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An oscillation result of a third order linear differential equation with entire periodic coefficients*

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Dedicated to the memory of Steven Bank

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We prove that the periodic equation $f''' - Kf' + e^z f = 0$ admits a solution with finite exponent of convergence if and only if $K = (n+1)^2/9$ where n is a non-negative integer satisfying a certain $(n+1) \times (n+1)$-determinant condition. Moreover, we obtain explicit representations for such solutions. Our result is somewhat similar to a result due to Bank, Laine and Langley [5] for a second order equation.

Keywords: differential equation; periodic coefficients oscillation result

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1. INTRODUCTION

We are concerned with the number of zeros of a third order linear differential equation with entire periodic coefficients. Our domain will be the entire complex plane and we shall employ Nevanlinna value

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distribution theory. For details of this theory we refer the reader to the book of Hayman [11].

Let $f(z)$ be a solution of an arbitrary linear differential equation with its zeros $a_1, a_2, a_3, \ldots$ ordered by increasing moduli. We define the *exponent of convergence* $\lambda(f)$ of f to be $\inf\{\lambda : \sum_{i=1}^{\infty} 1/|a_i|^{\lambda} < \infty\}$. The theory of *complex oscillation* of differential equations is to investigate how the quantity $\lambda(f)$ is affected by the coefficients of the equation and what value it takes including infinity. See Bank and Laine [2], and Laine [12].

Results concerning linear differential equations with entire periodic coefficients are particularly interesting. In fact, Bank and Laine [3] were able to find explicit representations for solutions f of such equations in the second order case, provided $\lambda(f) < \infty$. These representations depend upon whether $f(z)$ and $f(z+\omega)$ are linearly independent, where ω is the period of the coefficients. Their results have been generalized to some higher order equations by Bank and Langley [8], see Lemma D below.

In another paper [4], Theorem 2, Bank, Laine and Langley proved a specific result concerning the complex oscillation of a periodic second order differential equation.

THEOREM A. *Let $K \in \mathbf{C}$ and suppose that*

$$f'' + (e^z - K)f = 0 \tag{1.1}$$

has a non-trivial solution $f(z)$ such that $\lambda(f) < \infty$. Then

$$K = \frac{q^2}{16}, \tag{1.2}$$

where q is an odd positive integer. Conversely, if K is of the form (1.2), *then the equation* (1.1) *admits two linearly independent solutions f_1 and f_2 each with $\lambda(f_i) \le 1$, $i = 1, 2$.*

Of course, Theorem A is a special case of the following Theorem B, where $\lambda(f)$ is more restricted, see Bank, Laine and Langley [5], Theorem 3.3.

THEOREM B. *Let P be a polynomial of degree $n \ge 1$, and let Q be an entire function of order $\sigma(Q) < n$. Suppose that the equation*

$$f'' + (e^P + Q)f = 0 \tag{1.3}$$

admits a non-trivial solution $f(z)$ *with* $\lambda(f) < n$. *Then* f *has no zeros,* Q *is a polynomial and*

$$Q = -\frac{1}{16}(P')^2 + \frac{1}{4}P''.$$

Clearly, the equation (1.3) reduces to (1.1), provided $P(z) \equiv z$. We remark that (1.1) plays an important role in a number of recent papers, see Bank and Langley [6], [7], Chiang [9] and Wang [13], [14].

In [10], Theorem 3.2, Theorem B was extended to some third order equations. As a special case, we recall

THEOREM C. *Let* P *be a polynomial of degree* $n \geq 1$, *and* Q_1, Q *be entire functions each of order* $< n$. *Suppose that*

$$f''' + Q_1 f' + (e^P + Q)f = 0 \tag{1.4}$$

admits a solution f *such that* $\lambda(f) < n$. *Then* f *has no zeros,* Q_1 *and* Q *are polynomials such that*

$$Q_1 = -\frac{1}{9}(P')^2 + \frac{2}{3}P''$$

and

$$Q = \frac{1}{3}P^{(3)} - \frac{1}{9}P'P''.$$

In view of the relation between Theorem A and Theorem B, it is natural to ask, whether a similar result related to Theorem C holds in the case of $P(z) \equiv z$, i.e., provided the coefficients of (1.4) are periodic. We prove such a result below, including explicit representations of solutions. Observe that A. Baesch determines, in a forthcoming paper [1], all solutions f of

$$f^{(k)} + \sum_{j=1}^{k-2} A_j f^{(j)} + A_0(z)f = 0, \quad k \geq 3, \tag{1.5}$$

where $A_1, \ldots, A_{k-2}$ are constants and $A_0(z)$ is a nonconstant periodic entire function rational in e^z, such that $\log^+ N(r, 1/f) = o(r)$. She proves that this situation appears if and only if at least one of certain k^2 linear differential equations with polynomial coefficients admits a non-trivial polynomial solution. Our result below deals with a special case of (1.5) only. However, our characterization is of a more simple, constructive type. The open determinant problem described in Section 4 is perhaps of some independent interest.

2. THE MAIN RESULT

THEOREM 1. *Let $K \in \mathbf{C}$, and suppose that*

$$f''' - Kf' + e^z f = 0 \tag{2.1}$$

admits a non-trivial solution f such that

$$\log^+ N(r, 1/f) = o(r)$$

as $r \to \infty$. Then there exist two integers r and s, $r + s \geq 0$, such that

$$K = \frac{(r+s+1)^2}{9}. \tag{2.2}$$

Moreover, if $n = r + s > 0$, then n satisfies the following tridiagonal $(n+1) \times (n+1)$-determinant condition:

$$\det \mathbf{A} = 0, \tag{2.3}$$

where the non-zero diagonals of $\mathbf{A}$ are determined by

$$\begin{cases} a_{j,j-1} := (j-1)j(j+1) - 2jn - jn^2, & j = 1, \ldots, n, \\ a_{j,j} := -3j(j+1) + 2n + n^2, & j = 0, \ldots, n, \\ a_{j,j+1} := 3(j+1), & j = 0, \ldots, n-1. \end{cases} \tag{2.4}$$

Furthermore, f admits one of the following representations:

$$f_i(z) = e^{\frac{-s-1}{3}z} \psi(e^{z/3}) \exp(c_i e^{z/3}), \tag{2.5}$$

where $c_i^3 + 27 = 0$, $i = 1, 2, 3$, and

$$\psi(\zeta) = \sum_{j=-r}^{s} d_j \zeta^j, \quad d_{-r} d_s \neq 0. \tag{2.6}$$

Conversely, suppose K takes the form (2.2) and, if $n = r + s > 0$, then n satisfies (2.3) and (2.4). Then there exists a rational function of *the form* (2.6) *such that the three functions defined by* (2.5) *are linearly independent solutions of* (2.1) *each with* $\lambda(f_i) \leq 1$ *for* $i = 1, 2, 3$.

Remark The hypothesis $\log^+ N(r, 1/f) = o(r)$ as $r \to \infty$ that we have made above is in fact weaker than $\lambda(f) < \infty$, see Lemma D below.

The proof of Theorem A depends heavily on the explicit representation of solutions of periodic differential equations obtained by Bank and Laine [3], and a special non-linear second order differential equation in $E = f_1 f_2$, where f_1 and f_2 are two linearly independent solutions, see [3], p. 6. For higher order equations, no such useful differential equation in E has been found. Our argument depends on the following representation lemma obtained by Bank and Langley for higher order equations, see [8], Theorem 2:

LEMMA D. *Suppose that* $k \geq 3$, *that* A_0 *is a non-constant periodic entire function, rational in* e^z, *and that* $A_1, \ldots, A_{k-2}$ *are constants. Suppose finally that* f *is a non-trivial solution of*

$$y^{(k)} + \sum_{j=0}^{k-2} A_j(z) y^{(i)} = 0$$

such that

$$\log^+ N(r, 1/f) = o(r)$$

as $r \to \infty$. *Then there exists an integer* q *with* $1 \leq q \leq k$, *a constant* d, *and rational functions* $\psi(\zeta)$ *and* $S(\zeta)$, *analytic on* $0 < |\zeta| < \infty$ *such that*

$$f(z) = \psi(e^{z/q}) \exp\left(dz + S(e^{z/q})\right). \tag{2.7}$$

3. PROOF OF THEOREM 1

Under the hypothesis of Theorem 1 and by Lemma D (2.7), we may write f as

$$f(z) = e^{dz} G(e^{z/q}), \tag{3.1}$$

where $G(\zeta) = \psi(\zeta) \exp(S(\zeta))$, $1 \leq q \leq 3$, d is a constant and both ψ and S are rational and analytic on $0 < |\zeta| < \infty$.

By substituting $f(z)$ of (3.1) into (2.1) and denoting $\zeta = e^{z/q}$, we have

$$\zeta^3 G^{(3)}(\zeta) + (3dq+3)\zeta^2 G''(\zeta) + (3d^2q^2 + 3dq + 1 - q^2K)\zeta G'(\zeta) + q^3(\zeta^q + d^3 - Kd)G(\zeta) = 0. \tag{3.2}$$

We denote now

$$\psi(\zeta) = \sum_{j=-r}^{s} c_j \zeta^j \tag{3.3}$$

and

$$S(\zeta) = \sum_{j=-n}^{m} d_j \zeta^j. \tag{3.4}$$

Since f must be of infinite order, we have $(m, n) \neq (0, 0)$. We may also assume that $s \geq -r$ and $m \geq -n$. Then we have, for $m \geq 1$,

$$\frac{G'(\zeta)}{G(\zeta)} = \alpha\zeta^{m-1} + O(\zeta^{m-2}), \quad \frac{G''(\zeta)}{G(\zeta)} = \alpha^2\zeta^{2m-2} + O(\zeta^{2m-3})$$

and

$$\frac{G^{(3)}(\zeta)}{G(\zeta)} = \alpha^3\zeta^{3m-3} + O(\zeta^{3m-4})$$

as $\zeta \to \infty$ and $\alpha \neq 0$ is a constant. It follows from (3.2) that $3(m-1)+3 = q$, and since $1 \leq q \leq 3$, we deduce readily that $q = 3$ and $m = 1$ in (3.4). Therefore, we must have $m \leq 1$. Moreover, by considering $G_1(t) = G(1/t)$, we have, again from (3.2), the following equation:

$$t^3 G_1^{(3)}(t) + 3(1-dq)t^2 G_1''(t) + (3d^2q^2 - 3dq + 1 - Kq^2)tG_1'(t) - q^3(t^{-q} + d^3 - Kd)G_1(t) = 0. \tag{3.5}$$

Likewise, we deduce, for $n \geq 1$,

$$G_1'(t)/G_1(t) \sim \beta t^{n-1}, \quad G_1''(t)/G_1(t) \sim \beta^2 t^{2n-2}$$

and

$$G_1^{(3)}(t)/G_1(t) \sim \beta^3 t^{3n-3}$$

for some constant $\beta \neq 0$ as $t \to \infty$. It follows from (3.5) that $(3n-3)+3 = 0$ and hence $n = 0$. Therefore we must have $n \leq 0$. However, recalling that $(m, n) \neq (0, 0)$ and $m \geq -n$, we have $m = 1$, $n = 0$ and so G may be written as

$$G(\zeta) = \psi(\zeta)\exp(c\zeta) \tag{3.6}$$

for some non-zero constant c. From this expression, we have $G^{(j)}(\zeta)/G(\zeta) \sim c^j$ as $\zeta \to \infty$, $j = 1, 2, 3$. Substituting these estimates into (3.2) once more, we deduce that $c^3 + q^3 = 0$, i.e., $c^3 + 27 = 0$.

Substituting now (3.6) into (3.2), and making use of $q = 3$, we get

$$\begin{aligned}\zeta^3\psi'''(\zeta) &+ \left(3c\zeta^3 + (9d+3)\zeta^2\right)\psi''(\zeta) + \left(3c^2\zeta^3 + 2c(9d+3)\zeta^2\right.\\ &\left.+(27d^2 + 9d + 1 - 9K)\zeta\right)\psi'(\zeta) + \left(c^2(9d+3)\zeta^2\right.\\ &\left.+c(27d^2 + 9d + 1 - 9K)\zeta + 27(d^3 - Kd)\right)\psi(\zeta) = 0. \end{aligned} \tag{3.7}$$

Substituting (3.3) into (3.7), making use of $c^3 + 27 = 0$, and collecting the coefficient of the highest term ζ^{s+2} in (3.7), we get

$$(3d + 1 + s)3c^2 c_s = 0,$$

hence $d = (-s-1)/3$. Likewise, the coefficient of the lowest term ζ^{-r} is

$$\begin{aligned}&\left((-r)(-r-1)(-r-2) + 3(3d+1)(-r)(-r-1)\right.\\ &\quad\left.+(27d^2 + 9d + 1 - 9K)(-r) + 27(d^3 - Kd)\right)c_{-r} = 0,\end{aligned}$$

and so we must have

$$r^3 - 9dr^2 + 27d^2r - 27d^3 = 9(r - 3d)K.$$

Therefore,

$$K = \frac{r^3 - 9dr^2 + 27d^2r - 27d^3}{9(r-3d)} = \frac{1}{9}(r-3d)^2 = \frac{1}{9}(r+s+1)^2. \tag{3.8}$$

Gathering the results from above, we deduce that f takes the form

$$f(z) = e^{\frac{-s-1}{3}z}\psi(e^{z/3})\exp(ce^{z/3}), \tag{3.9}$$

where $c^3 + 27 = 0$ and

$$\psi(\zeta) = \sum_{j=-r}^{s} d_j\zeta^j, \quad d_{-r}d_s \neq 0.$$

It remains to verify the determinant condition (2.3). Setting $n = r + s$ and assuming that $n > 0$, we rewrite f as

$$f(z) = \Psi(e^{-z/3}) \exp(ce^{z/3} - z/3), \tag{3.10}$$

where

$$\Psi(\zeta) = \sum_{j=0}^{n} e_j \zeta^j, \quad e_j = d_{s-j} \text{ and } e_0 e_n = d_s d_{-r} \neq 0. \tag{3.11}$$

Substituting (3.10) into (2.1), and making use of (3.8), we obtain

$$\zeta^3 \Psi^{(3)}(\zeta) + 3(2\zeta^2 - c\zeta)\Psi''(\zeta) + \left((6 - 2n - n^2)\zeta - 6c + 3c^2/\zeta\right) \Psi'(\zeta) \\ - (n^2 + 2n)(1 - c/\zeta)\Psi(\zeta) = 0. \tag{3.12}$$

Then we substitute (3.11) into (3.12), and this gives

$$\sum_{j=-1}^{n-1} B_j \zeta^j = 0, \tag{3.13}$$

where

$$\begin{aligned} B_{-1} &= (n^2 + 2n)ce_0 + 3c^2 e_1, \\ B_j &= (j - n)(j + n + 2)(j + 1)e_j \\ &\quad - \left\{3(j + 1)(j + 2) - 2n - n^2\right\} ce_{j+1} + 3c^2(j + 2)e_{j+2} \end{aligned}$$

for $0 \leq j \leq n - 2$, and

$$B_{n-1} = -(2n^2 + n)(e_{n-1} + ce_n).$$

Therefore, we must have $B_j = 0$ for all $j = -1, \ldots, n - 1$. Let now $\mathbf{B}$ denote the tridiagonal determinant whose non-zero diagonals are determined by

$$\begin{cases} b_{j,j-1} := a_{j,j-1}, & j = 1, \ldots, n, \\ b_{j,j} := ca_{j,j}, & j = 0, \ldots, n, \\ b_{j,j+1} := c^2 a_{j,j+1}, & j = 0, \ldots, n - 1, \end{cases}$$

see (2.4). Then the above result can be rewritten as a matrix equation

$$\mathbf{B} \times \begin{pmatrix} e_0 \\ \vdots \\ e_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3.14}$$

As $e_0 e_n \neq 0$, the determinant $\det(\mathbf{B})$ must be zero for (3.14) to admit a non-trivial solution. Since $\det(\mathbf{B}) = c^{n+1}\det(\mathbf{A})$, this proves the necessary part of Theorem 1.

To prove the converse, it is immediately seen that $f_i(z) = \exp(c_i e^{z/3} - z/3)$, where $c_i^3 + 27 = 0$, $i = 1, 2, 3$, are linearly independent zero-free solutions of (2.1) for $K = 1/9$. Hence, we may assume that $K = (n+1)^2/9$ where $n = r + s > 0$ and r, s are two given integers. We define $\Psi(\zeta)$ by (3.11) where the coefficients e_j, $0 \leq j \leq n$ are as given after (3.13), satisfying (3.14). Therefore, by reversing the argument above, the function defined by (3.11) solves the equation (3.12), provided $c^3 + 27 = 0$. In particular, the function (3.10) then solves (2.1) and it can be written as $f(z) = e^{\frac{-s-1}{3}z}\psi(e^{z/3})\exp(ce^{z/3})$, where $\psi(\zeta) = \sum_{j=-r}^{s} d_j \zeta^j$, which is precisely (2.5).

4. CONCLUDING REMARKS

In Theorem A, a solution of (1.1) with $\lambda(f) < \infty$ exists for each possible n. The situation in Theorem 1 is different. In fact, the tridiagonal determinant condition (2.3) seems to be equivalent to $n \neq 3k + 2$, $k = 0, 1, 2, \ldots$. This has been verified numerically up to $n = 100$. Unfortunately, we have been unable to find a general proof. As the referee has pointed out, the condition (2.3) in fact implies that $n \neq 3k + 2$, $k = 0, 1, 2, \ldots$, by applying a simple congruence argument on the formulae below. The converse conclusion seems to be a non-trivial problem. By elementary linear algebra, the tridiagonal matrix $\mathbf{A}$ in (2.3) can be expressed as the product of three matrices (β_{ij}), (α_{ij}) and (γ_{ij}), where (α_{ij}) is a diagonal matrix, while (β_{ij}) is a lower triangular matrix such that $\beta_{ii} = 1$ for all i, and (γ_{ij}) an upper triangular matrix such that $\gamma_{ii} = 1$ for all i. Therefore, it suffices to consider the vanishing of $\det(\alpha_{ij})$. Now, it is easy to see that $\alpha_{0,0} = a_{0,0}$ and that the recursion formula

$$\alpha_{j+1,j+1} = a_{j+1,j+1} - \frac{a_{j+1,j}a_{j,j+1}}{\alpha_{j,j}}, \quad j = 0, \ldots, n-1$$

holds. By (2.4), this results in a continued fractional representation

$$\alpha_{j+1,j+1} = A_{j,j} + \frac{B_{j,j}}{\alpha_{j,j}}, \quad j = 0, \ldots, n-1,$$

where

$$A_{j,j} = -3(j+1)(j+2) + 2n + n^2,$$
$$B_{j,j} = -3(j+1)\left(j(j+1)(j+2) - 2(j+1)n - (j+1)n^2\right),$$

for the diagonal elements of (α_{ij}). Hence, the determinant condition (2.3) reduces to the question whether at least one of the diagonal elements $\alpha_{j,j}$, $j = 0, \ldots, n$, vanishes.

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