Estimates on the growth of meromorphic solutions of linear differential equations with density conditions

Y.-M. CHIANG

Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Hong Kong, China
machiang@ust.hk

This paper is dedicated to the seventieth birthday of Ilpo Laine

Abstract. We give an alternative and simpler method for getting pointwise estimate of meromorphic solutions of homogeneous linear differential equations with coefficients meromorphic in a finite disk or in the open plane originally obtained by Hayman and the author. In particular, our estimates generally give better upper bounds for higher order derivatives of the meromorphic solutions under consideration, are valid, however, outside an exceptional set of finite logarithmic density. The estimates again show that the growth of meromorphic solutions with a positive deficiency at ∞ can be estimated in terms of initial conditions of the solution at or near the origin and the characteristic functions of the coefficients.

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1. INTRODUCTION AND MAIN RESULTS

We consider meromorphic solutions of the differential equation

$$y^{(n)}(z) + \sum_{\nu=0}^{n-1} f_\nu(z)\, y^{(\nu)}(z) = 0, \tag{1.1}$$

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where the coefficients $f_\nu(z)$ $0 \le \nu \le n-1$ are meromorphic in $\mathbb{C}$. We apply freely the classical Nevanlinna Theory notation throughout this paper [6], [10]. We mention that Heittokangas, Korhonen and Rättyä obtained sharper estimates for analytic solutions when the coefficients are analytic functions in [8]. They also considered non-homogeneous equations in [9]. For an up-to-date account on the growth of meromorphic solutions of algebraic differential equations with meromorphic coefficients, we refer the reader to Hayman [7].

Bank asked, if as in the case when the (1.1) admits an entire solution [1], an meromorphic solution of (1.1) can be estimated in terms of growth of Nevanlinna characteristics of the meromorphic coefficients alone. In general, he showed that this statement is not true in [2] by constructing an example for any given real-valued increasing function $\Phi(r) \uparrow +\infty$ on $(0, +\infty)$, then one can construct a first order linear differential equation with entire coefficient of zero Nevanlinna order such that the differential equation admits a meromorphic function f with $T(r,\, f) > \Phi(r)$ as $r \to +\infty$. One would need some extra terms to bound the growth of meromorphic solutions. An example of such a result is given by Bank and Laine.

Theorem 1.1 ([3]). *Suppose that the coefficients of (1.1) are arbitrary meromorphic functions and that $y(z)$ is a meromorphic solution of (1.1). If*

$$\Phi(r) = \max_{0\le i\le n} \Big(\log r, T(r, f_i)\Big),$$

then for any $\sigma > 1$, there exist positive constants c, c_1 and r_0, such that for $r \ge r_0$,

$$T(r,y) \le c\Big\{rN(\sigma r, y) + r^2 \exp\Big(c_1 J(\sigma r)\log(rJ(\sigma r)\Big)\Big\}, \qquad (1.2)$$

where

$$J(r) = \overline{N}(r, 1/y) + \Phi(r).$$

We note that one needs counting function $\overline{N}(r,\, 1/y)$ of *distinct zeros* in the $J(r)$ above as part of the upper bound in (1.2).

Since the equation (1.1) is linear, so one can deduce from the expression

$$\frac{f^{(n)}(z)}{f(z)} + \sum_{\nu=1}^{n-1} f_\nu(z)\frac{f^{(\nu)}(z)}{f(z)} + f_0(z) = 0,$$

that

$$\overline{N}(r,\, f) \le \sum_{\nu=0}^{n-1} \overline{N}(r,\, f_\nu) \le \sum_{\nu=0}^{n-1} T(r,\, f_\nu),$$

indicating that the coefficients can only bound the *distinct* poles of f. Indeed, the example constructed by Bank [2] mentioned above has poles of rapidly increasing multiplicities, that is $N(r,\, f)/\overline{N}(r,\, f)$ is unbounded. Hayman and the author [5] showed that one can still bound the growth of a meromorphic solution f of (1.1) in terms of the characteristic functions of coefficients alone if the solution f has *relatively few poles.* In particular, this means that $\delta(\infty,\, f) > 0$. This follows from the following result.

Theorem 1.2 ([5]). *Suppose that* $0 < \rho < r < R$ *and suppose that the coefficients* f_ν, $0 \le \nu \le n-1$ *of (1.1) are analytic on the path* $\Gamma = \Gamma(\theta_0,\, \rho,\, t)$ *defined by the segment*

$$\Gamma_1 : z = \tau e^{i\theta_0}, \quad \rho \le \tau \le t,$$

followed by the circle

$$\Gamma_2 : z = te^{i\theta}, \quad \theta_0 \le \theta \le \theta_0 + 2\pi.$$

We suppose that $y(z)$ *is a solution of the equation (1.1) and define*

$$K = 2\max\Big\{1,\, \sup_{0\le\nu\le n} |y^{(\nu)}(z_0)|\Big\},$$

where $z_0 = \rho e^{i\theta_0}$. *We also define*

$$\begin{aligned} C &= C(f_\nu,\, \rho,\, r,\, R) \\ &= (n+2)\exp\Big\{\frac{20R}{R-r}\sum_{\nu=0}^{n} T(R,\, f_\nu) + \Big(\sum_{\nu=0}^{n} p_\nu\Big)\log\Big(\frac{R}{\rho}\Big)\Big\}, \end{aligned}$$

where p_ν *is the multiplicity of the order of pole of* f_ν *at* $z = 0$. *Then we have for* $|z| = t$, *where* t *is some number such that* $r < t < (3r+R)/4$,

$$|y^{(\nu)}(z)| < KC^{\nu}e^{(2\pi+1)CR}, \quad 0 \le \nu \le n.$$

One can easily deduce when $R = +\infty$, and for a transcendental meromorphic f with $\delta(\infty,\, f) > 0$, then for $0 < \varepsilon < \delta$, we have

$$T(r,\, y) \le \Big(\frac{1}{\delta - \varepsilon}\Big)\,(2\pi + 1)\, RC.$$

The main purpose of this paper is to use a different method to give a shorter proof of a slightly different statement to Theorem 1.2 that is valid outside some exceptional sets. On the other hand, the original Theorem 1.2 can deal with non-homogeneous (1.1), while our alternative can only deal with the (1.1). We prove

Theorem 1.3. *Let y be a meromorphic solution to the differential equation (1.1) with meromorphic coefficients f_ν, $\nu = 0, \cdots, n-1$ in $|z| = r < R \le +\infty$ such that f_ν has a pole of order $q_\nu \ge 0$ $(0 \le \nu \le n-1)$. Given a constant $C > 1$ and $0 < \eta < 3\,e/2$ and $r = |z|$ is outside a union of discs centred at the poles of y such that the sum of radii is not greater than $4\eta R$, then there is a $B = B(C) > 1$ and a path $\Omega = \Omega(\theta,\, \rho,\, r)$ consists of the line segment*

$$\Omega_1 : z = \tau e^{i\theta_0}, \quad 0 \le \rho \le \tau \le r$$

followed by the circle

$$\Omega_2 : z = re^{i\theta}, \quad \theta_0 \le \theta < \theta_0 + 2\pi,$$

on which the coefficients f_ν are analytic and we have, for z on Ω,

$$\sum_{j=0}^{n-1} |y^{(j)}(z)| \le K_1\, e^{(2\pi+1)\, D\, r} \le K_1\, e^{(2\pi+1)\, D\, R},$$

where

$$K_1 = \sum_{j=0}^{n-1} |y^{(j)}(z_0)|, \quad z_0 = \rho e^{i\theta_0} \tag{1.3}$$

and

$$\begin{aligned} D &:= D(f_\nu,\, \rho,\, r,\, R;\, \eta,\, B,\, C) \\ &= n\Bigg\{1 + \bigg(\frac{R^{H(\eta)(\frac{R+2\,er}{R-2\,er})}}{r}\bigg)^q \\ &\qquad\qquad \cdot \exp\Big[B\,(1 + H(\eta))\Big(\frac{R + 2\,er}{R - 2\,er}\Big)\, T(CR)\Big]\Bigg\} \end{aligned} \tag{1.4}$$

and where

$$T(r)=\max_{0\le\nu\le n-1}T(r,\,f_\nu),\quad q=\max_{0\le\nu\le n-1}q_\nu,\quad H(\eta)=2+\log\frac{3e}{2\eta}.$$

For any r', we choose r outside a union of discs centred at the zeros of y such that the sum of radii is not greater than $4\eta R$ such that $r'<r<R$ as described in the Theorem 1.3, we have

$$\begin{aligned}T(r',\,y)&\le N(r,\,y)+m(r,\,y)\\&\le N(r,\,y)+(2\pi+1)\,R\,D(f_\nu,\,\rho,\,r,\,R;\,\eta,\,B,\,C).\end{aligned}$$

This improves upon Bank and Laine's estimate mentioned above. Suppose that $\delta(\infty,\,y)>0$, we choose r outside a union of discs centred at the zeros of y such that the sum of radii is not greater than $4\eta R$ and sufficiently large such that $N(r,\,y)<(1-\delta+\varepsilon/2)\,T(r,\,y)$. Without loss of generality, we may also assume that r is so chosen such that $|\log K_1|<\frac{1}{2}\varepsilon T(r,\,y)$ so that $T(r,\,y)<(1-\delta+\varepsilon)\,T(r,\,y)+(2\pi+1)R\,D$. We can easily deduce

$$T(r',\,y)\le\Big(\frac{1}{\delta(\infty,\,y)-\varepsilon}\Big)\,(2\pi+1)R\,D.$$

Theorem 1.4. *Let $y(z)$ be a meromorphic solution to equation (1.1), and we choose $0<\eta<(1+\log 2)/(16\,e^{5/2})<1$. Then there is a constant $B>1$ such that for every $\varepsilon>0$ be given, there is a $r(\varepsilon)>0$ we have*

$$\begin{aligned}\log m(r,\,y^{(j)})&\le 5B\big(1+H(\eta)\big)\,T(3\,e^2r)\\&\qquad+\big[(5H(\eta)-1)q+1+\varepsilon\big]\,\log r\end{aligned}\tag{1.5}$$

$j=0,1,\ldots,n-1$ holds for all $r>r(\varepsilon)$ except perhaps for a set of positive logarithmic density $16\eta e^{5/2}/(1+\log 2)$.

We should compare this result with the following density-type result also obtained previously in [5]:

Theorem 1.5. *Let $y(z)$ be a meromorphic solution of (1.1) such that the f_ν are not all constant. Then we have*

$$\log m(r,\,y)<\Big(\sum_{\nu=0}^{n-1}T(r,\,f_\nu)\Big)\bigg[(\log r)\log\Big(\sum_{\nu=0}^{n-1}T(r,\,f_\nu)\Big)\bigg]^{\sigma},$$

where $\sigma > 1$ is a constant, to hold outside an exceptional set of finite logarithmic measure.

2. PRELIMINARIES

Let us write $\mathbf{y}(z) = (y_0, \cdots, y_{n-1})^T$ where $y_j(z)$, $j = 0, \cdots, n-1$ are complex functions of z. We define $\|\mathbf{y}\| = \sum_{j=0}^{n-1} |y_j|$. Suppose further that $\mathbf{A} = (a_{ij}(z))$ is a square matrix then we define $\|\mathbf{A}\| = \sum_{i,j} |a_{ij}|$. We note that

$$\left\| \int \mathbf{A}\, dt \right\| \le \int \|\mathbf{A}\|\ |dt|$$

(see e.g., [4, pp. 1–4]).

Lemma 2.1 ([11, pp. 21–22]). *Let $R > 0$ and $f(z)$ be analytic in $|z| \le 2\,e\,R$ with $f(0) = 1$, and let η be an arbitrary positive constant not exceeding $3\,e/2$. Then we have*

$$\log|f(z)| > -H(\eta)\log M(2\,e\,R,\ f) \tag{2.1}$$

for all z in $|z| \le R$ but outside a union of disks centred at the zeros of f such that the sum of radii is not greater than $4\eta\,R$, where

$$H(\eta) = 2 + \log\frac{3\,e}{2\,\eta}.$$

We also need the following quotient representation of meromorphic functions due to Miles [12] and Rubel [13, Chapter 14].

Lemma 2.2 ([12]). *Let f be a meromorphic function in the plane, and let $C > 1$ be a given constant, then there exist entire functions f_1 and f_2, and a constant $B = B(C) > 0$ such that*

$$f(z) = \frac{f_1(z)}{f_2(z)}, \qquad \text{and} \qquad T(r,\ f_j) \le B\,T(C\,r,\ f),$$

$j = 1, 2$ and $r > 0$. Here both the constants B and C are absolute constants, i.e., they are independent of the function f.

3. PROOF OF THEOREM 1.3

We state and prove our main lemma that leads to the proof of the Theorem 1.3.

Lemma 3.1. *Let y be a meromorphic solution to the differential equation (1.1) with meromorphic coefficients f_ν, $\nu = 0, \cdots, n-1$ in $|z| = r < R \leq +\infty$. Suppose that the coefficients f_ν, $\nu = 0, \cdots, n-1$ are analytic on the path $\Omega = \Omega(\theta, \rho, r)$ as defined in the Theorem 1.3. Suppose $z_0 = \rho e^{i\theta_0}$, then for all z on Ω,*

$$\sum_{j=0}^{n-1} |y^{(j)}(z)| \leq K_1 \exp\left[\left(\max_{\Omega} \sum_{\nu=0}^{n-1} (|f_\nu(z)| + 1)\right) (2\pi + 1)\, r\right],$$

where K_1 is given in (1.3).

Proof. It is well-known that equation (1.1) can be written in the matrix form

$$\mathbf{F}'(z) = \mathbf{A}(z)\, \mathbf{F}(z), \tag{3.1}$$

where $\mathbf{F} = (y, y', \cdots, y^{(n-1)})^T$, and

$$\mathbf{A}(z) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 \\ -f_0 & -f_1 & \cdots & \cdots & -f_{n-1} \end{pmatrix}.$$

A solution to the above matrix equation (3.1) is given by

$$\mathbf{F}(z) = \mathbf{F}(z_0) + \int_{z_0}^{z} \mathbf{A}(t)\, \mathbf{F}(t)\, dt. \tag{3.2}$$

We now apply Gronwall's inequality [4, pp. 35–36] to (3.1) to obtain

$$\begin{aligned} \|\mathbf{F}(z)\| &\leq \|\mathbf{F}(z_0)\| + \int_{z_0}^{z} \|\mathbf{A}(t)\|\, \|\mathbf{F}(t)\|\, |dt| \\ &\leq \|\mathbf{F}(z_0)\| \exp\left(\int_{z_0}^{z} \|\mathbf{A}(t)\|\, |dt|\right) \\ &\leq \|\mathbf{F}(z_0)\| \exp\left[\left(\max_{\Omega} \sum_{\nu=0}^{n-1} |f_\nu(z)| + (n-1)\right) (2\pi + 1) r\right] \\ &< \|\mathbf{F}(z_0)\| \exp\left[\left(\max_{\Omega} \sum_{\nu=0}^{n-1} (|f_\nu(z)| + 1)\right) (2\pi + 1) r\right], \end{aligned} \tag{3.3}$$

where we have parameterized the path Ω with respect to arc length. Clearly the length of Ω is $(2\pi+1)\,r$ at most. This proves Lemma 3.1. □

We are ready to prove the Theorem 1.3, which is a direct application of the Lemma 3.1 and the two lemmas stated in §2 .

Proof of the Theorem 1.3. Let $C>1$ be given. Then Miles' result in Lemma 2.2 asserts that we can choose a $B>0$ such that we can write the coefficients in $f_\nu = f_{\nu,1}/f_{\nu,2}$ from (1.1) with

$$T(r,\, f_{\nu,j}) \le B\,T(C\,r,\, f_\nu), \qquad j=1,\,2; \quad 0\le \nu \le n-1$$

for $r>0$. We first assume that $f_{\nu,2}(0)\ne 0$ for $0\le\nu\le n-1$, then it is easy to see that we may assume that $f_{\nu,2}(0)=1$ after dividing the numerator and the denominator by a suitable constant. Lemmas 2.1 and 2.2 imply that

$$\begin{aligned}
\log|f_\nu| &= \log|f_{\nu,1}| + \log|f_{\nu,2}|^{-1}\\
&\le \log M(r,\, f_{\nu,1}) + H(\eta)\log M(2\,e\,r,\, f_{\nu,2})\\
&\le \log^+ M(r,\, f_{\nu,1}) + H(\eta)\log^+ M(2\,e\,r,\, f_{\nu,2})\\
&\le \Big(\frac{R+r}{R-r}\Big)\,T(R,\, f_{\nu,1}) + H(\eta)\,\Big(\frac{R+2\,e\,r}{R-2\,e\,r}\Big)\,T(R,\, f_{\nu,2})\\
&\le B\,\Big(\frac{R+r}{R-r}\Big)\,T(CR,\, f_\nu) \qquad\qquad (3.4)\\
&\qquad + BH(\eta)\,\Big(\frac{R+2\,e\,r}{R-2\,e\,r}\Big)\,T(CR,\, f_\nu)\\
&\le B\big(1+H(\eta)\big)\Big(\frac{R+2\,e\,r}{R-2\,e\,r}\Big)\,T(CR, f_\nu)\\
&\le B\big(1+H(\eta)\big)\,\Big(\frac{R+2\,e\,r}{R-2\,e\,r}\Big)\,T(CR),
\end{aligned}$$

where

$$T(r) = \max_{0\le\nu\le n-1} T(r,\, f_\nu).$$

If, however, $f_{\nu,2}$ has a zero of order q_ν at $z=0$, we consider

$$f_\nu = \frac{f_{\nu,1}/z^{q_\nu}}{f_{\nu,2}/z^{q_\nu}}$$

in $0<\rho\le|z|$ in which the $f_{\nu,2}/z^{q_\nu}$ is clearly still analytic and not zero at the origin. We deduce from Lemmas 2.1 and 2.2 again that for $0 \le \nu \le n-1$

$$
\begin{aligned}
\log|f_\nu| &= \log|f_{\nu,1}| - q_\nu \log r + \log|f_{\nu,2}/z^{q_\nu}|^{-1}\\
&\le \log M(r,\, f_{\nu,1}) - q_\nu \log r + H(\eta) \log M\big(2\,e\,r,\, f_{\nu,2}/z^{q_\nu}\big)\\
&\le \log^+ M(r,\, f_{\nu,1}) - q_\nu \log r + H(\eta) \log^+ M\big(2\,e\,r,\, f_{\nu,2}/z^{q_\nu}\big)\\
&\le \Big(\frac{R+r}{R-r}\Big) T(R,\, f_{\nu,1}) - q_\nu \log r\\
&\qquad + H(\eta)\Big(\frac{R+2\,e\,r}{R-2\,e\,r}\Big) T(R,\, f_{\nu,2}/z^{q_\nu})\\
&\le \Big(\frac{R+r}{R-r}\Big) T(R,\, f_{\nu,1}) - q_\nu \log r\\
&\qquad + H(\eta)\Big(\frac{R+2\,e\,r}{R-2\,e\,r}\Big)\big(T(R,\, f_{\nu,2}) + q_\nu \log R\big) \qquad (3.5)\\
&\le B\Big(\frac{R+r}{R-r}\Big) T(CR,\, f_\nu) - q_\nu \log r\\
&\qquad + H(\eta)\Big(\frac{R+2\,e\,r}{R-2\,e\,r}\Big)\big(BT(CR,\, f_\nu) + q_\nu \log R\big)\\
&< B\big(1+H(\eta)\big)\Big(\frac{R+2\,e\,r}{R-2\,e\,r}\Big) T(CR,\, f_\nu) + q_\nu \log \frac{R^{H(\eta)(\frac{R+2\,e\,r}{R-2\,e\,r})}}{r}.
\end{aligned}
$$

Applying (2.1) to (3.4) or (3.5) ($0 \le \nu \le n-1$) and substituting them into (3.3) completes the proof. $\square$

4. PROOF OF THEOREM 1.4

We are now ready to prove Theorem 1.4. We choose $C = e$ in Theorem 1.3. Let $\alpha = 2\,e$. We define annuli by

$$\Lambda_j = \big\{z : \alpha^j \le |z| \le \alpha^{j+3/2}\big\}, \qquad j = 1,\, 2,\, \cdots .$$

We take $R = 3\,er$ in (1.4) in Theorem 1.3 and suppose z belongs to $\Omega \cap \Lambda_j$ where Ω is defined in Theorem 1.3. That is, we have, for each $0 \le j \le n-1$,

$$
\begin{aligned}
m(r,\, y^{(j)}) &\le \log K_1 + (2\pi+1)\, r\, n\Big\{1 + (3e)^{5H(\eta)q} r^{(5H(\eta)-1)q}\\
&\qquad\qquad \cdot \exp\Big[5B\big(1+H(\eta)\big)\, T(3\,e^2 r)\Big]\Big\}\\
&= \log K_1 + (2\pi+1)\, r\, D(f_\nu,\, \rho,\, r,\, 3er,\, \eta,\, B,\, e),
\end{aligned}
$$

where D is given in (1.4). Taking logarithm on both sides of the inequality once more yield the required estimate (1.5).

It remains to verify the size of the exceptional set of r, which follows from

Lemma 4.1. *Let η and $H(\eta)$ be defined in Lemma 2.1 and Theorem 1.3 respectively. Then the estimate (1.5) for a meromorphic solution $y(z)$ is valid for all r sufficiently large except on a set of positive logarithmic density at most $16\eta e^{5/2}/(1+\log 2)$.*

Proof. Let E_j be the union of exceptional circles lying in Λ_j and

$$E(r) = [1,\, r) \cap \big(\cup_{j=1}^{\infty} E_j\big).$$

Let $q = [(\log r)/(\log\alpha)]$, then Lemma 2.1 gives

$$\begin{aligned}\int_{E(r)} \frac{dt}{t} &\le \sum_{j=1}^{q} \int_{E_j} \frac{dt}{t} \le \sum_{j=1}^{q} \frac{4\eta(2\,er)}{\alpha^j} \le \sum_{j=1}^{q} \frac{4\eta(2e\alpha^{j+3/2})}{\alpha^j} \\ &\le \frac{\log r}{\log\alpha}\big(8\eta e\alpha^{3/2}\big) \le \Big(\frac{16\,\eta e^{5/2}}{1+\log 2}\Big)\log r.\end{aligned}$$

Thus

$$\limsup_{r\to\infty} \frac{1}{\log r}\int_{E(r)} \frac{dt}{t} \le \frac{16\eta e^{5/2}}{1+\log 2} < 1.$$

This completes the proof of the Lemma. ☐

This completes the proof of the theorem.

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