On The Zero-Free Solutions of Linear Periodic Differential Equations In The Complex Plane †

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Abstract The paper considers the solutions of linear periodic differential equations introduced by Bank and Laine [**6**]. In [**3**] Bank proposed a method to solve certain periodic differential equations with few zeros. The current paper offers alternative methods to deal with zero-free solutions and we obtain more precise results. Our results complement Bank's method. We also make precise a related result of Bank in [**4**].

1. Introduction

In [**6**], Bank and Laine considered the oscillation problem of

$$y'' + Ay = 0 \tag{1.1}$$

when the coefficient $A(z)$ has the form $A(z) = B(e^z)$, where

$$B(\zeta) = \frac{K_k}{\zeta^k} + \cdots + K_0 + \cdots + K_\ell \zeta^\ell, \tag{1.2}$$

and $K_\ell K_k \neq 0$. They found that any non-trivial solution f of (1.1) with a finite exponent of convergence of zeros, denoted by $\lambda(f) < +\infty$, must take the form

$$f(z) = \psi(e^{z/q}) \exp\left(\sum_{j=m'}^{m} d_j e^{jz/q} + dz\right), \tag{1.3}$$

where $q = 1$ or $2, d$ is a constant and $\psi(\zeta)$ is a polynomial:

$$\psi(\zeta) = c_s\zeta^s + \cdots + c_0 \tag{1.4}$$

with simple zeros only. In the case when $q = 2$, $d_j = 0$ for all even j in (1.3).

We remark that any non-trivial solution of (1.1) must have an infinite order of growth [**5**]. We assume the reader is familiar with the Nevanlinna theory and its notations [**11**, **12**, **13**].

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The question then is to determine for which $B(\zeta)$ given by (1.2) equation (1.1) with $A(z) = B(e^z)$ would admit a solution f with $\lambda(f) < +\infty$, and to write down such a solution explicitly. It is evident that the method used to obtain the representation (1.3) does not yield a solution to this problem (see page 102 of [**3**]). At present, the understanding of this problem is restricted to only a few cases.

Theorem A [7]. *Let K be a non-zero complex number. Suppose the equation*

$$f'' + (e^z - K)f = 0 \tag{1.5}$$

admits a non-trivial solution f with $\lambda(f) < +\infty$, then

$$f(z) = \psi(e^{z/2}) \exp\left(de^{z/2} - \frac{2s+1}{4}z\right) \tag{1.6}$$

where $\psi(\zeta) = \sum_{j=0}^{s} c_j\zeta^j$, $d^2 + 4 = 0$ and

$$K = \frac{(2s+1)^2}{16}. \tag{1.7}$$

Conversely, given K has the form (1.7), then f defined by (1.6) is a solution to (1.5) with $\lambda(f) < +\infty$.

We remark that if f_1 has the form (1.6) then the other linearly independent solution to (1.5) with finite exponent of convergence is given by

$$f_2(z) = \phi(e^{z/2}) \exp\left(de^{z/2} - \frac{2s+1}{4}z\right), \tag{1.8}$$

where $\phi(\zeta) = \sum_{j=0}^{s} c'_j\zeta^j$, $d^2 + 4 = 0$.

Theorem B [6]. *Let K be a non-zero complex number. Suppose the equation*

$$f'' + \left(-\frac{1}{4}e^{-2z} + \frac{1}{2}e^{-z} + K\right)f = 0 \tag{1.9}$$

admits a non-trivial solution f with $\lambda(f) < +\infty$, then there exists a positive integer s with $K = -s^2$ and f must be one of

$$f_1(z) = \left(\sum_{j=0}^{-(s-1)} B_j e^{jz}\right) \exp\left(\frac{1}{2}e^{-z} + sz\right)$$

or (1.10)

$$f_2(z) = \left(\sum_{j=0}^{-s} b_j e^{jz}\right) \exp\left(-\frac{1}{2}e^{-z} + sz\right).$$

Conversely, for each positive integer s, the equation (1.9) with $K = -s^2$ possesses two linearly independent solutions of forms (1.10).

To solve the general (1.1), Bank proposed a new method in [**3**] that could be applied to (1.1) with any given (1.2). As his theorems are too long to be quoted in full here, we refer the reader to his original paper for the results. Although Bank's method can be applied to the most general (1.1), the requirement that the majority of the coefficients K_i in (1.2) must be given explicitly means that it is difficult to apply the method to obtain comprehensive results comparable to those of Theorems A and B. We note that Baesch [**1**] has generalized Bank's results to higher-order equations.

The purpose of this paper is to offer alternative methods to (1.1) that admit a zero-free solution. The methods allow us to determine fairly completely the equation (1.1) when it admits a zero-free solution. In addition, there is a distinction between the cases when ℓ in (1.2) is odd or even. In the case when ℓ is odd, we show that all K_i except K_ℓ and K_0 must vanish (Theorem 1). When ℓ is even, we exhibit algorithms to find the constants in (1.3). In particular, we show that $K_0, \cdots, K_{\ell/2-1}$ are completely determined by $K_\ell, \cdots, K_{\ell/2}$ (Theorem 2). We next turn to solutions f of (1.1) with $0 < \lambda(f) < +\infty$. With the techniques developed in this paper we are able to sharpen a related result considered by Bank [**4**]. There Bank assumed (1.1) already admits a solution with $\lambda(f_1) < +\infty$, and he focuses on second linearly independent solution f_2 with $\lambda(f_2) < +\infty$. He obtains an arithmetical relation when the integer ℓ in (1.2) is even. We give a more precise result when ℓ is odd (Theorem 4). It does not seem to follow from the method used in [**4**].

This paper is organized so that the main results (Theorems 1-2) are stated and proved in Section 2. Examples that illustrate the main theorems are given in Section 3. Section 4 will discuss the method of comparing coefficients.

2. Main Results

Proposition 1. *Let f be a non-trivial solution of (1.1) with $\lambda(f) < +\infty$. If ℓ in (1.2) is an odd positive integer, then $k = 0$, and*

$$f(z) = \psi(e^{z/2}) \exp\Big(\sum_{j=0}^{\ell} d_j e^{jz/2} + dz\Big), \tag{2.1}$$

where $d_j = 0$ whenever j is even. While if ℓ in (1.2) is an even positive integer, then k is also even, and

$$f(z) = \psi(e^{z}) \exp\Big(\sum_{j=-k/2}^{\ell/2} d_j e^{jz} + dz\Big). \tag{2.2}$$

Proof It follows from (1.3) that there exist two polynomials $S(\zeta)$ and $R(\zeta)$ such that

$$\frac{f'}{f}(z) = S(e^{-z/q}) + R(e^{z/q}). \tag{2.3}$$

Thus

$$-B(e^z) = \frac{f''}{f}(z) = \frac{1}{q}\Big(- e^{-z/q} S'(e^{-z/q}) + e^{z/q} R'(e^{z/q})\Big) + \Big(S(e^{-z/q}) + R(e^{z/q})\Big)^2. \tag{2.4}$$

Suppose ℓ is an odd positive integer, then either [**2**, Theorem 2] or [**10**, Theorem 1] implies that $k = 0$. Thus $B(\zeta)$ is a polynomial. Let $z \to -\infty$ through real values in (2.4) shows that $S(\zeta) \equiv$ const. which we incorporate into $R(\zeta)$. Taking $z \to \infty$ in (2.4) and considering the highest exponents implies that $\ell q = 2m$ where $m = \deg R$. Since ℓ is odd, we deduce $q = 2$. This gives (2.1).

Suppose now that ℓ is a positive even integer. Let us suppose further that $q = 2$ in (2.1). Then the right-hand side of (2.4), as $z \to \infty$, is asymptotic to an odd exponent of e^z, while the left-hand side of (2.4) is asymptotic to an even exponent of e^z. This is a contradiction, and hence $q = 1$. Thus, $\ell = \ell q = 2m$ gives $m = \ell/2$. A similar argument applied to (2.4) after making a change of variable $t = 1/\zeta$ in (2.4) gives $m' = -k/2$. This gives (2.2).

Although most of the above conclusions are noted in various forms in [**2**],[**6**], [**10**], and [**3**], they have not been given in a unified and explicit form. The above proposition serves this purpose, and it is also deployed later in this paper. In addition, we note that the distinction between ℓ in (1.2) as odd and even cases is important, which will become evident in the main results below.

We remark that in view of (1.3), a solution f of (1.1) which is zero-free is equivalent to $\lambda(f) = 0$.

Theorem 1. *Let the two integers ℓ and k in (1.2) be such that $\ell \geq 0$ is odd and $\ell \geq |k| \geq 0$. Suppose the equation (1.1), with coefficient $A(z) = B(e^z)$, admits a zero-free solution f, then $k = 0$, $K_i = 0$, $i = 1, \cdots, \ell - 1$ and $K_0 \neq 0$. Moreover,*

$$K_0 = -\frac{\ell^2}{16} \tag{2.5}$$

and

$$f(z) = \exp(d_\ell e^{\ell z/2} - \ell z/4), \tag{2.6}$$

where $\ell^2 d_\ell^2 + 4K_\ell = 0$. Conversely, given K_0 in (2.5), then f defined by (2.6) always satisfy (1.1) with $A(z) = B(e^z), B(\zeta) = K_\ell \zeta^\ell + K_0$.

We immediately deduce

Corollary 1. *Let $\ell \geq 1$ be an odd integer and $k = 0$ in (1.2). Suppose (1.1), with $A(z) = B(e^z)$, is such that either $K_0 = 0$ or at least one of the coefficients $K_i \neq 0, i \in \{1, \cdots, \ell - 1\}$, then $\lambda(f) > 0$ for any non-trivial solution of f of (1.1).*

Proof of Theorem 1 Let f be a zero-free solution of (1.1). By Proposition 1 and (2.1), we can find a polynomial $P_1(\zeta)$ such that $f'(z)/f(z) = P_1(e^{z/2})$. Thus

$$-B(e^z) = -A(z) = \frac{1}{2} e^{z/2} P_1'(e^{z/2}) + P_1(e^{z/2})^2. \tag{2.7}$$

Substituting $\zeta = e^{z/2}$ into (2.7) yields

$$\frac{1}{2}\zeta P_1'(\zeta) + P_1(\zeta)^2 = -B(\zeta^2) = \frac{1}{2}\zeta P_1'(-\zeta) + P_1(-\zeta)^2,$$

since $B(\zeta^2)$ is even in ζ.

Suppose $P_1(\zeta) - P_1(-\zeta) \not\equiv 0$. It follows that

$$P_1(\zeta) + P_1(-\zeta) = \frac{1}{2}\zeta \frac{P_1'(\zeta) + P_1'(-\zeta)}{P_1(\zeta) - P_1(-\zeta)}. \tag{2.8}$$

But

$$\frac{P_1'(\zeta) + P_1'(-\zeta)}{P_1(\zeta) - P_1(-\zeta)}$$

is the logarithmic derivative of the polynomial $P_1(\zeta) - P_1(-\zeta)$. We deduce from (2.1) and (2.8) that the only zero of $P_1(\zeta) + P_1(-\zeta)$ must be at the origin. Thus $P_1(\zeta) - P_1(-\zeta) = d\zeta^m$ for some odd integer $m, 1 \leq m \leq \ell$, and d is a constant. It follows from (2.8) that $P_1(\zeta) + P_1(-\zeta)$ is a constant b, say. So

$$\begin{aligned} 2P_1(\zeta) &= \Big(P_1(\zeta) - P_1(-\zeta)\Big) + \Big(P_1(\zeta) + P_1(-\zeta)\Big) \\ &= d\zeta^m + b. \end{aligned}$$

Thus $f'(z)/f(z) = d/2e^{mz/2} + b/2$. Substituting this expression into (1.1) yields

$$-B(e^z) = \frac{f''}{f} = \frac{d^2}{4}e^{mz} + \left(\frac{dm}{4} + \frac{db}{2}\right)e^{mz/2} + \frac{b^2}{4}.$$

Hence $m = \ell$. Since ℓ is odd, the coefficient of $e^{\ell z/2}$ must be zero. That is, $b = -\ell/2$. Integration of f'/f gives precisely (2.5) and (2.6).

In the case when $P_1(\zeta) - P_1(-\zeta) \equiv 0$, then $P_1(\zeta)$ is an even polynomial. Thus we can replace $P_1(e^{z/2})$ by $H(e^z)$ in (2.7) where $H(\zeta)$ is a polynomial. We arrive at a contradiction in (2.7) since $B(\zeta)$ has an odd degree at infinity.

The above also proves the converse of the Theorem.

Remarks 1. We note that the above proof actually shows there exists another zero-free linearly independent solution and is given by

$$\exp\big(-d_\ell e^{\ell z/2} - \ell z/4\big).$$

2. In the proof of the original version of the paper the author substituted (2.1) into (1.1) with $\psi(e^{z/2}) \equiv 1$ and compared the coefficients of the resulting equation. The method is discussed in Section 4 below for the case when l is even. The referee of this paper illustrated the much simpler alternative above.

We next consider the case when ℓ is even. Proposition 1 implies that k must also be even.

Theorem 2. *Let ℓ and k be even integers with $\ell \geq k \geq 0$, and assume $A(z) = B(e^z)$ where $B(\zeta)$ is as in (1.2). Write $\zeta = e^{z/2}$ and*

$$B(\zeta^2) = B_0(\zeta^2) + o(\zeta^{\ell-k}) \tag{2.9}$$

as $\zeta \to \infty$. Suppose f is a zero-free solution of (1.1), then $f'(z)/f(z) = R(e^{z/2})$, where $R(\zeta)$ is a rational function with poles of orders ℓ and k at ∞ and 0 respectively and is determined by computing the Laurent series of

$$cB_0(\zeta^2)^{1/2} - \frac{\zeta^2 B_0'(\zeta^2)}{4B_0(\zeta^2)} \tag{2.10}$$

valid in a neighbourhood of infinity, and up to the term $\zeta^{-k}, c^2 + 1 = 0$. In particular, the coefficients $K_j, j = -k, \cdots, (\ell - k)/2 - 1$ are completely determined by $K_j, j = \ell, \cdots, (\ell - k)/2$.

Proof Let ℓ, k and $B(\zeta)$ be as in the Theorem. Let $\zeta = e^{z/2}$ and define $B_0(\zeta)$ by

$$\begin{aligned} A(z) = B(e^z) &= B(\zeta^2) \\ &= B_0(\zeta^2) + o(\zeta^{\ell-k}) \end{aligned}$$

as $\zeta \to \infty$. Thus,

$$B_0(\zeta^2) = K_\ell \zeta^{2\ell} + K_{\ell-1}\zeta^{2\ell-2} + \cdots + K_{(\ell-k)/2}\zeta^{\ell-k}.$$

By Proposition 1, $f'(z)/f(z) = R(e^{z/2})$, where $R(\zeta)$ is a rational function with poles of order $\ell/2$ and $k/2$ respectively at ∞ and 0. Then

$$\begin{aligned} \frac{1}{2}\zeta R'(\zeta) + R(\zeta)^2 &= -B(\zeta^2) \\ &= -B_0(\zeta^2) + o(\zeta^{\ell-k}). \end{aligned} \tag{2.11}$$

Taking the square root on both sides gives first the left-hand side

$$R(\zeta)\left(1+\frac{1}{2}\zeta\frac{R'(\zeta)}{R(\zeta)^2}\right)^{1/2} = R(\zeta)+\frac{1}{4}\frac{\zeta R'(\zeta)}{R(\zeta)}+o(1)$$
$$= R(\zeta)+\ell/4+O(\zeta^{-1}),$$

and then the right-hand side

$$cB_0(\zeta^2)^{1/2}+o(\zeta^{-k}),\quad c^2+1=0.$$

We obtain an asymptotic formula for

$$R(\zeta)=cB_0(\zeta^2)^{1/2}+W \tag{2.12}$$

where

$$W=-\ell/4+o(1),\quad W'=o(\zeta^{-1}),$$

as $\zeta\to\infty$. Substituting the formulae back into (2.11) we get

$$\frac{1}{2}\zeta\left(\frac{c}{2}B_0'(\zeta^2)2\zeta B_0^{-1/2}+W'\right)+c^2B_0(\zeta^2)+2cWB_0(\zeta^2)^{1/2}+W^2$$
$$= -B_0(\zeta^2)+o(\zeta^{\ell-k}),$$

as $\zeta\to\infty$. That is,

$$\frac{1}{2}\zeta^2c\frac{B_0'(\zeta^2)}{B_0(\zeta^2)^{1/2}}+\frac{1}{2}\zeta W'+2cWB_0(\zeta^2)^{1/2}+W^2=o(\zeta^{\ell-k}),$$

or

$$2cW+\frac{1}{2}\zeta^2c\frac{B_0'(\zeta^2)}{B_0(\zeta^2)}+\frac{\ell^2}{16}B_0(\zeta^2)^{-1/2}=o(\zeta^{-k})$$

because $B_0(\zeta^2)^{1/2}$ has a pole of order ℓ at ∞. Solving this equation for W and substituting the result into (2.12) yields

$$R(\zeta)=cB_0(\zeta^2)^{1/2}-\frac{1}{4}\frac{\zeta^2B_0'(\zeta^2)}{B_0(\zeta^2)}-\frac{\ell^2}{32c}B_0(\zeta^2)^{-1/2}+o(\zeta^{-k})$$
$$=cB_0(\zeta^2)^{1/2}-\frac{1}{4}\zeta^2\frac{B_0'(\zeta^2)}{B_0(\zeta^2)}+o(\zeta^{-k}),$$

since $B_0(\zeta^2)^{-1/2}=O(\zeta^{-\ell})$, as $\zeta\to\infty$ and $\ell\geq k\geq 0$ by our assumption. Now Proposition 1 implies that $R(\zeta)$ has a pole of order k at 0, we conclude from the above derivation that $R(\zeta)$ can be determined by computing the Laurent series of (2.10) down to the term ζ^{-k}, which is as described in (2.10).

In order to see that the coefficients $K_j, j=-k,\cdots,(\ell-k)/2-1$ are completely determined by those of $K_j, j=\ell,\cdots,(\ell-k)/2$, we only need to note from (2.11) that $B_0(\zeta^2)^{1/2}$ depends entirely on the coefficients $K_j, j=\ell,\cdots,(\ell-k)/2$, and that $R(\zeta)$ is given by (2.12) up to the term ζ^{-k}. This completes the proof of the theorem.

Remark We note that for the proof of the theorem in the original version of this paper, the author used the comparing coefficients method. The current asymptotic method suggested by the referee is much shorter. On the other hand, although (2.10) can be computed by a symbolic computation package, we show in section 4 how to use comparing coefficients method to find the f in (2.2).

As an application, we use Theorem 2 to obtain the following result.

Theorem 3. *Let $\ell > k = 0$ be an even integer. Suppose the equation*

$$f'' + \left(K_\ell e^{\ell z/2} + \sum_{j=0}^{\ell/2} K_j e^{jz}\right) f = 0, \tag{2.13}$$

$K_\ell \neq 0$, admits a zero-free solution f, then $K_j = 0, j = 1, \cdots, \ell/2 - 1$, and

$$f(z) = \exp\left(\frac{2c}{\ell} K_\ell^{1/2} e^{\ell z/2} + \left(\frac{c}{2}\frac{K_{\ell/2}}{K_\ell^{1/2}} - \frac{\ell}{4}\right) z\right), \tag{2.14}$$

where $c^2 + 1 = 0$.

We immediately obtain

Corollary 2. *Let ℓ and $k = 0$ be as in Theorem 3 and f be a non-trivial solution to (2.13). Suppose $K_j \neq 0$ for some $j \in \{1, \cdots, \ell/2 - 1\}$, then $\lambda(f) > 0$.*

Proof of Theorem 3 Let $B(\zeta)$ be as in (2.13). We get $B(\zeta^2) = K_\ell \zeta^{2\ell} + K_\ell \zeta^\ell + o(\zeta^\ell)$. Then (2.9) gives

$$B_0(\zeta) = K_\ell \zeta^\ell + K_{\ell/2}\zeta^{\ell/2}.$$

Thus,

$$\begin{aligned} B_0(\zeta^2)^{1/2} &= K_\ell^{1/2}\zeta^\ell \left(1 + \frac{K_{\ell/2}}{K_\ell}\zeta^{-\ell}\right)^{1/2} \\ &= K_\ell^{1/2}\zeta^\ell + \frac{1}{2}\frac{K_{\ell/2}}{K_\ell^{1/2}} + o(\zeta^{-\ell}) \end{aligned}$$

as $\zeta \to \infty$, and

$$\begin{aligned} \frac{1}{4}\zeta^2 \frac{B_0'(\zeta^2)}{B_0(\zeta^2)} &= \frac{1}{8}\left(2\zeta^2 \frac{B_0'(\zeta^2)}{B_0(\zeta^2)}\right) = \frac{1}{8}\left(\frac{2\ell K_\ell \zeta^{2\ell} + \ell K_{\ell/2}\zeta^\ell}{K_\ell \zeta^{2\ell} + K_{\ell/2}\zeta^\ell}\right) \\ &= \frac{\ell}{4}\left(\frac{1 + K_{\ell/2}/2\zeta^{-\ell}}{1 + K_{\ell/2}\zeta^{-\ell}}\right) \\ &= \frac{\ell}{4} + o(1). \end{aligned}$$

Thus, (2.10) becomes

$$cB_0(\zeta^2)^{1/2} - \frac{\zeta^2 B_0'(\zeta^2)}{4B_0(\zeta^2)} = cK_\ell^{1/2}\zeta^\ell + \left(\frac{c}{2}\frac{K_{\ell/2}}{K_\ell^{1/2}} - \frac{\ell}{4}\right) + o(1).$$

We determine $R(\zeta)$ by the above Laurent series up to $\zeta^{-k} = \zeta^0$, i.e., up to the constant term. That is,

$$R(\zeta) = cK_\ell^{1/2}\zeta^\ell + \left(\frac{c}{2}\frac{K_{\ell/2}}{K_\ell^{1/2}} - \frac{\ell}{4}\right).$$

Thus,

$$\frac{f'}{f}(z) = cK_\ell^{1/2} e^{\ell z/2} + \left(\frac{c}{2}\frac{K_{\ell/2}}{K_\ell^{1/2}} - \frac{\ell}{4}\right),$$

and this is precisely (2.14) after an integration on both sides.

Remark The original proof was again proved by the method of "comparison of coefficients".

We next consider the case in which (1.1) admits a solution f with $0 < \lambda(f) < +\infty$. In [**4**], Bank proved the following result.

Theorem C. *Let $\ell > 0$ be an even integer and $k = 0$ in (1.2) for $B(\zeta)$. Suppose (1.1), with $A(z) = B(e^z)$, admits a pair of linearly independent solutions f_1 and f_2 with $\lambda(f_i) < +\infty, i = 1, 2$ and that f_1 has the representation (2.2), then the representation for f_2 is given by*

$$f_2(z) = \Phi(e^z) \exp\Big(- \sum_{j=-k/2}^{\ell/2} d_j e^{jz} + dz\Big) \tag{2.15}$$

where $\Phi(\zeta)$ is a polynomial with only simple zeros and the constants $d_j, j = -k/2, \cdots, \ell/2$ and d are the same as those in (2.2). Moreover, we have the relation

$$2d + \ell/2 + \deg\psi(\zeta) + \deg\Phi(\zeta) = 0. \tag{2.16}$$

Although this result only describes solutions to (1.1) with finite exponent of convergence of zeros when ℓ is even, it can easily be extended to cover the case when ℓ is odd, as noted by Bank in [**4**, Section 5], that

$$4d + \ell/2 + \deg\psi(\zeta) + \deg\Phi(\zeta) = 0. \tag{2.17}$$

Theorem 4. *Let $\ell > 0$ be an odd integer and $k = 0$ in (1.2). Suppose (1.1), with $A(z) = B(e^z)$, admits a pair of linearly independent solutions f_1 and f_2 with $\lambda(f_i) < +\infty, i = 1, 2$, and f_1 has representation (2.1), then*

$$f_2(z) = \Phi(e^{z/2}) \exp\Big(- \sum_{j=0}^{\ell} d_j e^{jz/2} + dz\Big). \tag{2.18}$$

In addition to (2.17), we have $\deg\psi(\zeta) = \deg\Phi(\zeta)$.

We remark that given d is the same in (2.2) and (2.15), the $\deg\psi(\zeta)$ is not necessarily equal to $\deg\Phi(\zeta)$ when ℓ is even. For example, equation (1.9) admits two linearly independent solutions in (1.10), and $\deg\psi = s-1$ while $\deg\Phi = s$.

Proof: We obtain (2.18) from considering (1.1) with $A(z) = B(e^{2z}), g(z) = f(2z)$. Then $\deg B(\zeta^2)$ is 2ℓ, which is even, and we could then apply Theorem C to obtain (2.17).

Let $f(z) = \phi(e^{z/2}) \exp\Big(P(e^{z/2}) + dz\Big)$ be one of the two linearly independent solutions of f_1 and f_2, where $\deg\phi(\zeta) = s$. Substituting $\zeta = e^{z/2}$ in $f(z)$ as before yields

$$\frac{f'}{f}(z) = R(\zeta) = \frac{1}{2}\zeta P'(\zeta) + d + s/2 + o(1)$$

as $\zeta \to \infty$. Thus,

$$-B(\zeta^2) = \frac{f''}{f}(z) = \frac{1}{2}\zeta R'(\zeta) + R(\zeta)^2$$

and

$$\begin{aligned} cB(\zeta^2)^{1/2} &= R(\zeta)\Big(1+\zeta R'(\zeta)/2R(\zeta)^2\Big)^{1/2} \\ &= R(\zeta)+\frac{1}{4}\zeta\frac{R'(\zeta)}{R(\zeta)}+o(1) \\ &= \frac{1}{2}\zeta P'(\zeta)+d+s/2+\frac{1}{4}\zeta\frac{R'(\zeta)}{R(\zeta)}+o(1), \end{aligned} \tag{2.19}$$

as $\zeta \to \infty$, where $c^2+1=0$. It is easy to see that the constant term on the right-side of the (2.19) is $d+s/2+\ell/4$, while the left-side of the (2.19) gives no contribution of any constant term at infinity since $B(\zeta)$ has an odd degree. Hence $d+\ell/4+s/2=0$. This proves that $\deg\psi=\deg\Phi$ in (2.19).

Corollary 3. *Let $\ell>0$ be an odd integer and $k=0$ in (1.2). Suppose the differential equation (1.1) with its coefficient given by $A(z)=B(e^z)$ admits a pair of linearly independent solutions f_1 and f_2 with $\max\{\lambda(f_1),\lambda(f_2)\}<+\infty$. Then $\lambda(f_1),\lambda(f_2)$ equal to either 0 or 1 simultaneously.*

3. Examples

Although Theorem 1 shows that all $K_i=0$, $i=1,\cdots,\ell-1$ if (1.1) admits a zero-free solution when ℓ is odd, the converse of it is not true.

Example [10] Let $\ell>0$ be an odd integer. The equation

$$f''+\Big(K_\ell e^{\ell z}-\ell^2(2s+1)^2/16\Big)f=0$$

admits two linearly independent solutions of the forms

$$\psi_i(e^{z/2})\exp\Big(d_i e^{\ell z/2}-(2s+1)\ell z/4\Big),$$

where $\deg\psi_i(\zeta)=s\ell, d_i^2+1=0, i=1,2$. In fact, we can find a polynomial $\tilde{\psi}$ of degree s such that $\psi(\zeta)=\tilde{\psi}(\zeta^\ell)$. We also note that (2.17) becomes $4d+\ell+2s\ell=0$. This example shows that the distinction between zero-free solutions and solutions with $0<\lambda(f)<+\infty$.

Example [14] The equation

$$f''+\Big(-\frac{9}{4}e^{3z}-\frac{3}{2}\sqrt[3]{3}e^{2z}-\frac{3}{4}\sqrt[3]{9}e^{z}-\frac{25}{16}\Big)f=0$$

admits a solution

$$f(z)=e\Big(e^{z/2}-\frac{1}{\sqrt[3]{3}}\Big)\exp\Big(e^{3z/2}+\sqrt[3]{3}e^{z/2}-\frac{5}{4}z\Big)$$

where $\psi(\zeta)=e(\zeta-1/\sqrt[3]{3})$. Thus $\deg\psi=1$ and $\deg(\zeta^2+\sqrt[3]{3}\zeta)=3$ and $d=-5/4$. Thus (2.17) becomes

$$4(-5/4)+3+2(1)=0.$$

This example shows that there exists a (1.1), with $\ell>0$, which admits a non-trivial solution with $\lambda(f)>0$ but $K_i\neq 0$ for some $i\in\{1,\cdots,\ell-1\}$.

Example Suppose equation (1.1) with $A(z) = B(e^z)$,

$$B(\zeta) = K_4\zeta^4 + K_3\zeta^3 + K_2\zeta^2 + K_1\zeta + K_0$$

admits a zero-free solution. Then according to Theorem 2 we have

$$B_0(\zeta) = K_4\zeta^4 + K_3\zeta^3 + K_2\zeta^2,$$

and so

$$\begin{aligned} B_0(\zeta^2)^{1/2} &= K_4^{1/2}\zeta^4\big(1 + K_3/K_4\,\zeta^{-2} + K_2/K_4\,\zeta^{-4}\big)^{1/2} \\ &= K_4^{1/2}\zeta^4 + \frac{1}{2}\frac{K_3}{K_4^{1/2}}\zeta^2 + \left(\frac{1}{2}\frac{K_2}{K_4^{1/2}} - \frac{1}{8}\frac{K_3^2}{K_4^{3/2}}\right) + o(1). \end{aligned}$$

Similarly, $B_0(\zeta^2)^{-1/2} = o(1)$, and $\zeta^2 B_0'(\zeta^2)/4B_0(\zeta^2) = \ell/4(1+o(1)) = 1 + o(1)$, as $\zeta \to \infty$. Hence according to Theorem 2

$$\frac{f'}{f}(z) = R(e^{z/2}) = cK_4^{1/2}e^{2z} + \frac{c}{2}\frac{K_3}{K_4^{1/2}}e^z + \left(\frac{c}{2}\left(\frac{K_2}{K_4^{1/2}} - \frac{K_3^2}{4K_4^{3/2}}\right) - 1\right),$$

and thus

$$f(z) = \exp\left\{\frac{c}{2}K_4^{1/2}e^{2z} + \frac{c}{2}\frac{K_3}{K_4^{1/2}}e^z + \left(cK_4^{1/2}\left(\frac{K_2}{2K_4} - \frac{K_3^2}{8K_4^2}\right) - 1\right)z\right\},$$

in which $c^2 + 1 = 0$. We deduce

$$d_2 = \frac{c}{2}K_4^{1/2},\ d_1 = \frac{c}{2}\frac{K_3}{K_4^{1/2}} \quad \text{and} \quad d = cK_4^{1/2}\left(\frac{K_2}{2K_4} - \frac{K_3^2}{8K_4^2}\right) - 1.$$

Substituting f back into (1.1) gives

$$K_1 = \frac{c}{2}\frac{K_3}{\sqrt{K_4}} - \frac{1}{4}\frac{K_3^2}{K_4} + K_2K_3, \quad K_0 = -\left(cK_4^{1/2}\left(\frac{1}{2}\frac{K_2}{K_4} - \frac{K_3}{8K_4}\right) - 1\right)^2,$$

in (1.2). This example clearly shows that K_4, K_3 and K_2 completely determine d_2, d_1 and d, and K_1 and K_0.

4. The method of comparing coefficients

Let us consider again the coefficient of (1.1) with $\ell > 0$ being an even integer. By theorem 2 although symbolic computational software can be used to compute any zero-free solution f of (1.1) and the relations between the constants K_j in (1.2), the method is less effective when used to discover new properties of (1.1), especially about new relations of $K_j, j = \ell, \cdots, 0$. Thus, we provide a second approach to the problem here. The comparing coefficient method is not only instrumental in the initial stage of this work, but it was used successfully to consider certain higher-order equations in [**8**] and [**9**].

Suppose (1.1) admits a zero-free solution f with $A(z) = B(e^z)$ where $B(\zeta)$ is given in (1.2) with $\ell > k \geq 0$ both even. Then according to Proposition 1 (2.2) with $\psi(\zeta) \equiv 1, f(z) = \exp\Big(P(e^z) + dz\Big)$ where $P(\zeta) = \sum_{-k/2}^{\ell/2} d_j\zeta^j$. Substituting P into (1.1) yields

$$\zeta^2P''(\zeta) + \Big(\zeta P'(\zeta)\Big)^2 + (2d+1)\zeta P'(\zeta) + \left(d^2 + \sum_{j=-k}^{\ell} K_j\zeta^j\right) = 0. \tag{4.1}$$

We immediately deduce

$$\left(\frac{\ell}{2}\right)^2 {d_{\ell/2}}^2 + K_\ell, \quad \left(\frac{-k}{2}\right)^2 {d_{-k/2}}^2 + K_{-k} = 0, \tag{4.2}$$

by comparing the coefficients of ζ^ν with $\nu = \ell$ and $-k$ respectively in (4.1). Similarly, we have

($\mathbf{A}$) ν even: $\underline{\ell/2+1 \leq \nu \leq \ell-1 \text{ and } -(k-1) \leq \nu \leq (k/2+1) \text{ when } k \geq 4}$

$$\left(\frac{\nu}{2}\right)^2 {d_{\nu/2}}^2 + 2 \sum_{\substack{i+j=\nu \\ i<j}} ij\, d_i\, d_j + K_\nu = 0 \tag{4.3}$$

$\underline{\ell \leq \nu \leq \ell/2, -k/2 \leq \nu \leq -1}$,

$$\nu(2d+\nu)d_\nu + \left(\frac{\nu}{2}\right)^2 {d_{\nu/2}}^2 + 2 \sum_{\substack{i+j=\nu \\ i<j}} ij\, d_i\, d_j + K_\nu = 0; \tag{4.4}$$

($\mathbf{B}$) ν odd: $\underline{\ell/2+1 \leq \nu \leq \ell-1, \text{ and } -(k-1) \leq \nu \leq (k/2+1) \text{ when } k \geq 4}$

$$2 \sum_{\substack{i+j=\nu \\ i<j}} ij\, d_i\, d_j + K_\nu = 0, \tag{4.5}$$

$\underline{\ell \leq \nu \leq \ell/2, \text{ and } -k/2 \leq \nu \leq -1}$

$$\nu(2d+\nu)d_\nu + 2 \sum_{\substack{i+j=\nu \\ i<j}} ij\, d_i\, d_j + K_\nu = 0. \tag{4.6}$$

In particular, we have

$$d^2 + 2 \sum_{\substack{i+j=0 \\ i<j}} ij\, d_i\, d_j + K_0 = 0 \tag{4.7}$$

for the constant term. The above algorithm can be verified by finite induction.

Example Suppose equation (1.1), with

$$A(z) = 36e^{6z} + 72e^{5z} + 24e^{4z} + (12+6i)e^{3z} - 5e^{2z} + (6-i)e^{z} - \frac{3i}{2}e^{-z} + \frac{1}{4}e^{-2z},$$

admits a solution f with $\lambda(f) = 0$. Then Proposition 1 immediately implies

$$f(z) = \exp\left(\sum_{j=-1}^{3} d_j e^{jz} + dz\right)$$

where $d_j, j = -1, \cdots, 3$ and d are constants to be determined.

Applying the algorithm (4.2)–(4.7) above, we easily arrive at the following equations

$$9d_3^2 + K_6 = 0, \tag{4.8}$$

$$12d_2\, d_3 + K_5 = 0, \tag{4.9}$$

$$4d_2^2 + 6d_1\, d_3 + K_4 = 0, \tag{4.10}$$

$$4d_1\, d_2 + 3(2d+3)d_3 + K_3 = 0, \tag{4.11}$$

$$d_1^2 + 2(2d+2)d_2 - 6d_{-1}\, d_3 + K_2 = 0, \tag{4.12}$$

$$-4d_2\, d_{-1} + (2d+1)\, d_1 + K_1 = 0, \tag{4.13}$$

$$d^2 - 2d_1\, d_{-1} + K_0 = 0, \tag{4.14}$$

$$(1-2d)\, d_{-1} + K_{-1} = 0, \tag{4.15}$$

$$d_{-1}^2 + K_{-2} = 0, \tag{4.16}$$

where $K_6 = 36, K_5 = 72, K_4 = 24, K_3 = -(12+6i), K_2 = -5, K_1 = -6-i, K_0 = 0, K_{-1} = -3i/2, K_{-2} = 1/4$. Equation (4.8) gives $d_3 = \pm 2i$. We consider the value $d_3 = 2i$ first and treat $-2i$ later. Substituting d_3 into (4.9) yields $d_2 = 3i$. Substituting the values for d_3 and d_2 into (4.10) yields $d_1 = -i$. Substituting the values for d_3, d_2, d_1 and K_3 into (4.11) gives $d = -1$. Substituting all the above values for d_3, d_2, d_1, d and K_2 into (4.12) gives $d_{-1} = i/2$. It is now a routine exercise to check that the values for d_3, d_2, d_1, d all satisfy the equations (4.13)–(4.16). Thus

$$f(z) = \exp\left(2ie^{3z} + 3ie^{2z} - ie^{z} + i/2e^{-z} - z\right) \tag{4.17}$$

is the desired solution.

If we had chosen $d_3 = -2i$ in (4.8) instead, then similar calculations show that the equations (4.8)–(4.16) are inconsistent. In fact, this will give a different

$$A(z) = 36e^6 + 72e^{5z} + 24e^{4z} - (12-6i)e^{3z} - (5-12i)e^{2z} - (6+3i)e^{z} - 5i/2e^{-z} + 1/4e^{-2z} - 3.$$

Hence (4.17) is, except for a constant multiple, the only zero-free solution for (1.1) with this particular choice of $A(z)$. This conclusion is in line with the known result of Fact(**B**) in [**3**, page 108] that the equation (1.1), with both ℓ and k even and $k > 0$ in (1.2), cannot process two linearly independent solutions each with finite exponent of convergence of zeros.

Finally, we note that a set of equations similar to (4.2)–(4.7) exist when $\ell > 0$ is odd. One can use these formulae to prove Theorem 1.

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