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Oscillation Results of Certain Higher Order Linear* Differential Equations with Periodic Coefficients in the Complex Plane

Yik-Man Chiang

Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong

and

Shupei Wang

Department of Mathematics, University of New Orleans, New Orleans, Louisiana 70148

Baesch (*Results in Math*. **29**, 1996, 42–55) has given a characterization of homogeneous linear differential equations with certain analytic periodic coefficients which admits a solution with finite exponent of convergence. However, her method seems too general and in most cases too complicated for applications. We give, in this paper, a direct approach to the problem and obtain several such characterizations which do not seem to follow from those of Baesch. In particular, an explicit, necessary and sufficient condition to the problem is given for certain third order equations. The results again do not seem to follow from those of Baesch. Our method is based on that of Y. M. Chiang, I. Laine, and S. Wang (*Complex Variables*, **34** (1997), 25–34) which in turn depends on basic representations of solutions given by Bank, Laine, and Langley. © 1997 Academic Press

1. INTRODUCTION

In this paper we consider linear differential equations of the form

$$f^{(n)} + K_{n-2} f^{(n-2)} + \cdots + K_1 f' + K(e^z) f = 0, \qquad n \geq 2, \quad (1.1)$$

where $K_{n-2}, \ldots, K_1$ are constants, and $K(\zeta)$ is a non-constant rational function analytic on $0 < |\zeta| < +\infty$, taking the form

$$K(\zeta) = \sum_{i=p}^{m} K_{0,i} \zeta^i, \qquad m \geq p, K_{0,m} K_{0,p} \neq 0, \quad (1.2)$$

and our notation is that of [10].

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For equations of the form (1.1), it is well known that every non-trivial solution of (1.1) must be an entire function which has an infinite order of growth, and that for any non-zero constant a and any solution $f \not\equiv 0$, it holds $\lambda(f-a) = +\infty$. Here $\lambda(f-a)$ stands for the exponent of convergence of the zero sequence of $f-a$. The value zero, however, does play an exceptional role in the sense that there exists an equation of the form (1.1) which possesses a non-trivial solution with $\lambda(f) < +\infty$. See the Examples in Sections 3 and 6. So, two natural questions arise:

(I) Characterize those equations of the form (1.1) which possess a non-trivial solution satisfying $\lambda(f) < +\infty$.

(II) Find the representations of those solutions of (1.1) which satisfy $\lambda(f) < +\infty$, if such solutions exist.

In fact representations of any solution to (1.1) with finite exponent of convergence were obtained in [4, 6]; see Theorem B below. In [3], Bank developed a method which allows one to test any equation of the form (1.1) with $n = 2$ for the existence of a solution satisfying $\lambda(f) < +\infty$, and to specify any such solution (based on Theorem B). This method has recently been generalized to higher order equations of the form (1.1); see Baesch [1]. Although the above two questions seem to be completely solved in [1], the method developed there seems to be too complicated for most equations of the form (1.1), since it involves a procedure of determining whether any one of certain n^2 auxiliary differential equations with polynomial coefficients will admit a polynomial solution. Thus an alternative approach is desirable.

It is the purpose of this paper to reconsider the two questions (I) and (II) above. However, our approach is based on the method in [8], which is different from those of Bank and Baesch; see also [5]. As a consequence, we are able to obtain new higher order results (Theorems 2.1 and 2.2) which do not seem to follow from [1]. In particular, we obtain a complete characterization to a special case of (1.1) when $k = 3$ (Theorems 3.1 and 3.2).

In [8], a complete characterization was obtained to certain equations of the form (1.1) with $n = 3$ which possess a non-trivial solution satisfying $\lambda(f) < +\infty$. Indeed a necessary and sufficient condition is given in terms of the vanishing of a determinant (Theorem A and Theorem 3.1) and the result does not seem to follow from [1]. The following result was proved in [8].

THEOREM A. *Let $K \in \mathbf{C}$, and suppose that*

$$f''' - Kf' - e^z f = 0 \tag{1.3}$$

admits a non-trivial solution f such that $\lambda(f) < +\infty$. Then there exists an integer $k \geq 0$, such that

$$K = \frac{(k+1)^2}{9}. \tag{1.4}$$

Moreover, f admits one of the following representations,

$$f_i(z) = e^{((-k-1)/3)z}\psi(e^{z/3})\exp\left(c_i e^{z/3}\right), \tag{1.5}$$

where $c_i^3 + 27 = 0$, $i = 1, 2, 3$, and

$$\psi(\zeta) = \sum_{j=0}^{k} d_j \zeta^j, \qquad d_0 d_k \neq 0. \tag{1.6}$$

It was also proved in [8] that the integer k in (1.4) exists if and only if a certain determinant condition holds, and this in turn gives an equivalent criterion on the existence of (1.6) and hence on the existence of (1.5). Such a determinant condition is conjectured to be equivalent to k being a certain integer sequence, which again does not seem to follow from [1]. A more general determinant related to a general third order equation will be given in Theorem 3.1 below.

The proof of Theorem A is based on Theorem B below, due to Bank and Langley [6].

THEOREM B. *Suppose that Eq.* (1.1) *admits a non-trivial solution f satisfying $\lambda(f) < +\infty$. Then there exist an integer q with $1 \leq q \leq n$, a constant d, and rational functions $R(\zeta)$ and $S(\zeta)$, analytic on $0 < |\zeta| < +\infty$, such that*

$$f(z) = e^{dz}R(e^{z/q})\exp\left(S(e^{z/q})\right). \tag{1.7}$$

In fact, Bank and Langley proved in [6] that Theorem B still holds under a weaker hypothesis that $\log^+ N(r, 1/f) = o(r)$ as $r \to +\infty$. Notice that $\log^+ N(r, 1/f) = o(r)$ as $r \to +\infty$ implies $\lambda(f) < +\infty$. We shall, however, prefer to use the exponent of convergence $\lambda(f)$ as the measure of the density of zeros of f in the description of our results below; see also p. 8 of [11]. Since the proofs of our results are based on Theorem B, most of the theorems obtained in this paper are therefore still valid under the weaker hypothesis that $\log^+ N(r, 1/f) = o(r)$ as $r \to +\infty$.

The main objective of the present paper is to show that the method developed in [8] for Eq. (1.3) can be generalized to higher order equations of the form (1.1). Using this method we shall give a more precise description on the forms of the rational functions $S(\zeta)$ and $K(\zeta)$. We show that there is a strong relation between n—the order of the Eq. (1.1), and the forms of $R(\zeta)$ and $S(\zeta)$ in (1.7). One of our results generalizes a result of Gao [9] in the second order case and a result of Bank and Langley [6] in the higher order case. In Theorem 2.2 we consider $K(\zeta) = \zeta + K_0$ in (1.1), and in this case, we are able to obtain explicit forms for those solutions of (1.1) which satisfy $\lambda(f) < +\infty$ and a generalization of (1.4). These results will be stated in Section 2. We then discuss in Section 3 the third order equation of the form $f''' + K_1 f' + (e^z + K_0)f = 0$, where K_1 and K_0 are constants. A complete characterization of such equations which admit a solution satisfying $\lambda(f) < +\infty$ will be given in Theorem 3.1. It generalizes Theorem A. In Theorem 3.2 we show that all non-trivial solutions of this equation must satisfy $\lambda(f) = +\infty$ provided $K_1 \geq 0$. Examples will be given after Theorem 3.2 to show that this condition is necessary.

The authors dedicate this paper to the memory of Lee A. Rubel.

2. RESULTS FOR HIGHER ORDER EQUATIONS

Our first result in this paper is a refinement of Theorem B in Section 1.

THEOREM 2.1. *Suppose* $f \not\equiv 0$ *is a solution of Eq.* (1.1) *satisfying* $\lambda(f) < +\infty$. *Then*

$$f(z) = e^{dz} R(e^{z/q}) \exp\big(S(e^{z/q})\big), \tag{2.1}$$

where d is a constant, q is an integer, with $1 \leq q \leq n$, and $S(\zeta)$ and $R(\zeta)$ are rational functions taking the forms

$$S(\zeta) = \sum_t^s a_j \zeta^j, \qquad s \geq t,\ a_s a_t \neq 0, \tag{2.2}$$

and

$$R(\zeta) = \sum_0^k b_j \zeta^j, \qquad k \geq 0,\ b_k b_0 \neq 0. \tag{2.3}$$

Moreover, recalling m and p as in (1.2), *we have*

(A) *If $m \geq 1$, and m and n are relatively prime, then $q = n$, $p \geq 0$, $s = m$, and $t \geq 0$. Let $p' \geq 1$ be the smallest integer such that $K_{0,p'} \neq 0$ in* (1.2). *Then in addition to $t \geq 0$ we have $t \leq np'$ provided $k < np'$.*

(B) *If $p \leq -1$, and p and n are relatively prime, then $q = n$, $m \leq 0$, $t = p$, and $s \leq 0$. Let $m' \leq -1$ be the largest integer such that $K_{0,m'} \neq 0$ in* (1.2). *Then in addition to $s \leq 0$ we have $s \geq m'n$ provided $k > m'n$.*

(C) *Suppose* $m \geq 1$ *and which is divisible by* n. *Then* q *must also divide* $s \geq 1$. *Similarly, if* $p \leq -1$ *and is divisible by* n, *then* q *must also divide* $t \leq -1$. *If, however,* n *is divisible by* m, *then* s *divides* q.

Theorem 2.1 generalizes a theorem in [9] where a similar result was obtained for second order equations of the form (1.1). We remark that if $K(\zeta)$ in (1.2) contains only a single non-constant term $K_m \zeta^m$ or $K_p \zeta^p$, then only the corresponding parts of (A) and (B) of the above theorem are valid. We also note that in part (C) above it is possible that both K_m $(m \geq 1)$ and K_p $(p \leq -1)$ are non-zero while both m and p are divisible by n. We shall give examples concerning these cases in Section 6.

THEOREM 2.2. *Let* $n \geq 2$ *be an integer, and let* $K_{n-2}, \ldots, K_1$, K_0 *be constants. Suppose the following equation*

$$f^{(n)} + K_{n-2} f^{(n-2)} + \cdots + K_1 f' + (e^z + K_0) f = 0 \tag{2.4}$$

admits a non-trivial solution f *satisfing* $\lambda(f) < +\infty$. *Then there exists a non-negative integer* k *so that*

$$d^n + K_{n-2} d^{n-2} + \cdots + K_1 d + K_0 = 0, \tag{2.5}$$

where

$$d = -\frac{2k + n - 1}{2n}. \tag{2.6}$$

Moreover, in this case, the solution f *takes the form*

$$f(z) = e^{dz} R(e^{z/n}) \exp(\alpha e^{z/n}), \tag{2.7}$$

where d *is the constant in* (2.6), α *is a constant satisfying* $\alpha^n + n^n = 0$, *and* $R(\zeta)$ *is a polynomial in* ζ, *having the form*

$$R(\zeta) = \sum_{i=0}^{k} b_o \zeta^i, \qquad b_0 b_k \neq 0. \tag{2.8}$$

We remark that in the case when $k = 0$ in (2.8), $R(\zeta)$ is understood to be a non-zero constant.

As a corollary to Theorem 2.2, we obtain the following result.

COROLLARY 2.3. *Let* K_0 *be a constant and* $n \geq 2$ *be an integer. Suppose that the equation*

$$f^{(n)} + (e^z + K_0) f = 0 \tag{2.9}$$

admits a non-trivial solution f such that $\lambda(f) < +\infty$. *Then there exists an integer* $k \geq 0$ *such that*

$$K_0 = -\left(\frac{2k + n - 1}{2n}\right)^n. \tag{2.10}$$

It follows from (2.10) that if K_0 is non-real then any non-trivial solution to (2.9) must have $\lambda(f) = +\infty$. Corollary 2.3 generalizes Theorem 2 in [5] where the same result was obtained for equations of the form (2.9) with $n = 2$. We also remark that equations of the form (2.9) with K_0 given by (2.10) cannot be treated by the method developed in [2]. Although Corollary 2.3 gives a necessary condition for (2.9) to admit a non-trivial solution with $\lambda(f) < +\infty$, it appears to the authors that no non-trivial solution to (2.9) could have finite exponent of convergence when $n \geq 3$. In other words, we conjecture that every non-trivial solution of Eq. (2.9) must satisfy $\lambda(f) = +\infty$, as long as $n \geq 3$. We will verify this conjecture for $n = 3$; see Corollary 3.3 in the next section. We remark that the argument used in the proof for $n = 3$ does not seem to apply to the cases when $n \geq 4$. Actually, when $n = 3$ in (2.4), a result which is more precise than Theorem 2.2 can be obtained; see Theorem 3.1 in the next section.

3. RESULTS FOR THIRD ORDER EQUATIONS

Our main result in this section is to give a necessary and sufficient condition to third order equations of the form (2.4) which possess a non-trivial solution with $\lambda(f) < +\infty$. This result generalizes Theorem A in Section 1.

THEOREM 3.1. *Let* K_1 *and* K_0 *be two constants, and suppose that*

$$f''' + K_1 f' + (e^z + K_0) f = 0 \tag{3.1}$$

admits a non-trivial solution f with $\lambda(f) < +\infty$. *Then there exists an integer* $k \geq 0$ *such that*

$$K_1 = -\frac{(k+1)^2}{9} + \frac{3}{k+1} K_0, \tag{3.2}$$

and

$$\det \mathbf{A}_k(K_1) = 0. \tag{3.3}$$

Here $\mathbf{A}_k(K_1)$ *is a* $(k+1)\times(k+1)$ *matrix, defined by*

$$\begin{pmatrix} Y_1 & Z_1 & 0 & & & \cdots & 0 \\ X_2 & Y_2 & Z_2 & 0 & & \cdots & 0 \\ 0 & X_3 & Y_3 & Z_3 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & X_k & Y_k & Z_k \\ 0 & \cdots & \cdots & 0 & 0 & X_{k+1} & Y_{k+1} \end{pmatrix}, \tag{3.4}$$

where

$$X_j = (j-1)^3 - (k+1)^3 + 9K_1(j-k-2) \tag{3.5}$$

$$Y_j = -3j(j-1) - 1 - 9K_1 \tag{3.6}$$

$$Z_j = 3j \tag{3.7}$$

for $1 \le j \le k+1$. *Moreover, f admits one of the representations as in* (1.5).

Conversely, suppose K_1, K_0 *and a non-negative integer k satisfy* (3.2) *and* (3.3). *Then there exist three linearly independent solutions of* (3.1) *of the forms* (1.5), *each with finite exponent of convergence of its zero sequence*.

Clearly, Theorem 3.1 characterizes completely those equations of the form (3.1) which admit a non-trivial solution satisfying $\lambda(f) < +\infty$. A simple argument on (3.3) leads us to the following result.

THEOREM 3.2. *Let* $K_1 \ge 0$ *and* K_0 *be two constants*. *Then every non-trivial solution f of* (3.1) *satisfies* $\lambda(f) = +\infty$.

We remark that the condition $K_1 \ge 0$ in Theorem 3.2 is necessary. Let α be a constant satisfying $\alpha^3 + 27 = 0$. Then the function $f(z) = (1 - \alpha e^{z/3})\exp(\alpha e^{z/3} - \frac{2}{3}z)$, which satisfies $\lambda(f) < +\infty$, is a solution of $f''' - \frac{4}{9}f' + e^z f = 0$. On the other hand, the function $g(z) = (1 - \frac{1}{2}\alpha e^{z/3})\exp(\alpha e^{z/3} - \frac{2}{3}z)$ solves the equation $f''' - \frac{7}{9}f' + (e^z - \frac{2}{9})f = 0$. Notice that $\lambda(g) < +\infty$.

The following result follows immediately from Theorem 3.2, which verifies the conjecture stated after Corollary 2.3 for $n = 3$.

COROLLARY 3.3. *Let* K_0 *be a constant*. *Then every non-trivial solution f of the equation*

$$f''' + (e^z + K_0)f = 0$$

satisfies $\lambda(f) = +\infty$.

4. PROOFS OF THEOREMS 2.1 AND 2.2

Proof of Theorem 2.1. Suppose that Eq. (1.1) admits a non-trivial solution f with $\lambda(f) < \infty$. Then it follows from Theorem B that f takes the form (1.7). Since $R(\zeta)$ and $S(\zeta)$ in (1.7) are analytic rational functions on $0 < |\zeta| < +\infty$, we may assume that

$$S(\zeta) = \sum_{t}^{s} a_j \zeta^j, \qquad s \geq t,\, a_s a_t \neq 0, \tag{4.1}$$

and

$$R(\zeta) = \sum_{0}^{k} b_j \zeta^j, \qquad k \geq 0,\, b_k b_0 \neq 0. \tag{4.2}$$

Here the factor e^{dz} in (1.7) makes the form (4.2) possible. Set

$$G(\zeta) = R(\zeta)\exp(S(\zeta)). \tag{4.3}$$

Then, from (1.7) we have

$$f(z) = \zeta^{dq} G(\zeta), \qquad \zeta = e^{z/q}. \tag{4.4}$$

Differentiating (4.4) yields

$$f^{(v)}(z) = \sum_{j=0}^{v} B(v,j)\zeta^{dq+j} G^{(j)}(\zeta), \qquad \zeta = e^{z/q}, \tag{4.5}$$

where $B(v,0) = d^v$, $B(v,v) = 1/q^v$, and

$$B(v+1,j) = \frac{1}{q}((dq+j)B(v,j) + B(v,j-1)), \qquad 0 \leq j \leq v+1. \tag{4.6}$$

Here we set $B(v,-1) = B(v,v+1) = 0$ for later use. Substituting (4.5) into Eq. (1.1) we obtain

$$\sum_{j=1}^{n} c_j \zeta^j G^{(j)}(\zeta) + (c_0 + K(\zeta^q))G(\zeta) = 0, \tag{4.7}$$

where $c_n = B(n,n) = 1/q^n$, $c_{n-1} = B(n,n-1)$, and

$$c_j = B(n,j) + \sum_{i=j}^{n-2} K_i B(i,j), \qquad 1 \leq j \leq n-2,$$

and

$$c_0 = B(n,0) + \sum_{i=1}^{n-2} K_i B(i,0).$$

We first prove Part (A). Suppose that m (see (1.2)) and n are relatively prime, and that $m \geq 1$. Since m is not divisible by n, it follows immediately from [6, Theorem 3] that $K_{0,i} = 0$ for all $i \leq -1$. Hence, $p \geq 0$. On the other hand, we get from (4.3), (4.1), and (4.2) that

$$\begin{aligned} \frac{G'(\zeta)}{G(\zeta)} &= S'(\zeta) + \frac{R'(\zeta)}{R(\zeta)} \\ &= s a_s \zeta^{s-1} + o\left(\zeta^{s-1}\right), \end{aligned}$$

as $\zeta \to \infty$. Hence, it follows from induction that for $1 \leq j \leq n$,

$$\frac{G^{(j)}(\zeta)}{G(\zeta)} = (s a_s)^j \zeta^{j(s-1)} + o\left(\zeta^{j(s-1)}\right), \tag{4.8}$$

as $\zeta \to \infty$.

Substituting (4.8) into Eq. (4.7) and comparing the leading terms in the resulting equation yields the relations

$$c_n (s a_s)^n + K_{0,m} = 0 \qquad \text{and} \qquad ns = mq. \tag{4.9}$$

Thus, we see that mq is divisible by n. But n and m are relatively prime by hypothesis, so it follows from Euclid's lemma (see Burton [7, p. 28]) that n must also divide q. However, we know from Theorem B that $1 \leq q \leq n$. Hence $q = n$, and $s = m$.

To prove $t \geq 0$, we set $G_1(\eta) = G(1/\eta)$. Then we see, from (4.7), that $G_1(\eta)$ satisfies the equation

$$\begin{aligned} &\tilde{c}_n \eta^n G_1^{(n)}(\eta) + \tilde{c}_{n-1} \eta^{n-1} G_1^{(n-1)}(\eta) + \cdots + \tilde{c}_1 \eta G_1'(\eta) \\ &\quad + \left(c_0 + \tilde{K}(\eta^q)\right) G_1(\eta) = 0. \end{aligned} \tag{4.10}$$

Here $\tilde{c}_n, \ldots, \tilde{c}_1$ ($\tilde{c}_n = (-1)^n c_n$) are some constants and

$$\tilde{K}(\eta) = K(1/\eta) = \sum_{i=p}^{m} K_{0,i} \eta^{-i}.$$

Let $S_1(\eta) = S(1/\eta)$ and $R_1(\eta) = R(1/\eta)$. Suppose that $t \leq -1$ in (4.1). Then we obtain, as (4.8), that the asymptotic representations

$$\frac{G_1^{(j)}(\eta)}{G_1(\eta)} = (-t a_t)^j \eta^{j(-t-1)} + o\left(\eta^{j(-t-1)}\right) \tag{4.11}$$

hold for $1 \le j \le n$, as $\eta \to \infty$. Substitute (4.11) into Eq. (4.10). By comparing the leading terms in the resulting equation and noting that $K(\eta) = O(1)$ as $\eta \to \infty$ yields the relation that $-tn = 0$. This is impossible. Hence, $t \ge 0$.

To prove $t \le np'$ in Part (A), we need a more detailed analysis on the function $G(\zeta)$ in (4.3) and its derivatives. Differentiating (4.3) we obtain that for $u \ge 1$,

$$G^{(u)}(\zeta) = \left(\sum_{i=0}^{u} D(u,i) R^{(i)}(\zeta) \right) e^{S(\zeta)}, \tag{4.12}$$

where $D(u,u) = 1$ and

$$D(u+1,i) = D'(u,i) + S'D(u,i) + D(u,i-1), \qquad 0 \le i \le u+1. \tag{4.13}$$

Here we set $D(u, u+1) = D(u, -1) = 0$. It follows from (4.13) and a simple induction that $D(u,i)$ is a differential polynomial in $S'(\zeta)$ and its derivatives of *weight* equal to $u - i$ and with constant coefficients.

Substituting (4.12) into Eq. (4.7) yields a differential equation in $R(\zeta)$ as follows (note that we have already proved $q = n$):

$$\sum_{j=1}^{n} c_j \zeta^j \left(\sum_{i=0}^{j} D(j,i) R^{(i)}(\zeta) \right) + \left(c_0 + K(\zeta^n)\right) R(\zeta) = 0.$$

By changing the order of summation, we obtain

$$\sum_{i=1}^{n} \left(\sum_{j=i}^{n} c_j D(j,i) \zeta^j \right) R^{(i)}(\zeta) + \left(\sum_{j=1}^{n} c_j D(j,0) \zeta^j + c_0 + K(\zeta^n) \right) R(\zeta) = 0.$$

We may rewrite the above equation in the form

$$\sum_{i=1}^{n} P_i(\zeta) R^{(i)}(\zeta) + \left(P_0(\zeta) + K(\zeta^n)\right) R(\zeta) = 0, \tag{4.14}$$

where

$$P_i(\zeta) = \sum_{j=i}^{n} c_j D(j,i) \zeta^j, \qquad 0 \le i \le n. \tag{4.15}$$

We distinguish two cases:

Case (i). $R(\zeta)$ is a constant, not identically zero. In this case, Eq. (4.14) reduces to

$$P_0(\zeta) + K(\zeta^n) = 0. \tag{4.16}$$

An elementary analysis on (4.13) shows that for $j \geq 1$,

$$D(j,0) = \sum_I d_I (S')^{k_1} (S'')^{k_2} \cdots (S^{(\mu)})^{k_\mu}, \tag{4.17}$$

where d_I is a constant and the summation is taken over all multi-indices $I = (k_1, \dots, k_\mu)$ such that $k_1 + 2k_2 + \cdots + \mu k_\mu = j$.

Recall that we have proved $t \geq 0$. Hence, by substituting (4.1) into (4.17) we obtain that, as $\zeta \to 0$,

$$D(j,0) = t(t-1)\cdots(t-(j-1))a_t \zeta^{t-j} + o(\zeta^{t-j}), \qquad 1 \leq j \leq t. \tag{4.18}$$

Thus, from (4.15) and (4.18) we obtain that, as $\zeta \to 0$,

$$P_0(\zeta) = \sum_{j=0}^{n} c_j D(j,0) \zeta^j = c_0 + C\zeta^t + o(\zeta^t), \tag{4.19}$$

where C is a constant. We may assume that $C \neq 0$, for if no such C can be found then we deduce from (4.16) that $K(\zeta)$ must be a constant, a contradiction.

Substituting (4.19) and (1.2) into (4.16) (with $\zeta \to 0$), we obtain from the resulting equation that $t \leq np'$.

Case (ii). $R(\zeta)$ is a non-constant polynomial. Then we obtain as in (4.18) that as $\zeta \to 0$,

$$\begin{aligned} D(j,i) &= S^{(j-i)}(\zeta) + o(\zeta^{t-(j-i)}) \\ &= t(t-1)\cdots(t-(j-i)+1)a_t \zeta^{t-(j-i)} + o(\zeta^{t-(j-i)}). \end{aligned} \tag{4.20}$$

Recall that $D(i,i) = 1$ for all $i \geq 0$. Hence, from (4.15) we obtain that for $0 \leq i \leq n$,

$$\begin{aligned} P_i(\zeta) &= c_i D(i,i)\zeta^i + \sum_{j=i+1}^{n} c_j D(j,i) \zeta^j \\ &= c_i \zeta^i + \sum_{j=i+1}^{n} c_j \zeta^j \big[t(t-1)\cdots(t-(j-i)+1) a_t \zeta^{t-(j-i)} \\ &\qquad + o(\zeta^{t-(j-i)}) \big] \\ &= c_i \zeta^i + \left[\sum_{j=i+1}^{n} c_j t(t-1)\cdots(t-(j-i)+1) a_t \right] \zeta^{t+i} + o(\zeta^{t+i}) \\ &= c_i \zeta^i + B_i \zeta^{t+i} + o(\zeta^{t+i}), \end{aligned} \tag{4.21}$$

as $\zeta \to 0$, where B_i is a constant. Hence, it follows from (4.2) and (4.21) that for $1 \le i \le n$,

$$\begin{aligned} P_i(\zeta)\frac{R^{(i)}(\zeta)}{R(\zeta)} &= \frac{1}{b_0}P_i(\zeta)R^{(i)}(\zeta) + o(\zeta^{t+i}) \\ &= \frac{1}{b_0}\left[c_i\zeta^i + B_i\zeta^{t+i} + o(\zeta^{t+i})\right] \\ &\quad \times\left[\sum_{j=i}^{k} j(j-1)\cdots(j-i+1)b_j\zeta^{j-i}\right] + o(\zeta^{t+i}) \\ &= \frac{c_i}{b_0}\left[\sum_{j=i}^{k} j(j-1)\cdots(j-i+1)b_j\zeta^{j}\right] \\ &\quad + \frac{B_i}{b_0}\left[\sum_{j=i}^{k} j(j-1)\cdots(j-i+1)b_j\zeta^{t+j}\right] + o(\zeta^{t+i}) \\ &= \frac{c_i}{b_0}\left[\sum_{j=i}^{k} j(j-1)\cdots(j-i+1)b_j\zeta^{j}\right] + F_i\zeta^{t+i} \\ &\quad + o(\zeta^{t+i}), \end{aligned} \tag{4.22}$$

as $\zeta \to 0$, where F_i are some constants possibly zero.

As in Case (i), substituting (4.19), (4.22), and (1.2) into Eq. (4.14) yields $t \le np'$ provided $k < np'$. This proves Part (A) of Theorem 2.1.

The proof of Part (B) is easily accomplished by considering the function G_1 as defined earlier and Eq. (4.10) instead with similar reasoning as in Part (A).

Let us now assume n divides m. Then, from (4.9) we obtain $ns = mq$. But m is divisible by n so we may write $m = n_1 n$ for some constant n_1. Hence $ns = n_1 nq$, i.e., $s = n_1 q$ and we obtain that s is divisible by q. Similarly, by considering the differential equation (4.10) instead, we obtain that t is divisible by q if p is divisible by n. A similar reasoning applies to the case when n is divisible by m. This completes the proof of Part (C) and also Theorem 2.1.

Proof of Theorem 2.2. Suppose now Eq. (2.4) possesses a non-trivial solution f with $\lambda(f) < +\infty$. Note that Eq. (2.4) is of the form (1.1) with $K(\zeta) = \zeta + K_0$, and that the degree of $K(\zeta)$ is 1 which is relatively prime with n. Hence, from Theorem 2.1(A) we have $s = 1$ and $t \ge 0$ in (2.2). Thus $S(\zeta)$ in (2.2) is a linear polynomial in ζ. Therefore, again by Theorem 2.1(A), the solution f takes the form

$$f(z) = e^{dz}R(e^{z/n})\exp(\alpha e^{z/n}), \tag{4.23}$$

where $\alpha(\neq 0)$ and d are two constants, and $R(\zeta)$ is a polynomial taking the form (2.3). This gives (2.7).

Next, we will determine the two constants d and α in (4.23). Write f as in (4.4) with $G(\zeta) = R(\zeta)\exp(\alpha\zeta)$. Then $R(\zeta)$ solves (4.14) with $K(\zeta) = \zeta + K_0$. That is,

$$\sum_{i=1}^{n} P_i(\zeta) R^{(i)}(\zeta) + \big(P_0(\zeta) + \zeta^n + K_0\big) R(\zeta) = 0, \tag{4.24}$$

where $P_0(\zeta)$ and $P_i(\zeta)$ take the form (4.15). Recall the formula (4.13), since we now have $S(\zeta) = \alpha\zeta$ (see (4.23)), it follows from (4.13) that $D(u,i)$ are all constants. Therefore, substituting (2.3) and (4.15) into Eq. (4.24) and collecting likewise terms in the resulting equation, we obtain

$$\sum_{j=0}^{n+k} H_j \zeta^j = 0, \tag{4.25}$$

where H_j are constants such that

$$H_{n+k} = \big(c_n D(n,0) + 1\big) b_k, \tag{4.26}$$

$$\begin{aligned} H_{n+k-1} = {} & \big(c_{n-1} D(n-1,0) + k c_n D(n,1)\big) b_k \\ & + \big(c_n D(n,0) + 1\big) b_{k-1}, \end{aligned} \tag{4.27}$$

$$\cdots$$

$$H_0 = (c_0 + K_0) D(0,0) b_0. \tag{4.28}$$

It is clear that all the coefficients H_j in (4.25) must vanish. Note that $c_n = B(n,n) = 1/n^n$ and $D(n,0) = \alpha^n$. Hence, it follows from $H_{n+k} = 0$ in (4.26) that $\alpha^n + n^n = 0$, which gives the value for α. To get the constant d in (4.23), we need to compute c_{n-1} and $D(n,1)$. First, note that the recurrence formula (4.13) reduces to

$$D(u+1,i) = \alpha D(u,i) + D(u,i-1), \qquad 0 \le i \le u+1,$$

and from which we obtain

$$\begin{aligned} D(n,1) &= \alpha D(n-1,1) + D(n-1,0) \\ &= \alpha D(n-1,1) + \alpha^{n-1} \\ &= \alpha^2 D(n-2,1) + 2\alpha^{n-1} \\ &\cdots \\ &= \alpha^{n-1} D(1,1) + (n-1)\alpha^{n-1} = n\alpha^{n-1}. \end{aligned} \tag{4.29}$$

On the other hand, since $c_{n-1} = B(n, n-1)$, it then follows from (4.6) (note that $q = n$) that

$$\begin{aligned} c_{n-1} = B(n, n-1) &= \frac{dn + n - 1}{n} B(n-1, n-1) \\ &\quad + \frac{1}{n} B(n-1, n-2) \\ &= \frac{dn + n - 1}{n^n} + \frac{dn + n - 2}{n^2} B(n-2, n-2) \\ &\quad + \frac{1}{n^2} B(n-2, n-3) \\ &\quad \cdots \\ &= \frac{dn + n - 1}{n^n} + \frac{dn + n - 2}{n^n} + \cdots + \frac{dn}{n^n} \\ &= \frac{2dn + n - 1}{2n^{n-1}}. \end{aligned} \tag{4.30}$$

Now set $H_{n+k-1} = 0$. Then (4.27) and (4.26) imply that

$$c_{n-1} D(n-1, 0) + k c_n D(n, 1) = 0. \tag{4.31}$$

Substituting (4.29) and (4.30), together with the facts that $c_n = 1/n^n$ and $D(n-1, 0) = \alpha^{n-1}$, into (4.31) yields

$$d = -\frac{2k + n - 1}{2n},$$

which proves (2.6). Finally, we set $H_0 = 0$ and from this and (4.28) we obtain that

$$(c_0 + K_0) D(0, 0) = 0. \tag{4.32}$$

Notice that $D(0, 0) = 1$ and that

$$c_0 = B(n, 0) + \sum_{i=1}^{n-2} K_i B(i, 0) = d^n + \sum_{i=1}^{n-2} K_i d^i, \tag{4.33}$$

since $B(i, 0) = d^i$. Hence, (2.5) follows immediately from (4.32) and (4.33). This completes the proof of Theorem 2.2.

5. PROOFS OF THEOREMS 3.1 AND 3.2

Proof of Theorem 3.1. Applying Theorem 2.2 to $n = 3$ we obtain from (2.6) that

$$d = -\frac{k+1}{3}. \tag{5.1}$$

Then, substituting (5.1) and $n = 3$ into (2.5) gives (3.2). Next, we prove (3.3). By (5.1), (2.7), and the fact that $n = 3$, we see that the solution f takes the form

$$f(z) = e^{-((k+1)/3)z} R(e^{z/3}) \exp(\alpha e^{z/3}), \tag{5.2}$$

where $k \geq 0$ is an integer, α is a constant satisfying $\alpha^3 + 27 = 0$, and $R(\zeta)$ is as defined in (2.8). We rewrite (5.2) into the form

$$f(z) = \zeta \Psi(\zeta) \exp(\alpha \zeta^{-1}), \qquad \zeta = e^{-z/3}, \tag{5.3}$$

where

$$\Psi(\zeta) = \sum_{i=0}^{k} e_i \zeta^i. \tag{5.4}$$

Here in (5.4), $e_i = b_{k-i}$, $i = 0, 1, \ldots, k$, where the b_j's are constants in (2.8). Hence, $e_0 e_k \neq 0$ by (2.8). Substituting (5.3) into Eq. (3.1) we obtain the following equation in Ψ:

$$\begin{aligned} &\zeta^3 \Psi'''(\zeta) + 3(2\zeta^2 - \alpha\zeta)\Psi''(\zeta) \\ &\quad + \left((7 + 9K_1)\zeta - 6\alpha + 3\alpha^2 \zeta^{-1}\right)\Psi'(\zeta) \\ &\quad + \left(1 + 9K_1 - 27K_0 - (1 + 9K_1)\alpha\zeta^{-1}\right)\Psi(\zeta) = 0. \end{aligned} \tag{5.5}$$

Then, we substitute (5.4) into (5.5) and this gives an algebraic equation in ζ as

$$\sum_{j=1}^{k+1} U_j \zeta^{j-2} = 0, \tag{5.6}$$

where

$$U_j = X_j e_{j-2} + \alpha Y_j e_{j-1} + \alpha^2 Z_j e_j, \qquad 1 \leq j \leq k+1, \tag{5.7}$$

where X_j, Y_j, Z_j are constants defined by (3.5), (3.6), and (3.7), respectively. (Note that, in order to get these constants, we need to replace K_0 by K_1 in (5.5) by using (3.2).) Since we must have $U_j = 0$ for all j, we obtain from (5.7) that

$$\tilde{\mathbf{A}}_k(K_1) \times \begin{pmatrix} e_0 \\ \vdots \\ e_k \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{5.8}$$

where

$$\tilde{\mathbf{A}}_k(K_1) = \begin{pmatrix} \alpha Y_1 & \alpha^2 Z_1 & 0 & & & \cdots & 0 \\ X_2 & \alpha Y_2 & \alpha^2 Z_1 & 0 & & \cdots & 0 \\ 0 & X_3 & \alpha Y_3 & \alpha^2 Z_3 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & X_k & \alpha Y_k & \alpha^2 Z_k \\ 0 & \cdots & \cdots & 0 & 0 & X_{k+1} & \alpha Y_{k+1} \end{pmatrix}.$$

Since $e_0 e_k \neq 0$, from (5.8) we must have $\det \tilde{\mathbf{A}}_k(K_1) = 0$. But an elementary calculation shows that

$$\det \tilde{\mathbf{A}}_k(K_1) = \alpha^{k+1} \det \mathbf{A}_k(K_1).$$

Hence, (3.3) follows.

Conversely, suppose that K_0, K_1 and an integer $k \geq 0$ satisfy (3.2) and (3.3). Then, by reversing the above argument, we can define a polynomial $\Psi(\zeta)$ of the form (5.4), with coefficients $e_0, e_1, \dots, e_k$ forming a non-trivial solution of (5.8). With this $\Psi(\zeta)$, the functions defined by

$$f_j(z) = \zeta \Psi(\zeta) \exp\left(\alpha_j \zeta^{-1}\right), \qquad \zeta = e^{-z/3}, j = 1, 2, 3,$$

where α_j $(j = 1, 2, 3)$ are the distinct roots of $\alpha^3 + 27 = 0$, are linearly independent solutions of Eq. (3.1), each with finite exponent of convergence of its zero sequence. This proves Theorem 3.1.

Proof of Theorem 3.2. Suppose $K_1 \geq 0$ in Eq. (3.1), and assume that the equation admits a non-trivial solution with $\lambda(f) < \infty$. Then, Theorem 3.1 holds. However, we see from (3.5), (3.6), and (3.7) that if $K_1 \geq 0$, then $X_j < 0$, $Y_j < 0$, and $X_j > 0$ hold for all $1 \leq j \leq k + 1$. A simple inductive argument shows that in this case the determinant of the matrix in (3.4) can never vanish for any $k \geq 0$. This contradicts (3.3).

6. EXAMPLES TO THEOREM 2.1

In this section we give three examples to illustrate Theorem 2.1.

EXAMPLE 1. It is easy to verify that $f(z) = \exp(e^{5z} + e^{4z} + e^{3z} + e^{2z})$ solves the equation

$$f'' + (-25e^{10z} - 40e^{9z} - 46e^{8z} - 44e^{7z} - 25e^{6z} - 37e^{5z} - 20e^{4z} - 9e^{3z} - 4e^{2z})f = 0.$$

For this equation, we have that $m = 10$ which is a multiple of $n = 2$, and that $q = 1$. Hence, both $s = 5$ and $t = 2$ are multiples of q. This shows that part (C) of Theorem 2.1 can occur when $n = 2$.

EXAMPLE 2. The function $f(z) = \exp(e^{4(z/2)} + e^{2(z/2)})$ solves

$$f''' - f' + (-8e^{6z} - 12e^{5z} - 30e^{4z} - 19e^{3z} - 9e^{2z})f = 0.$$

In this example we have $n = 3$ which divides $m = 6$ and that $q = 2$ divides $s = 4$.

In the following example both $K_{0,m}$ $(m \geq 1)$ and $K_{0,p}$ $(p \leq -1)$ in (1.2) are non-zero.

EXAMPLE 3. The function $f(z) = \exp(e^{6(z/3)} + e^{-3(z/3)})$ solves

$$f^{(4)} - 2f'' - 3f' - K(e^z)f = 0,$$

where

$$K(\zeta) = 16\zeta^8 + 96\zeta^6 - 32\zeta^5 + 104\zeta^4 - 72\zeta^3 + 26\zeta^2 - 8\zeta - 6\zeta^{-1} + 5\zeta^{-2} + 6\zeta^{-3} + 4\zeta^{-4}.$$

In this equation, $n = 4$ divides both $m = 8$ and $p = -4$, and $q = 3$ divides both $s = 6$ and $t = -3$.

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