \le k \le n+1)$ and $A = A^{+}_{2k-1}$ $(1 \le k \le n+1)$.*

Moreover, the eigenvalues $A = A^{-}_{2k}$ $(1 \le k \le n)$, $A = A^{+}_{2k}$ $(1 \le k \le n)$, $A = A^{-}_{2k-1}$ $(1 \le k \le n+1)$ and $A = A^{+}_{2k-1}$ $(1 \le k \le n+1)$ are solutions of certain determinants $|D_n(j)| = 0$ $(j = 1, 2, 3, 4)$ respective, whose respective entries are suitably defined.

E. T. Whittaker (1873-1956)

Figure: (MathTutor, 1915)

- Supervisor: A. R. Forsyth
- Students: G. H. Hardy, W. Hodge, G. N. Watson, A. Eddington

E. L. Ince (FRSE: 1891-1941)

Figure: (MathTutor, 1923)

Instability intervals (gaps) of the Hill operator I

- In the case of specific potentials, like the Mathieu potential

$$\nu(x) = -B\cos 2x,$$

where $0 \neq B$ is real, or more general trigonometric polynomials

$$\nu(x) = \sum_{-N}^{N} c_k e^{ikx}, \quad c_k = \overline{c_{-k}}, \quad 0 \leq k \leq N < \infty,$$

one comes to two category of questions:

1. (Notation change) The *left-end point* λ_n^- and *right-end points*
2. Is the n-th intervals of instability *closed*, i.e.,

$$\gamma_n = \lambda_n^+ - \lambda_n^- = 0,$$

   or, equivalently, is the multiplicity of λ_n^+ *double*?
3. If $\gamma_n \neq 0$, what could we say about $\gamma_n = \gamma_n(\nu) \to ?$ as $n \to \infty$?

Instability intervals (gaps) of the Mathieu operator II

- Ince (1922) answered in a *negative* way for question (1) the Mathieu-operator has only simple eigenvalues both for periodic and semi-periodic boundary conditions, i.e., infinitely many gaps.
- Harrell (1981), Avron & Simon (1981) gave

$$\gamma_n = \lambda_n^+ - \lambda_n^- = \frac{8}{[(n-1)!]^2} \cdot \left(\frac{|B|}{8}\right)^n \left(1 + o\left(\frac{1}{n^2}\right)\right).$$

as $n \to \infty$ which was improved by Anahtarci & Djakov (2012). ($[1 - \pi^2/4n^3 + O(1/n^4)$).
- Levy & Keller (1963) gave the asymptotics of $\gamma_n = \gamma_n(B)$, i.e., for fixed n and real $B \neq 0$, when $B \to 0$,

$$\gamma_n = \lambda_n^+ - \lambda_n^- = \frac{8}{[(n-1)!]^2} \cdot \left(\frac{|B|}{8}\right)^n (1 + O(B)).$$

- Djakov & Mityagin (2007): WH Eqn contains modular forms studied by Kac & Wakimoto, Milne and Zagier in 1990's.

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Summary

- We have introduced (semi)finite-gap problems for Hill's equations.
- We have reviewed some classical and recent results for Mathieu and Whittaker-Hill operatprs
- We have found exact solutions in terms of ${}_1F_1$ as basis. (N. Katz's *rigid local systems theory* can offer a deeper monodromy/geometric insight: on going project)
- We relate complex oscillatory and non-oscillatory solutions to those semi-finite gap solutions (Picard-type viewpoint).
- `Very little is known about the real nature of the eigenvalues` $\lambda = A$ `with respect to` B `and` C`.`

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Whittaker–Hill Equation, Ince Polynomials, and Molecular Torsional Modes

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DOI 10.1002/qua.22255

Figure: (Wiley)

Roncaratti and Aquilanti (H_2O_2 : $\lambda(B, C)$)

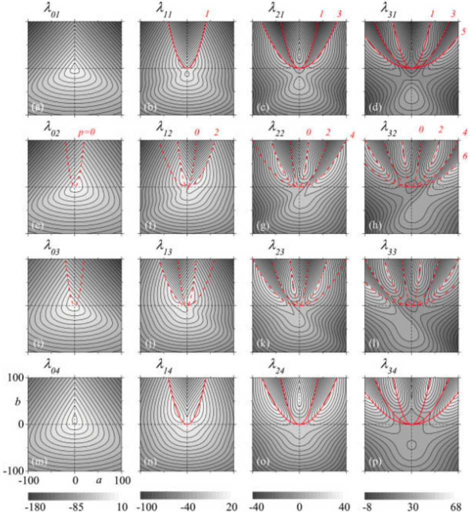

FIGURE 2. Surfaces $\lambda_{n\tau}(a, b)$ where λ's are the eigenvalues of Whittaker–Hill equation; n and τ are quantum numbers; a and b are the torsional potential parameters. Valley bottoms and ridges follow parabolic curves defined by the parameter p (see text). Each color bar describes the color scale for the pictures in the column above it. [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

Ince's contour plot

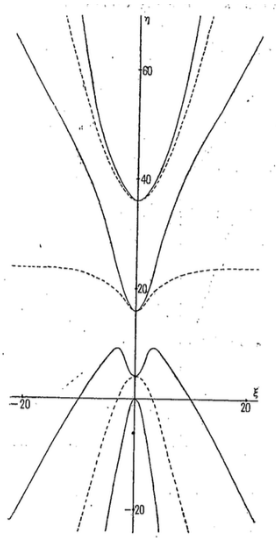

Figure: Proc. Lond. Math Soc. (1923)

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Figure: Clear Water Bay, Hong Kong Thank you for your attention !!