7.1 Prove that every uniformly convergent sequence of bounded functions is uniformly bounded. Rudin's Ex. 1

Proof Suppose that $f_n \to f$ uniformly on some set E and that for each n, there exists M_n such that $|f_n(x)| \le M_n$ for any $x \in E$. Let N be such that

$$|f_n(x) - f(x)| < 1,$$

whenever $n \ge N$ and $x \in E$. Then for $n \ge N$ and $x \in E$,

$$|f_n(x)| \le |f_n(x) - f(x)| + |f(x) - f_N(x)| + |f_N(x)| < 2 + M_N.$$

Put

$$M = \max\{M_1, \cdots, M_{N-1}, 2 + M_n\}.$$

Then, for any $x \in E$,

$$|f_n(x)| \le M, \qquad n = 1, 2, 3, \ldots.$$

This means that $\{f_n\}$ is uniformly bounded. ■

7.2 If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E, prove that $\{f_n + g_n\}$ converges uniformly on E. If, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, prove that $\{f_n g_n\}$ converges uniformly on E. Rudin's Ex. 2

Proof Let $\epsilon > 0$ be given.

Suppose that $f_n \to f$ uniformly and $g_n \to g$ uniformly for some f and g. Then, there exist N_1 and N_2, such that $n \ge N_1$ implies $|f_n(x) - f(x)| < \epsilon/2$ for any $x \in E$, and $n \ge N_2$ implies $|g_n(x) - g(x)| < \epsilon/2$ for any $x \in E$. Hence, if $n \ge \max\{N_1, N_2\}$, then

$$\big|[f_n(x) + g_n(x)] - [f(x) + g(x)]\big| \le |f_n(x) - f(x)| + |g_n(x) - g(x)| < \epsilon/2 + \epsilon/2 = \epsilon,$$

for any $x \in E$. This means that $\{f_n + g_n\}$ converges uniformly to $f + g$ on E.

If for each n, f_n and g_n are bounded on E, by Problem 1, we know that $\{f_n\}$ and $\{g_n\}$ are uniformly bounded. In other words, there are M and L, such that for any $x \in E$,

$$|f_n(x)| \le M, \quad |g_n(x)| \le L, \qquad n = 1, 2, 3, \ldots.$$

Let N be such that $n \ge N$ implies that for any $x \in E$,

$$|f_n(x) - f(x)| < 1.$$

Then, for any $x \in E$, we have

$$|f(x)| \le |f(x) - f_N(x)| + |f_N(x)| < 1 + M.$$

Since $f_n \to f$ uniformly and $g_n \to g$ uniformly on E, there exist K_1 and K_2 such that $n \ge K_1$ implies that for any $x \in E$

$$|f_n(x) - f(x)| < \frac{\epsilon}{2(L+1)},$$

and $n \geq K_2$ implies that for any $x \in E$

$$|g_n(x) - g(x)| < \frac{\epsilon}{2(M+1)}.$$

Hence, if $n \geq \max\{K_1, K_2\}$, for any $x \in E$, we have

$$\begin{aligned}|f_n(x)g_n(x) - f(x)g(x)| &\leq |f_n(x)g_n(x) - f_n g(x)| + |f_n(x)g(x) - f(x)g(x)| \\ &= |f_n(x)|\,|g_n(x) - g(x)| + |g(x)|\,|f_n(x) - f(x)| \\ &< M \cdot \frac{\epsilon}{2(M+1)} + L \cdot \frac{\epsilon}{2(L+1)} < \epsilon.\end{aligned}$$

This means that $f_n g_n \to fg$ uniformly on E. ■

7.3 Construct sequences $\{f_n\}$, $\{g_n\}$ which converge uniformly on some set E, but such that $\{f_n g_n\}$ does not converge uniformly on E. Rudin's Ex. 3

Proof Let

$$f_n(x) = \frac{1}{n}, \qquad x \in (0,1),$$

and

$$g_n(x) = \frac{1}{x}, \qquad x \in (0,1).$$

It is easy to check that $f_n \to 0$ uniformly on $(0,1)$, and $g_n(x) \to \frac{1}{x}$ uniformly on $(0,1)$. For any fixed $x \in (0,1)$,

$$f_n(x)g_n(x) = \frac{1}{nx} \to 0.$$

In other words, $\{f_n g_n\}$ converges to 0 pointwise. However, $\{f_n g_n\}$ does not converge to 0 uniformly, since

$$f_n(n)g_n(n) = 1, \qquad n = 1, 2, 3, \ldots.$$