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Properties of Analytic Functions with Small Schwarzian Derivative

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Let $f(z) = z + \sum_2^\infty a_n z^n$ be an analytic function defined in the unit disc. Gabriel has proved that if the Schwarzian derivative $S(f,z)$ of f is bounded by a constant, then f is a starlike function. We show that under the assumption that if $S(f,z)$ and a_2, the second coefficient of f, are small then f is a strongly starlike function of order α. Some conditions found are best possible in certain sense. Moreover if $S(f,z)$ is bounded by a smaller constant and together a_2 is also small, then f is a convex function.

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1. INTRODUCTION

Let $f(z) = z + \sum_2^\infty a_n z^n$ be an analytic function defined in the unit disc $\Delta = \{z : |z| < 1\}$. We denote the collection of such functions by N. If in addition f is univalent, then we say $f \in S$. Suppose $f'(z) \neq 0$ in Δ, then we define

$$S(f,z) := \left(\frac{f''}{f'}\right)'(z) - \frac{1}{2}\left(\frac{f''}{f'}(z)\right)^2$$

to be the *Schwarzian Derivative* of f. The Schwarzian derivative has a remarkable property that it is invariant with respect to Möbius transformations; i.e., $S(M \circ f,z) \equiv S(f,z)$ for any Möbius transformation $M(z)$, and $S(M,z) \equiv 0$ if and only if $M(z)$ is a Möbius transformation.

Our starting point is the following results of Nehari and Gabriel.

THEOREM A (Nehari [8]) *If $f \in N$ and it satisfies*

$$|S(f,z)| \leq \frac{\pi^2}{2} \qquad \textit{for all} \quad z \in \Delta,$$

then $f(z)$ is univalent. The result is sharp.

The constant $\pi^2/2$ is best possible as shown by the example $(\exp(i\pi z) - 1)/i\pi$. We also have

THEOREM B (Gabriel [3]) *Suppose $f \in N$ and that*

$$|S(f,z)| \leq 2c_0 \approx 2.73 \qquad \textit{for all} \quad z \in \Delta,$$

where c_0 *is the smallest positive root of the equation* $2\sqrt{x}-\tan\sqrt{x}=0$, *then* $f(z)$ *maps* Δ *onto a starlike domain.*

Recall that $f \in S$ is starlike (with respect to the origin) if and only if $\Re(zf'/f)>0$ for all $z\in\Delta$. We denote the class of starlike functions by S^*. It is however not known whether the constant $2c_0$ in Theorem B is the best possible. For details about univalent functions we refer to [2] and [4].

Remark Notice that we have not assumed that $f'(z)\neq 0$ in both Theorems A and B. However, the Schwarzian derivatives are still well-defined. For suppose $f'(z_0)=0$ for some $z_0\in\Delta$, then $S(f,z_0)$ will have a pole of order 2 at z_0. This contradicts the assumption that $|S(f,z)|$ is uniformly bounded in Δ.

Similar results also exist for functions analytic in $\mathbb{C}\backslash\Delta$, see [3] for details. We can reformulate the above results as follows. We define

$$\frac{\pi^2}{2}=\Omega(N;S):=\sup\{2\tilde{\delta} : g\in N;\ |S(g,z)|\leq 2\tilde{\delta}\Rightarrow g \text{ univalent}\}$$

to be the *Schwarzian radius of univalence of the class* N, and let

$$\Omega(N;S^*):=\sup\{2\tilde{\delta} : g\in N; |S(g,z)|\leq 2\tilde{\delta}\Rightarrow g \text{ starlike}\}$$

to be the *Schwarzian radius of starlikeness of the class* N. Gabriel's result indicates that $2c_0\leq\Omega(N;S^*)$.

Let $f\in N$ and $|\arg zf'/f|\leq\alpha\pi/2$ $(0<\alpha\leq 1)$. Then f is said to belong to the class of *strongly-starlike functions of order* α, denoted by $S^*(\alpha)$. Clearly we have $S^*(1)=S^*$. Recall that if $f\in N$ then $f(z)$ is *convex of order* η if and only if $\Re(1+zf''/f')>\eta$ for all $z\in\Delta$. The class is denoted by $K(\eta)$. Similarly $K(0)=K$ is the class of usual convex functions.

In this note we would like to consider the following problems:

$$\Omega(N;S^*(\alpha)):=\sup\{2\tilde{\delta} : g\in N;\ |S(g,z)|\leq 2\tilde{\delta}\Rightarrow g\in S^*(\alpha)\}$$

and

$$\Omega(N;K(\eta)):=\sup\{2\tilde{\delta} : g\in N;\ |S(g,z)|\leq 2\tilde{\delta}\Rightarrow g\in K(\eta)\}.$$

Those are the *Schwarzian radius of strongly-starlikeness of* N and the *Schwarzian radius of* $K(\eta)$. The main results are stated in Section 2 and their proofs are given in Sections 5 and 6. We shall show that the constant $\Omega(N;S^*(\alpha))$ does not exist as soon as $\alpha<1$ and we obtain a partial result about $\Omega(N;K)$. Some examples are given in Section 3. A brief discussion on the method used is given in Section 4. We shall also consider another related subclass of N in Section 7 (where a connection with quasiconformal extensions is considered). In Section 8, we give a further example to illustrate a possible sharp bound for $\Omega(N;S^*)$.

2. STATEMENTS OF THE MAIN RESULTS

THEOREM 1 *Let* $f(z)\in N$, $0<\alpha\leq 1$ *and* $|a_2|=\eta<\sin(\alpha\pi/2)$. *Suppose*

$$\sup_{z\in\Delta}|S(f,z)|=2\delta(\eta)$$

where $\delta(\eta)$ satisfies the inequality

$$\sin^{-1}\left(\frac{\delta e^{\delta/2}}{2}\right) + \sin^{-1}\left(\eta + \frac{(1+\eta)\delta e^{\delta/2}}{2}\right) \leq \frac{\alpha\pi}{2}. \tag{1}$$

Then $f(z) \in S^*(\alpha)$.

Remarks (1) The inequality (1) guarantee the existence of such a $\delta(\eta)$ since we have assumed $\sin^{-1}\eta < \alpha\pi/2$ in the hypotheses of the Theorem.

(2) When $\alpha = 1$ and $a_2 = 0$, Theorem 1 gives a poor estimate for 2δ for starlike functions when compared to Theorem B. The best 2δ that we can derive from (1) in this case is approximately 1.8.

THEOREM 2 *Let $f \in N$ and $|a_2| = \eta < \frac{1}{3}$. Suppose*

$$\sup_{z\in\Delta}|S(f,z)| = 2\delta(\eta)$$

where $\delta(\eta)$ satisfies the inequality

$$6\delta + 5(1+\eta)\delta e^{\delta/2} < 2. \tag{2}$$

Then

$$f(z) \in K\left(\frac{2 - 6\eta - 5(1+\eta)\delta e^{\delta/2}}{2 - 2\eta - (1+\eta)\delta e^{\delta/2}}\right). \tag{3}$$

In particular if $a_2 = 0$ and $2\delta \leq 0.6712$, then f is convex.

Remark Note that (3) holds if (2) holds so that the quotient appearing in (3) is positive.

3. EXAMPLES

First we consider the following example

$$g(z) = \frac{z}{1+cz}, \qquad |c| \leq 1.$$

We require $|c| \leq 1$, since $g(z)$ is univalent in Δ. Note that it has the following series expansion

$$g(z) = z - cz^2 + c^2z^3 - \cdots \in S.$$

If $|c| \leq \sin(\alpha\pi/2)$ for some $0 < \alpha \leq 1$, then $g \in S^*(\alpha)$ since $S(g,z) \equiv 0$ and $g(z)$ satisfies the hypothesis of Theorem 1 with $2\delta \equiv 0$.

In fact

$$\frac{zg'(z)}{g(z)} = \frac{1}{1+cz}.$$

So

$$\left|\arg\frac{zg'}{g}\right| = \left|\arg\frac{1}{1+cg}\right| = |\arg(1+cz)| \leq \sin^{-1}|c|.$$

Hence $|\arg zg'/g| \le \alpha\pi/2$ if and only if $|c| \le \sin(\alpha\pi/2)$. This shows, at least for this particular g, that it is necessary to assume $|a_2| \le \sin(\alpha\pi/2)$ in order for $g \in S^*(\alpha)$. So the hypothesis in the Theorem 1 is nearly the best possible. Note also the example actually shows that $\Omega(N; S^*(\alpha))$ does not exist as soon as $\alpha < 1$.

Next we consider an example similar to the above. This time we let

$$h(z) = \frac{z}{1-cz} = z + cz^2 + c^2z^3 + \cdots, \qquad |c| = 1.$$

Here $h(z)$ maps the unit disc onto a rotation of right-half plane passing through $-\frac{1}{2}\bar{c}$, and so it is clearly a convex function with $|a_2| = |c| = 1$. However the hypothesis in the above theorem requires $|a_2| < \frac{1}{3}$. Hence it is not sharp.

4. PRELIMINARY DISCUSSION AND A MAIN LEMMA

To prove the theorems, we use some classical methods in the area of second order differential equations. Consider the following equation

$$y'' + Ay = 0 \tag{4}$$

where $A(z)$ is an analytic function.

Let $A := \frac{1}{2}S(f,z)$, where $f \in N$ and $f'(z) \neq 0$ in Δ. Then there exist linearly independent solutions f_1, f_2 of (4) such that $f(z) = f_1(z)/f_2(z)$. Conversely if A is analytic in Δ and if f_1 and f_2 are linearly independent solutions of (4), then $f(z) := f_1(z)/f_2(z)$ satisfies the equation $S(f,z) = 2A$. We shall use integral representations of $f_1(z)$ and $f_2(z)$ to facilitate certain estimates which in turn give estimates on $\Re(zf'/f)$ and $\Re(1 + zf''/f')$. Therefore our method depends heavily on the following integral inequality of Gronwall. For details see Hille [5]. In Gabriel's paper [3], estimations were made by using $f = f_1/f_2$ only and in a somewhat different direction where integral representations of f_1 and f_2 were not involved.

LEMMA C (Gronwall T. H.; see Hille [5], p. 19) *Suppose $A(t)$ and $g(t)$ are non-negative continuous real functions for $t \ge 0$. Let $k > 0$ be a constant. Then the inequality*

$$g(t) \le k + \int_0^t g(s)A(s)\,ds$$

implies for all $t > 0$ that

$$g(t) \le k \exp\left(\int_0^t A(s)\,ds\right).$$

5. PROOF OF THEOREM 1

Suppose $u(z)$ and $v(z)$ are linearly independent solutions of the differential equation (4) with $A(z) \equiv \frac{1}{2}S(f,z)$ where $u(0) = v'(0) = 0$ and $u'(0) = v(0) = 1$. This is always possible since the *Wronskian* $W(u,v)$ of $u(z)$ and $v(z)$ of a second order differential equation is identically equal to a constant which we may assume to be -1. Thus we have $u(z) = z + \cdots$ and $v(z) = 1 + \cdots$.

By the discussion preceding Lemma C, we cna find two linearly independent solutions f_1 and f_2 of (4) such that

$$f(z) = \frac{f_1(z)}{f_2(z)} = \frac{au(z) + bv(z)}{cu(z) + dv(z)}, \qquad ad - bc \neq 0. \tag{5}$$

This representation depends on three arbitrary constants only. But $A(z) = \frac{1}{2}S(f, z)$ is a third order differential equation in f, hence the constants can be determined uniquely and every solution of (4) can be obtained from (5) by a suitable choice of these constants.

Since $f(z) = z + \cdots$, we deduce that $b = 0$. We can divide through the right hand side of equation (5) by a and therefore we may assume $a = 1$ at the beginning. Since $f'(0) = 1$, so $d = 1$. Also we have $c = -a_2$ because $v'(0) = 0$. Hence

$$f(z) = \frac{u(z)}{cu(z) + v(z)}.$$

Differentiating f yields

$$\begin{aligned} f'(z) &= \frac{u'(z)v(z) - v'(z)u(z)}{(cu(z) + v(z))^2} \\ &= -\frac{W(u, v)}{(cu(z) + v(z))^2} = \frac{1}{(cu(z) + v(z))^2}. \end{aligned}$$

Hence

$$\frac{zf'(z)}{f(z)} = \frac{z}{u(z)(cu(z) + v(z))}.$$

We will show that $|\arg zf'/f| \leq \alpha\pi/2$. Integrating (4) by parts, we may write $u(z)$ in the following form:

$$u(z) = z + \int_0^z (\zeta - z)A(\zeta)u(\zeta)\, d\zeta. \tag{6}$$

The path of integration is taken along the radius $\zeta(t) = te^{i\theta}$, $t \in [0, r]$, $z = re^{i\theta}$. Thus we have the estimate

$$\begin{aligned} |u(z)| &\leq r + \int_0^r |te^{i\theta} - re^{i\theta}|\, |A(te^{i\theta})|\, |u(te^{i\theta})|\, dt \\ &< 1 + \int_0^r (r - t)|A(te^{i\theta})|\, |u(te^{i\theta})|\, dt. \end{aligned}$$

Now $|A(z)| < \delta = \delta(\eta)$ by assumption, where $\delta(\eta)$ satisfies (1). Thus applying Lemma C, we deduce

$$|u(z)| \leq \exp\left(\int_0^r (r-t)|A(te^{i\theta})|\,dt\right)$$

$$< \exp\left(\delta(\eta)\int_0^r (r-t)\,dt\right)$$

$$= \exp\left(\frac{\delta r^2}{2}\right). \tag{7}$$

Substituting (7) back into (6) yields

$$|u(z)-z| \leq \int_0^r (r-t)|A(te^{i\theta})|\,|u(te^{i\theta})|\,dt$$

$$< \delta(\eta)\exp(\delta(\eta))\int_0^r (r-t)\,dt$$

$$= \delta(\eta)\exp(\delta(\eta))\frac{r^2}{2}.$$

Hence

$$\left|\frac{u(z)}{z} - 1\right| < \frac{\delta(\eta)\exp(\delta(\eta))r}{2} < \frac{\delta(\eta)\exp(\delta(\eta))}{2}. \tag{8}$$

Similarly, $v(z)$ can also be written in the form

$$v(z) = 1 + \int_0^z (\zeta - z)A(\zeta)v(\zeta)\,d\zeta.$$

Combining this with (6), we have

$$cu(z) + v(z) = 1 + cz + \int_0^z (\zeta - z)A(\zeta)(cu(\zeta) + v(\zeta))\,d\zeta. \tag{9}$$

So we can estimate $cu(z) + v(z)$ as above,

$$|cu(z) + v(z)| \leq 1 + |c|r + \int_0^r (r-t)|A(te^{i\theta})|\,|cu(te^{i\theta}) + v(te^{i\theta})|\,dt.$$

Since $|A(z)| < \delta(\eta)$ where $\delta(\eta)$ satisfies (1), we obtain, by applying Lemma C again that

$$|cu(z) + v(z)| < (1 + |c|)\exp\left(\int_0^r (r-t)|A(te^{i\theta})|\,dt\right)$$

$$< (1 + |c|)\exp(\delta(\eta)/2). \tag{10}$$

We substitute (10) back into (9) and note that $|c| = \eta < \sin(\alpha\pi/2)$,

$$|cu(z) + v(z) - 1| \leq |c|r + \int_0^r (r-t)|A(te^{i\theta})|\,|cu(te^{i\theta}) + v(te^{i\theta})|\,dt$$
$$< \eta + (1+\eta)\delta(\eta)\exp(\delta(\eta)/2)\int_0^r (r-t)\,dt$$
$$< \eta + (1+\eta)\delta(\eta)\frac{\exp(\delta(\eta)/2)}{2}. \tag{11}$$

It follows from (8) and (10) that

$$\left|\arg\frac{zf'}{f}\right| = \left|\arg\frac{f}{zf'}\right| = \left|\arg\frac{u(z)(cu(z)+v(z))}{z}\right|$$
$$\leq \left|\arg\frac{u(z)}{z}\right| + |\arg(cu(z)+v(z))|$$
$$\leq \sin^{-1}\left(\frac{\delta e^{\delta/2}}{2}\right) + \sin^{-1}\left(\eta + \frac{(1+\eta)\delta e^{\delta/2}}{2}\right) \leq \frac{\alpha\pi}{2}.$$

The last inequality follows from the hypothesis (1). Hence $f \in S^*(\alpha)$ and this completes the proof of the Theorem. ∎

Another observation is that we can estimate $\arg f/z$ the same way that we have done to $\arg zf'/f$. Since

$$|\arg f/z| = \left|\arg\frac{u(z)}{z(cu(z)+v(z))}\right| \leq |\arg u(z)/z| + |\arg(cu(z)+v(z))|$$
$$\leq \sin^{-1}\left(\frac{\delta e^{\delta/2}}{2}\right) + \sin^{-1}\left(\eta + \frac{(1+\eta)\delta e^{\delta/2}}{2}\right) \leq \frac{\alpha\pi}{2},$$

and this estimate is exactly the same as (1) of Theorem 1. Hence we obtain

COROLLARY 1 *Let* $f \in N$, $0 < \alpha \leq 1$ *and* $|a_2| = \eta < \sin(\alpha\pi/2)$. *If*

$$\sup_{z\in\Delta}|S(f,z)| = 2\delta(\eta),$$

where $\delta(\eta)$ *satisfies the inequality* (1), *then* $|\arg f(z)/z| \leq \alpha\pi/2$.

6. PROOF OF THEOREM 2

We shall use the same idea as in the proof of Theorem 1. Since

$$f(z) = \frac{u(z)}{cu(z)+v(z)},$$

it is easy to obtain

$$1 + \frac{zf''}{f'} = 1 - 2z\frac{cu'(z)+v'(z)}{cu(z)+v(z)}. \tag{12}$$

We first prove that $\Re(1 + zf''/f') > 0$, for all $z \in \Delta$. In view of (12), it is sufficient to prove that

$$\left|\frac{cu'(z) + v'(z)}{cu(z) + v(z)}\right| < \frac{1}{2}.$$

We note that $\eta + \frac{1}{2}(1+\eta)\delta e^{\delta/2} < 1$ since both η and $\delta(\eta)$ satisfy (2). From (9), we have the expression

$$\frac{cu'(z) + v'(z)}{cu(z) + v(z)} = \frac{c - \int_0^z A(\zeta)(cu(\zeta) + v(\zeta))\,d\zeta}{1 + cz + \int_0^z (\zeta - z)A(\zeta)(cu(\zeta) + v(\zeta))\,d\zeta}.$$

Since $|A(z)| < \delta$ by hypothesis, we deduce from (10) (after applying Lemma C), that

$$\begin{aligned}
&\left|cz + \int_0^z (\zeta - z)A(\zeta)(cu(\zeta) + v(\zeta))\,d\zeta\right| \\
&\quad < |c| + (1 + |c|)\int_0^r (r - t)|A(te^{i\theta})|e^{\delta t^2/2}\,dt \\
&\quad \le \eta + (1+\eta)\delta e^{\delta/2}\int_0^r (r - t)\,dt \\
&\quad < \eta + (1+\eta)\delta e^{\delta/2}/2 < 1. \qquad (13)
\end{aligned}$$

The last inequality follows from assumption (2). Thus

$$\begin{aligned}
\left|\frac{cu'(z) + v'(z)}{cu(z) + v(z)}\right| &= \frac{\left|c - \int_0^z A(\zeta)(cu(\zeta) + v(\zeta))\,d\zeta\right|}{\left|1 + cz + \int_0^z (\zeta - z)A(\zeta)(cu(\zeta) + v(\zeta))\,d\zeta\right|} \\
&\le \frac{\left|c - \int_0^z A(\zeta)(cu(\zeta) + v(\zeta))\,d\zeta\right|}{1 - \left|cz + \int_0^z (\zeta - z)A(\zeta)(cu(\zeta) + v(\zeta))\,d\zeta\right|} \\
&\le \left(\left|c - \int_0^z A(\zeta)(cu(\zeta) + v(\zeta))\,d\zeta\right|\right) \\
&\quad \times \left(\sum_0^\infty \left|cz + \int_0^z (\zeta - z)A(\zeta)(cu(\zeta) + v(\zeta))\,d\zeta\right|^n\right) \\
&< (\eta + (1+\eta)\delta e^{\delta/2})\left(\sum_0^\infty (\eta + (1+\eta)\delta e^{\delta/2}/2)^n\right) \\
&= \frac{\eta + (1+\eta)\delta e^{\delta/2}}{1 - \eta - \dfrac{(1+\eta)\delta e^{\delta/2}}{2}} = \frac{2(\eta + (1+\eta)\delta e^{\delta/2})}{2 - 2\eta - (1+\eta)\delta e^{\delta/2}}.
\end{aligned}$$

The above geometric progression converges because of (13). Now $\Re(1 + zf''/f') > 0$ follows from (2). Moreover

$$\begin{aligned}\Re\left(1+\frac{zf''}{f'}\right) &= \Re\left(1-2z\frac{cu'(z)+v'(z)}{cu(z)+v(z)}\right)\\ &\geq 1-2\left(\frac{2(\eta+(1+\eta)\delta e^{\delta/2})}{2-2\eta-(1+\eta)\delta e^{\delta/2}}\right)\\ &= \frac{2-6\eta-5(1+\eta)\delta e^{\delta/2}}{2-2\eta-(1+\eta)\delta e^{\delta/2}}.\end{aligned}$$

If we now put $-a_2 = c = 0$ in the above argument, it follows from (2) that

$$5\delta\exp(\delta/2) < 2,$$

where δ can be computed. Numerical calculation suggests that it suffices to assume $\delta < 0.3365$. Hence $|S(f,z)| < 0.6712$ implies that f is convex univalent. ■

We summarize the above relations in terms of Schwarzian radii:

$$0.6712 < \Omega(N;K) < 2c_0 < \Omega(N;S^*) < \Omega(N;S) = \pi^2/2.$$

7. AN APPLICATION OF THE SECOND COEFFICIENT OF f

Theorem 2 is proved under the assumptions that both the Schwarzian derivative and a_2 of f are small and are related by (2). Thus if we want to delete the assumption (2) so that a bound imposed on $|S(f,z)|$ alone is sufficient to guarantee that f is convex, more restrictions on f are expected. We shall show that if f has a quasiconformal extension, then the assumption (2) can be dropped. Let us recall that a homeomorphism f defined on a domain D is a *quasiconformal mapping* if (i) f is absolutely continuous on almost all vertical and horizontal lines in D, and (ii) that the *complex dilatation* $\mu(z)$ of $f(z)$ is less than 1. If in addition that $|\mu| < k < 1$, then f is a *k-quasiconformal mapping* (see [7] Chapter 1). Note that we have not used the standard notation for quasiconformal mappings here. Let $S_k(\infty)$ be a subclass of N if f is univalent in Δ and has a k-quasiconformal extension to $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ in $\mathbb{C}\backslash\Delta$ with $f(\infty) = \infty$. Note that we have used f again to denote the extension of $f(z)$ in $\mathbb{C}/\Delta$. We discuss the problem of Schwarzian radius of convexity of f in $S_k(\infty)$. That is, we consider

$$\Omega(S_k(\infty);K) := \sup\{2\tilde{\delta} : g \in S_k(\infty);\ |S(g,z)| \leq 2\tilde{\delta} \Rightarrow g(\Delta) \text{ is convex}\}.$$

We find the following result if k is also suitably restricted.

THEOREM 3 *Let $f \in S_k(\infty)$ where $k \leq 0.108$ and suppose that*

$$\sup_{z\in\Delta} |S(f,z)| = 2\delta(\eta) \leq 0.217.$$

Then $f(\Delta)$ is a convex domain; i.e., $0.217 < \Omega(S_k(\infty);K)$, $k \leq 0.108$.

LEMMA D (Kühnau [6]) *Suppose $f \in S_k(\infty)$ then $|a_2| \leq 2k$. The bound is sharp.*

Proof of Theorem 3 The first half of the proof is identical to that of Theorem 2. We shall also use the same notations here as in Theorem 2. By our hypotheses,

$$(1-|z|^2)^2|S(f,z)| < |S(f,z)| < 2\delta < 2 \qquad \text{for all} \quad z \in \Delta.$$

It follows from the well-known criterion of Ahlfors and Weill [1] that f admits a δ-quasiconformal extension to $\overline{\mathbb{C}}$. But since $f(\infty)=\infty$, it follows from Lemma D that $|a_2| \le 2\delta$. Now f is convex if and only if (2) is valid. Replacing η by 2δ in (2), we have

$$6\eta + 5(1+\eta)\delta\exp(\delta/2) < 12\delta + 5(1+2\delta)\delta\exp(\delta/2).$$

Hence we only need to solve the last inequality for δ so that it is less than 2. Numerical calculations show that this is true if $2\delta \le 0.217$. So f is convex univalent in Δ. ■

8. A FURTHER EXAMPLE

Consider the function

$$g(z) = \frac{1}{\sqrt{\delta}}\tan(\sqrt{\delta}z).$$

It is easy to show that $S(f,z) = 2\delta$. Notice that we have $g''(0)=0$. Now

$$\frac{zg'(z)}{g(z)} = \frac{\sqrt{\delta}z}{\sin(\sqrt{\delta}z)\cos(\sqrt{\delta}z)} = \frac{2\sqrt{\delta}z}{\sin(2\sqrt{\delta}z)}.$$

We require to show that $\Re(zg'/g) > 0$, for all $z \in \Delta$ when δ is small. This is equivalent to finding the largest disc $|w| < r$ such that $\Re(\sin(w)/w) > 0$ where $2\sqrt{\delta}z = w = \xi + i\mu$.

Let

$$H(\xi,\mu) = \Re\left(\frac{\sin w}{w}\right) = \frac{\xi\sin\xi\cosh\mu - \mu\cos\xi\sinh\mu}{\xi^2+\mu^2},$$

and

$$F(\xi,\mu) = \xi\sin\xi\cosh\mu - \mu\cos\xi\sinh\mu.$$

We apply the method of the Lagrange's multiplier to $F(\xi,\mu)$ subject to $\xi^2+\mu^2 = r^2$ for some $r>0$. So let

$$\phi(\xi,\mu) = F(\xi,\mu) + \lambda(\xi^2+\mu^2-r^2).$$

We proceed to solve the resulting equations

$$\phi_\xi = \cosh\mu(\xi\cos\xi + \sin\xi) - \mu\sin\xi\sinh\mu + 2\lambda\xi = 0, \tag{14}$$

$$\phi_\mu = \xi\sin\xi\sinh\mu + \cos\xi(\mu\cosh\mu + \sinh\mu) + 2\lambda\mu = 0, \tag{15}$$

$$\phi_\lambda = \xi^2+\mu^2 - r^2 = 0.$$

Multiply (14) by μ and (15) by ξ, and equating them. Then

$$\xi^2 + \mu^2 = \frac{\mu}{\tanh\mu} - \frac{\xi}{\tan\xi}. \tag{16}$$

Hence

$$H(\xi,\mu) = \frac{\sinh\mu}{\mu}(\xi\sin\xi + \cos\xi).$$

As $\sinh\mu/\mu$ is an even function and so is always positive, we have $H(\xi,\mu) < 0$ if and only if $\xi\sin\xi + \cos\xi < 0$. This is equivalent to finding the smallest positive root of the equation $\xi\tan\xi = -1$. Substitute this into (16). We obtain

$$\xi^2 + \mu^2 = \frac{\mu}{\tanh\mu} - \frac{\xi}{\left(-\frac{1}{\xi}\right)} = \frac{\mu}{\tanh\mu} + \xi^2.$$

Hence the problem has been reduced to solving the following transcendental equations

$$\xi\tan\xi = -1,$$

and

$$\mu\tanh\mu = 1.$$

Numerical calculation gives

$$2.79 < \xi < 2.8$$

and

$$1.119 < \mu < 1.2.$$

So

$$3.037 < r = (\xi^2 + \mu^2)^{1/2} < 3.046.$$

Thus $\Re(\sin w/w)$ will first become negative when w lies in the above annulus. Hence if we require δ such that $|2\sqrt{\delta}z| = |w| \le 3.037$; i.e., $\delta < 2.3$, then g is a starlike function.

We can similarly consider the convexity case. Now that

$$1 + \frac{zg''(z)}{g'(z)} = 1 + 2\sqrt{\delta}z\tan\sqrt{\delta}z.$$

Again let $w = \sqrt{\delta}z$, it is sufficient to find the largest $r > 0$ such that $\Re(w\tan w) > -\frac{1}{2}$. We have

$$\Re(w\tan w) = \frac{\xi\tan\xi(1-\tanh^2\mu) - \mu\tanh\mu(1+\tan^2\xi)}{1+\tan^2\xi\tanh^2\mu}.$$

Unlike the starlike case, this time it is more difficult to find out precisely the first $r > 0$ such that $\Re(w\tan w) > -\frac{1}{2}$. But if we assume $\xi = 0$ then it amounts to solve

$$-\mu\tanh\mu = -\tfrac{1}{2}.$$

The approximate solution is $0.7715 < \mu < 0.773$. Hence if $\mu > 0.7715$, $\Re(w \tan w)$ could be less than $-\frac{1}{2}$; i.e., if $\delta > 05952$, g need not be a convex function.

Summarizing the above results, we deduce

PROPOSITION 4 *Let* $g(z) = (1/\sqrt{\delta})\tan(\sqrt{\delta}z)$. *If*

(i) $$|S(g,z)| = 2\delta \approx 4.6 \quad \textit{for all} \quad z \in \Delta$$

where $\delta = (x^2 + y^2)^{1/2}$ *and* x, y *are the first positive roots of the transcendental equations*

$$2\sqrt{\delta}x\tan(\sqrt{\delta}x) = -1 \qquad \textit{and} \qquad 2\sqrt{\delta}y\tanh(\sqrt{\delta}y) = 1$$

respectively. Then f *is a starlike function;*

(ii) *if for some* $z_0 \in \Delta$ *such that* $|S(g, z_0)| > 1.2$, *then* g *need not be convex univalent.*

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Note added in proof: Author's current address is Dept. of Mathematics, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong.

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