7

Sequences and Series of Functions

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7.1 DISCUSSION OF MAIN PROBLEM

7.1 **Convergence Pointwise**: Suppose $\{f_n\}$, $n = 1, 2, 3, \dots$, is a sequence of functions defined on a set E, and suppose that the sequence of numbers $\{f_n(x)\}$ converges for every $x \in E$. We can then define a function f by

$$f(x) = \lim_{n\to\infty} f_n(x), \qquad x \in E.$$

We say that f is the limit of $\{f_n\}$, or $\{f_n\}$ converges to f pointwise on E.

Similarly, if $\sum f_n(x)$ converges for every $x \in E$, and if we define

$$f(x) = \sum_{n=1}^{\infty} f_n(x), \qquad x \in E,$$

we say that the function f is the sum of the series $\sum f_n$.

- **Problems**: The main problem which arises is to determine whether important properties of functions are preserved under the limit operations above. For instance, if the functions f_n are continuous, or differentiable, or integrable, is the same true of the limit function f? What are the relations between f_n' and f', say, or between the integrals of f_n and that of f?

  To say that f is continuous at a limit point x means

$$\lim_{t\to x} f(t) = f(x).$$

Hence, to ask whether the limit of a sequence of continuous functions is continuous is the same as to ask whether

$$\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t),$$

i.e., whether the orders in which limit processes are carried out can be inter-changed: on the left, we first let $n \to \infty$, then $t \to x$; on the right side, $t \to x$ first, then $n \to \infty$.

7.2 **Example** For $m = 1, 2, 3, \ldots$, $n = 1, 2, 3, \ldots$, let

$$s_{m,n} = \frac{m}{m+n}.$$

Then, for each fixed n,

$$\lim_{m\to\infty} s_{m,n} = 1,$$

so that

$$\lim_{n\to\infty}\lim_{m\to\infty} s_{m,n} = 1.$$

On the other hand, for every fixed m,

$$\lim_{n\to\infty} s_{m,n} = 0,$$

so that

$$\lim_{m\to\infty}\lim_{n\to\infty} s_{m,n} = 0.$$

7.3 **Example** For real x, let

$$f_n(x) = \frac{x^2}{(1+x^2)^n}, \qquad n = 1, 2, 3, \ldots.$$

and consider

$$f(x) = \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}.$$

Since $f_n(0) = 0$, we have $f(0) = 0$. For $x \neq 0$, the last series converges to $1 + x^2$. Hence

$$f(x) = \begin{cases} 0, & x = 0, \\ 1 + x^2 & x \neq 0. \end{cases}$$

This example shows that a convergent series of continuous functions may have a discontinuous sum.

7.5 **Example** For real x, let

$$f_n(x) = \frac{\sin nx}{\sqrt{n}}, \qquad n = 1, 2, 3, \ldots,$$

and

$$f(x) = \lim_{n\to\infty} f_n(x) = 0$$

Then $f'(x) = 0$, and

$$f'_n(x) = \sqrt{n}\cos nx,$$

so that $\{f'_n\}$ does not converge to f'.

7.6 **Example** For $0 \le x \le 1$, let

$$f_n(x) = nx(1-x^2)^n, \qquad n = 1, 2, 3, \ldots.$$

For $0 < x \le 1$, it is clear that

$$\lim_{n\to\infty} f_n(x) = 0.$$

For $x = 0$, $f_n(0) = 0$. Hence

$$\lim_{n\to\infty} f_n(x) = 0, \qquad 0 \le x \le 1.$$

It is easy to calculate

$$\int_0^1 f_n(x)\,\mathrm{d}x = \frac{n}{2n+2}.$$

Thus,

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,\mathrm{d}x = \frac{1}{2} \neq 0 = \int_0^1 \left[\lim_{n\to\infty} f_n(x)\right]\mathrm{d}x.$$

7.2 UNIFORM CONVERGENCE

7.7 **Uniform Convergence**: A sequence of functions $\{f_n\}$ converges uniformly on E to a function f if for every $\epsilon > 0$, there is an integer N such that $n \ge N$ implies

$$|f_n(x) - f(x)| < \epsilon$$

for all $x \in E$.

A series $\sum f_n(x)$ converges uniformly on E if the sequence $\{s_n\}$ of partial sums,

$$s_n(x) = \sum_{i=1}^{n} f_i(x),$$

converges uniformly on E.

- It is clear that if $\{f_n\}$ converges uniformly to f on E, then $\{f_n\}$ converges pointwise to f on E.

7.8 **Theorem** **(Cauchy Criterion)** The sequence of functions $\{f_n\}$, defined on E, converges uniformly on E if and only if for every $\epsilon > 0$ there exists an integer N such that $m, n \ge N$ implies

$$|f_n(x) - f_m(x)| < \epsilon$$

for all $x \in E$.

Proof Suppose the Cauchy condition holds. By Theorem 3.11, the sequence $\{f_n(x)\}$ converges for each fixed x to a limit which we may call $f(x)$. Hence the sequence $\{f_n\}$ converges to f on E. Let $\epsilon > 0$ be given. There is an integer N such that $m, n \geq N$ implies

$$|f_n(x) - f_m(x)| < \epsilon/2.$$

If we let $m \to \infty$ in the inequality, by Theorem 3.19, we have

$$|f_n(x) - f(x)| \leq \epsilon/2 < \epsilon, \qquad n \geq N,$$

for all $x \in E$. Thus $\{f_n\}$ converges uniformly to f.

Conversely, suppose $\{f_n\}$ converges uniformly to f on E. Then, for any $\epsilon > 0$, there is an integer N such that $n \geq N$ implies

$$|f_n(x) - f(x)| < \epsilon/2$$

for all $x \in E$. Hence, if $n, m \geq N$, we have

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f_m(x) - f(x)| < \epsilon$$

for all $x \in E$. ∎

7.9 **Theorem** Suppose $\{f_n\}$ converges to f on E. Put

$$M_n = \sup_{x \in E} |f_n(x) - f(x)|.$$

Then $\{f_n\}$ converges uniformly to f on E if and only if $M_n \to 0$ as $n \to \infty$.

Proof The proof is straightforward.

7.10 **Theorem** **(Weierstrass M-Test)** Suppose $\{f_n\}$ is a sequence of functions defined on E satisfying

$$|f_n(x)| \leq M_n, \qquad n = 1, 2, 3, \ldots,$$

for all $x \in E$. Then $\sum f_n$ converges uniformly on E if $\sum M_n$ converges.

Proof If $\sum M_n$ converges, then, for any $\epsilon > 0$, there is an integer N such that $m \geq n \geq N$ implies

$$\sum_{i=n}^{m} M_i < \epsilon.$$

Hence, if $m \geq n \geq N$,

$$\left| \sum_{i=n}^{m} f_i(x) \right| \leq \sum_{i=n}^{m} M_i < \epsilon,$$

for all $x \in E$. It follows from Theorem 7.8 that $\{f_n\}$ converges uniformly to f on E. ∎

7.3 UNIFORM CONVERGENCE AND CONTINUITY

7.11 **Theorem** Suppose $f_n \to f$ uniformly on a set E in a metric space. Let x be a limit point of E, and suppose that

$$\lim_{t\to x} f_n(t) = A_n, \qquad n = 1, 2, 3, \ldots.$$

Then $\{A_n\}$ converges, and

$$\lim_{t\to x} f(t) = \lim_{n\to\infty} A_n.$$

In other words,

$$\lim_{t\to x} \lim_{n\to\infty} f_n(t) = \lim_{n\to\infty} \lim_{t\to x} f_n(t).$$

Proof Let $\epsilon > 0$ be given. Since $\{f_n\}$ converges uniformly on E, there exists an integer N such that $n, m \geq N$ implies

$$|f_n(t) - f_m(t)| < \epsilon,$$

for all $t \in E$. Letting $t \to x$, we have

$$|A_n - A_m| \leq \epsilon,$$

form $n, m \geq N$. Hence, $\{A_n\}$ is a Cauchy sequence, and therefore converges, say to A.

From the inequality

$$|f(t) - A| \leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A|,$$

we now give the estimates for the terms on the right hand side. In fact, since $f_n \to f$ uniformly, we can choose n sufficiently large such that

$$|f(t) - f_n(t)| < \epsilon/3$$

for all $t \in E$, and such that

$$|A_n - A| < \epsilon/3.$$

For this large n, by a condition in the theorem, we can choose a neighborhood V of x such that

$$|f_n(t) - A_n| < \epsilon/3$$

if $t \in V \cap E$, $t \neq x$. Thus, we know that for $t \in V \cap E$, $t \neq x$,

$$|f(t) - A| < \epsilon.$$

That is, $\lim_{t\to x} f(t) = A = \lim_{n\to\infty} A_n$. ∎

7.12 **Theorem** If $\{f_n\}$ is a sequence of continuous functions on E, and if $f_n \to f$ uniformly on E, then f is continuous.

Proof By Theorem 7.11, for every $t \in E$, we have

$$\lim_{t\to x} f(t) = \lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t) = \lim_{n\to\infty} f_n(x) = f(x).$$

Thus, f is continuous on E ■

7.13 **Theorem** Suppose K is compact, and

(a) $\{f_n\}$ is a sequence of continuous functions on K,

(b) $\{f_n\}$ converges pointwise to a continuous function f on K,

(c) $f_n(x) \geq f_{n+1}(x)$ for all $x \in K$, $n = 1, 2, 3, \ldots$.

Then $f_n \to f$ uniformly on K.

Proof Put $g_n = f_n - f$. Then g_n is continuous, $g_n \to 0$, and $g_n \geq g_{n+1}$. We have to prove that $g_n \to 0$ uniformly on K.

Let $\epsilon > 0$ be given. Write

$$K_n = \{x \in K : g_n(x) \geq \epsilon\}, \qquad n = 1, 2, 3, \ldots.$$

Since g_n is continuous, by Theorem 4.8, $K_n \subset K$ is closed for each n. By Theorem 2.35, K_n is compact. Since $g_n \geq g_{n+1}$, we know that $K_n \supset K_{n+1}$. Fix $x \in K$, since $g_n(x) \to 0$, we see that $x \notin K_n$ if n is sufficiently large. Hence $x \notin \bigcap K_n$. In other words, $\bigcap K_n$ is empty. Hence K_N is empty for some N, by Theorem 2.36. It follows that $0 \leq g_n(x) < \epsilon$ for all $x \in K$ and all $n \geq N$. This proves the theorem. ■

- Example Consider $f_n(x) = \dfrac{1}{nx+1}$ on $(0, 1)$. It is clear that f_n is continuous on $(0, 1)$, $f_n(x) \to 0$ on $(0, 1)$, and $f_n \geq f_{n+1}$. However, $\{f_n\}$ does not converge uniformly to 0 on $(0, 1)$. In fact, for $\epsilon = 1/2 > 0$, no matter how large n is, we can always find a point $x \in (0, 1)$ such that

$$\left|\frac{1}{nx+1}\right| \geq \epsilon.$$

7.14 **Metric Space** $\mathscr{C}(X)$: If X is a metric space, we define $\mathscr{C}(X)$ to be the set of all complex-valued, continuous, bounded functions with domain X.

For each $f \in \mathscr{C}(X)$, we define its supremum norm:

$$\|f\| = \sup_{x\in X} |f(x)|.$$

In fact, $\|\cdot\|$ is a norm defined on $\mathscr{C}(X)$. Since $f \in \mathscr{C}(X)$ has to be bounded, $\|f\| < \infty$. If $\|f\| = 0$ only if $f(x) = 0$ for every $x \in X$, that is, only if $f = 0$. If c is a complex number, then $\|cf\| = \sup_{x\in X} |cf(x)| = |c|\,\|f\|$. If $h = f + g$, then $|h(x)| \leq |f(x)| + |g(x)| \leq \|f\| + \|g\|$ for all $x \in X$, which implies $\|f + g\| \leq \|f\| + \|g\|$.

Thus, together with the supremum norm $\|\cdot\|$, $\mathscr{C}(X)$ is a metric space.

- A sequence $\{f_n\}$ converges to f with respect to the metric of $\mathscr{C}(X)$ if and only if $f_n \to f$ uniformly on X.

7.15 **Theorem** The above metric makes $\mathscr{C}(X)$ into a complete metric space.

Proof Let $\{f_n\}$ be a Cauchy sequence in $\mathscr{C}(X)$. For any $\epsilon > 0$, there exists an integer N such that $\|f_n - f_m\| < \epsilon$ if $n, m \geq N$. Since $|f_n(x) - f_m(x)| \leq \|f_n - f_m\|$ for all $x \in X$, by Theorem 7.8, there is a function f with domain X to which $\{f_n\}$ converges uniformly. Since f_n is continuous for every n, by Theorem 7.12, f is continuous. Moreover, since there is an n such that $|f(x) - f_n(x)| < 1$ for all $x \in X$, we know that $|f(x)| \leq |f_n(x) - f(x)| + |f_n(x)| < 1 + \|f_n\|$ which implies that $f(x)$ is bounded. Thus, $f \in \mathscr{C}(X)$. Since $f_n \to f$ uniformly on X, we know that $\|f_n - f\| \to 0$ as $n \to \infty$. Therefore, $\mathscr{C}(X)$ is a complete metric space. ■

7.4 UNIFORM CONVERGENCE AND INTEGRATION

7.16 **Theorem** Let α be monotonically increasing on $[a, b]$. Suppose $f_n \to f$ uniformly on $[a, b]$. If $f_n \in \mathscr{R}(\alpha)$ on $[a, b]$ for $n = 1, 2, 3, \ldots$, then $f \in \mathscr{R}(\alpha)$ on $[a, b]$, and

$$\int_a^b f \, d\alpha = \lim_{n\to\infty} \int_a^b f_n \, d\alpha.$$

In other words,

$$\int_a^b \left[\lim_{n\to\infty} f_n\right] d\alpha = \lim_{n\to\infty} \int_a^b f_n \, d\alpha.$$

Proof We only need to prove the theorem for real functions. Put

$$\epsilon_n = \sup_{x\in[a,b]} |f_n(x) - f(x)|.$$

Then

$$f_n - \epsilon_n \leq f \leq f_n + \epsilon.$$

These inequalities give

$$\int_a^b (f_n - \epsilon_n) \, d\alpha \leq L(f, \alpha) \leq U(f, \alpha) \leq \int_a^b (f_n + \epsilon_n) \, d\alpha.$$

This sequence of inequalities gives

$$0 \leq U(f, \alpha) - L(f, \alpha) \leq 2\epsilon_n[\alpha(b) - \alpha(a)],$$

which implies $f \in \mathscr{R}(\alpha)$ on $[a, b]$, since $\epsilon_n \to 0$ by the hypothesis.

From the following inequality

$$\left|\int_a^b f_n \, d\alpha - \int_a^b f \, d\alpha\right| \leq \epsilon_n[\alpha(b) - \alpha(a)],$$

we let $n \to \infty$ and obtain the desired limit. ∎

- **Corollary**: If $f_n \in \mathscr{R}(\alpha)$ on $[a,b]$ and

$$\sum_{n=1}^{\infty} f_n(x) = f(x)$$

uniformly on $[a,b]$. Then

$$\int_a^b f \, \mathrm{d}\alpha = \sum_{n=1}^{\infty} \int_a^b f_n \, \mathrm{d}\alpha.$$

7.5 UNIFORM CONVERGENCE AND DIFFERENTIATION

7.17 **Theorem** Suppose $\{f_n\}$ is a sequence of differentiable functions on $[a,b]$ such that $\{f_n(x_0)\}$ converges for some $x_0 \in [a,b]$. If $\{f_n'\}$ converges uniformly on $[a,b]$, then $\{f_n\}$ converges uniformly on $[a,b]$, and

$$f'(x) = \lim_{n\to\infty} f_n'(x).$$

Proof To show that $\{f_n\}$ converges uniformly on $[a,b]$, we consider the following estimates: for any $x \in [a,b]$,

$$\begin{aligned} |f_n(x) - f_m(x)| &\le |f_n(x) - f_m(x) - f_n(x_0) + f_m(x_0)| + |f_n(x_0) - f_m(x_0)| \\ &\le |f_n'(t) - f_m'(t)| \cdot |x - x_0| + |f_n(x_0) - f_m(x_0)| \end{aligned}$$

where t is a number between x and x_0.

Let $\epsilon > 0$ be given. Choose N such that $n, m \ge N$ imply

$$|f_n(x_0) - f_m(x_0)| < \epsilon/2,$$

and

$$|f_n'(t) - f_m'(t)| < \frac{\epsilon}{2(b-a)},$$

for all $t \in [a,b]$. Hence, for $n, m \ge N$, and $x \in [a,b]$,

$$|f_n(x) - f_m(x)| < \frac{\epsilon}{2(b-a)} \cdot |x - x_0| + \epsilon/2 \le \epsilon.$$

Thus, $\{f_n\}$ converges uniformly on $[a,b]$.

To prove the desired limit in the theorem, we let f be the limit of $\{f_n\}$. For a fixed $x \in [a,b]$, put

$$\phi_n(t) = \frac{f_n(t) - f_n(x)}{t - x}, \qquad \phi(t) = \frac{f(t) - f(x)}{t - x},$$

where $t \in [a,b]$, $t \neq x$. It is clear that

$$\lim_{t\to x} \phi_n(t) = f_n'(x), \qquad n = 1,2,3,\ldots.$$

By Theorem 7.11, if we can show that $\phi_n \to \phi$ uniformly on $[a,b] \setminus \{x\}$, then

$$\lim_{t\to x} \phi(t) = \lim_{n\to\infty} f_n'(x),$$

which is the desired limit since $f'(x) = \lim_{t\to x} \phi(t)$.

To show that $\{\phi_n\}$ converges uniformly to ϕ on $[a,b] \setminus \{x\}$, we have the following estimates: by the Mean Value Theorem, there exists $\tilde{t}$ between t and x, such that

$$\begin{aligned} |\phi_n(t) - \phi_m(t)| &= \left| \frac{f_n(t) - f_n(x)}{t-x} - \frac{f_m(t) - f_m(x)}{t-x} \right| \\ &= \left| \frac{[f_n(t) - f_m(t)] - [f_n(x) - f_m(x)]}{t-x} \right| \\ &= \left| f_n'(\tilde{t}) - f_m'(\tilde{t}) \right| < \frac{\epsilon}{2(b-a)}, \end{aligned}$$

if $n, m \geq N$, $t \neq x$. Hence $\{\phi_n\}$ converges uniformly, for $t \neq x$. Since $\{f_n\}$ converges to f, we know that $\phi_n(t) \to \phi(t)$ pointwise for $t \neq x$. Thus $\{\phi_n\}$ converges uniformly to ϕ, for $t \neq x$. ∎

7.18 There exists a real continuous function on the real line which is nowhere differentiable. The details are skipped.

7.6 EQUICONTINUOUS FAMILIES OF FUNCTIONS

- **Problem**: we know that every bounded sequence of complex numbers contains a convergent subsequence, and the question arises whether something similar is true for sequences of functions. To make the question more precise, we shall define two kinds of boundedness.

7.19 **Pointwise Bounded, Uniformly Bounded**: Let $\{f_n\}$ be a sequence of functions defined on E.

$\{f_n\}$ is pointwise bounded on E, if for a fixed point, there exists a finite-valued function ϕ defined on E such that

$$|f_n(x)| < \phi(x), \qquad n = 1,2,3,\ldots.$$

$\{f_n\}$ is uniformly bounded on E, if there exists a number M such that

$$|f_n(x)| < M, \qquad n = 1,2,3,\ldots.$$