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COMPLEX OSCILLATION THEORY AND SPECIAL FUNCTIONS

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RÉSUMÉ. Ceci est une announce de nôtre resultat [10] dans lequel nous montrons que les représentations des solutions non-oscillatoires de certaines équations differentielles ordinaires proviennent des fonctions hypergéométriques confluents avec un nombre fini de zéros et aussi des polynômes orthogonaux.

Nous caractérisons les distributions nulles et les représentations des solutions de deux classes des équations differentielles ordinaires et indiquons que la solution du problème dans les autres cas est relié à un problème de Heine.

1. Introduction. Let $f(z)$ be an entire function and denote its order by $\sigma(f)$. The exponent of convergence of f is $\lambda(f) = \limsup_{r\to+\infty} \log n(r,f)/\log r$, where $n(r,f)$ denotes the number of the zeros of f in $|z| < r$. It is an easy consequence of Weierstrass' theorem that $\lambda(f) \le \sigma(f)$ (see [15]).

The *complex oscillation problem* (see [18]) we are interested in is to study the quantity $\lambda(f)$ in relation to $\sigma(f)$ for entire functions f satisfying

$$f''(z) + A(z)f(z) = 0, \tag{1.1}$$

and $A(z)$ is an entire transcendental function of finite order. One of the main problems is to seek sufficient conditions on $A(z)$ that guarantee that each solution f of (1.1) to satisfy

$$\lambda(f) = +\infty. \tag{1.2}$$

Some basic oscillation problems are considered in [5], followed by [6] which deals with periodic coefficients, and [7] which deals with meromorphic solutions. We refer to [18, Chapter 5] and the references therein for subsequent works.

The equation (1.1) considered by Bank and Laine in [6] has periodic coefficient

$$A(z) = B(e^z), \quad B(\zeta) = K_k\zeta^{-k} + \cdots + K_0 + \cdots + K_\ell\zeta^\ell, \quad \ell > 0, k \ge 0, \tag{1.3}$$

where $K_\ell K_k \neq 0$. They established the following theorem when (1.2) is violated.

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THEOREM 1.1. *Let $A(z)$ be an entire function given by (1.3). If (1.1) admits a non-trivial solution f with $\lambda(f) < +\infty$, then there exist constants d, d_j, and a polynomial $\psi(\zeta)$ with only simple roots, such that if ℓ is odd in (1.3), then $k = 0$, and*

$$f(z) = \psi(e^{z/2}) \exp\Big(\sum_{j=0}^{\ell} d_j e^{jz/2} + dz\Big), \tag{1.4}$$

where $d_j = 0$ for even j. If ℓ is even, then k is also even, and

$$f(z) = \psi(e^{z}) \exp\Big(\sum_{j=-k/2}^{\ell/2} d_j e^{jz} + dz\Big). \tag{1.5}$$

It appears to be a difficult problem to characterize the choices of $B(\zeta)$ in (1.3) so that equation (1.1) would admit a solution f with $\lambda(f) < +\infty$, and to write down such a solution explicitly [4], [9], [12]. The major difficulty is to verify the existence of the polynomial $\psi(\zeta)$ that appears in (1.4) and (1.5).

This announcement summarizes our findings in [10] where we show that the above mentioned problem is closely related to special functions and in some cases to orthogonal polynomials. We have shown that for the well-studied classes of (1.3), equation (1.1) can be transformed to equations satisfied by two classes of special functions—the Bessel functions and Coulomb wave functions. We solve these two subclasses of (1.1) completely—general and explicit solutions $f(z)$ to (1.1) are found regardless of whether $\lambda(f) < +\infty$ or $= +\infty$. Moreover, we reproduce the polynomial component in the solution $f(z)$ via the confluent hypergeometric functions. This extends our understanding of the polynomial components found by Bank, Laine and Langley [6], [8]. This establishes a connection between the finding of $\psi(\zeta)$ for (1.4)–(1.5) for (1.1) in the above question and to a class of orthogonal polynomials—Bessel polynomials. The key is the following transformation due to Lommel [21, p. 97]

$$x = \alpha t^{\beta}, \quad y(x) = t^{\gamma} u(t),$$

where α, β and γ are constants. We have identified the transformations needed to transform (1.1) to the above-mentioned equations. We then establish another connection to (1.1) with the remaining (1.3) not covered in the previous consideration of the *Heine problem* for differential equations with polynomial coefficients [20]. The study here is based on the solutions (1.4) and (1.5). In the case when $\ell > 0$, $k = 0$ and ℓ is odd, we shall give a characterization of the coefficients $\{K_j\}$ in (1.3) based on the Heine problem. This extends the results obtained in [9] where only zero-free solutions were studied.

We also study the closely related *subnormal solution* problem first studied by Frei [11], then by Wittich [22] and then by Bank and Laine [6].

The subnormal solution problem can be transformed to be an oscillation problem of (1.1) with specific (1.3) in one of the two above-mentioned subclasses of (1.3). Hence the corresponding subnormal solution problem can be solved completely to the same effect as the oscillation problem.

We follow notations in Watson [21] for Bessel functions and in Grosswald [13] for the Bessel polynomials. The notations of the Coulomb wave functions are as in [1].

We first review what was obtained by Bank, Laine and Langley in [8].

THEOREM 1.2. *Let K be a complex number. The equation*

$$f'' + (e^z - K)f = 0 \tag{1.6}$$

admits a non-trivial solution f with $\lambda(f) < +\infty$ if and only if

$$K = (2n+1)^2/16, \tag{1.7}$$

where n is a non-negative integer, and

$$f(z) = \psi(e^{z/2}) \exp[de^{z/2} - (2n+1)z/4], \tag{1.8}$$

where ψ is a polynomial of degree n, and $d^2 + 4 = 0$. Conversely, when K is given in (1.7), then f of (1.8) is a solution to (1.6) with $\lambda(f) < +\infty$.

Our findings show that the polynomial component $\psi(\zeta)$ in (1.8) and more generally in (1.4) and (1.5) are very special polynomials for two special classes of (1.3). One of these is essentially given in equation (1.6). Moreover, we shall show that the general solution of this equation is related to Bessel functions. The other class that has this property turns out to be the equation of the form

$$f'' + (K_{-2}e^{-2z} + K_{-1}e^{-z} + K_0)f = 0, \tag{1.9}$$

to be discussed in the next section. The equations (1.6) (and its generalization (1.10)) and (1.9) seem to be the only two subclasses among the general case of (1.1) with coefficient (1.3) that are related to classical confluent hypergeometric functions and hence can be solved completely. We now state our first main result.

THEOREM 1.3. *Let K_0 and K_1 be complex constants. Then any two linearly independent solutions of the equation*

$$f'' + (K_1 e^z - K_0)f = 0 \tag{1.10}$$

are given by

$$y_\pm(z) = A_\pm J_{2\sqrt{K_0}}(\pm 2\sqrt{K_1}e^{z/2}) + B_\pm Y_{2\sqrt{K_0}}(\pm 2\sqrt{K_1}e^{z/2}). \tag{1.11}$$

The solutions (1.11) have a finite exponent of convergence if and only if

$$2\sqrt{K_0} = n + 1/2,$$

where n is an integer, and, if $n \geq 0$, then

$$y_{\pm}(z) = \theta_n(\pm 2i\sqrt{K_1}e^{z/2}) \exp(\mp 2i\sqrt{K_1}e^{z/2} + dz), \tag{1.12}$$

where $\theta_n(x)$ is the reverse Bessel polynomial of degree n and $d = (-2n-1)/4$. If, however, $n = -m$, $m \geq 1$, then we replace all the n in (1.12) by $m - 1$.

We note that (1.6) was already considered by Lommel in 1871 [19].

2. **Subnormal solutions.** The oscillation problem of the second class of equation that we have found exact solutions are the ones given in the form (1.9). The following special case

$$f'' + [K + (2e^{-z} - e^{-2z})/4]f = 0 \tag{2.1}$$

is considered by Bank and Laine in [6] in connection with subnormal solutions of the differential equation (2.4) below first considered by Frei.

Let $P(\zeta)$ and $Q(\zeta)$ be two polynomials at least one of which is non-constant. Then it is known that the growth of an entire solution $g(z)$ of the equation

$$g''(z) + P(e^z)g'(z) + Q(e^z)g(z) = 0 \tag{2.2}$$

can reach infinite order. We say that $g(z)$ is *subnormal* if it satisfies

$$\limsup_{r \to +\infty} \frac{\log \log M(r, g)}{r} = 0. \tag{2.3}$$

That is, the subnormal solutions of (2.2) are those solutions that grow slower than the maximum speed allows (*i.e.*, infinite order). Wittich [22] showed that any subnormal solution can be written in the form $g(z) = e^{dz}S(e^z)$ for some polynomial $S(\zeta)$. However, Frei [11] considered the special case

$$g''(z) + e^{-z}g'(z) + Kg(z) = 0, \tag{2.4}$$

and she showed that equation (2.4) will have a subnormal solution if and only if $K = -n^2$ for some integer n. Moreover, the polynomial $S(\zeta)$ mentioned above has exact degree n.

By applying the transformation $f(z) = g(z)\exp(-e^{-z}/2)$, Bank and Laine [6] showed that equation (2.4) can be transformed to (2.1). It follows from this transformation that the solution $f(z)$ of (2.1) that satisfies $\lambda(f) < +\infty$ if and only if $g(z)$ is a subnormal solution to (2.4). Indeed, they proved:

THEOREM 2.1. *Let K be a non-zero complex number. Suppose equation (2.1) admits a non-trivial solution f that satisfies $\lambda(f) < +\infty$, then there exists a positive integer n with $K = -n^2$ and f must be equal to either*

$$f_1(z) = \Big(\sum_{j=0}^{-(n-1)} B_j e^{jz}\Big) \exp(nz + e^{-z}/2)$$

or

$$f_2(z) = \Big(\sum_{j=0}^{-n} b_j e^{jz}\Big) \exp(nz - e^{-z}/2).$$

Conversely, for each positive integer n, equation (2.1) with $K = -n^2$ possesses two linearly independent solutions of the above forms.

The equations (2.2) and (2.4) and their generalizations were also considered in [2], [14]. In particular, we mention the works of Gundersen, Langley and Ozawa (see [18] and the references therein).

We show that equation (1.9) is closely related to the *Coulomb Wave equation*

$$y'' + [1 - 2\eta x^{-1} - L(L+1)x^{-2}]y = 0, \tag{2.5}$$

where L, η are complex constants. Two linearly independent solutions of (2.5) are given by $F_L(\eta, x)$ and $G_L(\eta, x)$ known as the *regular* and *irregular Coulomb Wave functions* respectively [1]. We have:

THEOREM 2.2. *Let K_{-2}, K_{-1} and K_0 be complex numbers such that $K_{-2}K_{-1} \neq 0$. Then two linearly independent solutions of (1.9) are given by*

$$f_\pm(z) = A_\pm F_L(\eta_\pm, \alpha_\pm e^{-z}) + B_\pm G_L(\eta_\pm, \alpha_\pm e^{-z}), \tag{2.6}$$

where

$$\alpha_\pm^2 = K_{-2} \quad -2\eta_\pm\alpha_\pm = K_{-1}, \quad L = \tfrac{1}{2}(-1 + 2i\sqrt{K_0}).$$

The solutions (2.6) have $\lambda(f_\pm) < +\infty$ if and only if there are non-negative integers n_+ and n_-, $n_+ > n_- \geq 0$ such that

$$i(2\sqrt{K_0} \pm K_{-1}/\sqrt{K_2}) = 2n_\pm + 1,$$

or equivalently

$$iK_{-1}/\sqrt{K_{-2}} = n_+ - n_- \quad \text{and} \quad 2i\sqrt{K_0} = n_+ + n_- + 1,$$

and, by writing $n = n_+$, $\hat{n} = n_+ - n_-$, we have

$$f_+(z) = e^{z/2}e^{-(1-a_+/2)z}y_n(e^z\,;\, a_+, b_+)\exp(-b_+e^{-z}/2), \tag{2.7}$$

$$f_-(z) = e^{z/2}e^{-(1-a_-/2)z}y_{(n-\hat{n})}(e^z\,;\, a_-, b_-)\exp(-b_-e^{-z}/2), \tag{2.8}$$

where $a_\pm = 2(1 + i\eta_\pm)$ and $b_\pm = -2i\alpha_\pm$, and the y_j in (2.7) and (2.8) are generalized Bessel polynomials.

We deduce immediately, as a simple corollary, the complete solution of the oscillation problem of equation (2.1).

THEOREM 2.3. *Let K be a complex constant. Then any two linearly independent solutions of equation (2.1) are given by*

$$f_{\pm}(z) = A_{\pm}F_L\big(\pm\tfrac{i}{2}, \pm\tfrac{i}{2}e^{-z}\big) + B_{\pm}G_L\big(\pm\tfrac{i}{2}, \pm\tfrac{i}{2}e^{-z}\big),$$

where $L = \frac{1}{2}(-1+2i\sqrt{K})$. Moreover, we have $\lambda(f_{\pm}) < +\infty$ if and only if $L = n+\frac{1}{2}$ or equivalently $K = -(L+\frac{1}{2})^2 = -n^2$ for some positive integer n, and

$$f_{+}(z) = y_n(e^z\,;1,1)\exp(-e^{-z}/2), \tag{2.9}$$

$$f_{-}(z) = e^z y_{n-1}(e^z\,;3,-1)\exp(e^{-z}/2). \tag{2.10}$$

Here the y_j in (2.9) and (2.10) are generalized Bessel polynomials.

3. **Heine problems.** We next consider the remaining classes of equation (1.1) not covered by the previous discussion. We recall that the solution f to (1.1) must satisfy (1.2) if both the integers ℓ and k in (1.3) are not equal to zero [3], [12]. Thus no representation of the forms (1.4) or (1.5) are possible. Hence it suffices to consider the case where $\ell > 0$ and $k = 0$. We choose to consider ℓ being odd below.

Here we shall take an approach to the problem yet again different from that of Bank [4]. We shall suppose that (1.1) has a solution $f(z)$ with $\lambda(f) < +\infty$. Hence $f(z)$ must admit either the representation (1.4) or (1.5) depending on whether ℓ is odd or even respectively [9, Prop. 1]. Write $f(z)$ in the standard form

$$f(z) = \psi(e^{z/q})\exp\big(P(e^{z/q}) + dz\big) \tag{3.11}$$

where

$$P(\zeta) = \sum_{j=1}^{\ell/(3-q)} d_j\zeta^j,$$

$q = 1$ if ℓ is even or $q = 2$ when ℓ is odd [9, Prop. 1], d is a constant and $\psi(\zeta)$ is a polynomial of degree n. We may assume that $\psi(0) \neq 0$. One can show that the $\psi(\zeta)$ satisfies a second order differential equation with polynomial coefficients.

We can then apply a theorem of Heine (see [20]) and prove the following.

THEOREM 3.1. *Let ℓ, $\ell \geq 3$, be odd and the coefficients $K_\ell, \ldots, K_{\frac{\ell+1}{2}}$ be given in (1.3). Then there are at most*

$$2\binom{n+(\ell-1)/2}{n}$$

choices for the remaining coefficients $K_{\frac{\ell-1}{2}}, \ldots, K_1$ so that the differential equation (1.1) can admit a solution $f(z)$ given by (3.11) where $\psi(\zeta)$ is a polynomial of degree n. The coefficient K_0 is determined by ℓ and n.

We note that the Heine theorem is a fundamental result in electrostatic models [16] and it also has applications in quantized physical models (see *e.g.* [17]).

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