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1

Measure Theory

1.1 CONSTRUCTION OF THE LEBESGUE MEASURE

10.1 **Length:** By Redin's Exercise 29 of Chapter 2 (pp. 45), every open set in \mathbb{R} is the union of an at most countable collection of disjoint segments. Hence, if U is a bounded open set, we write $U = \bigsqcup_{i=1}^{\infty} (a_i, b_i)$, where $\{(a_i, b_i)\}$ are disjoint, and we define the *length*

$$\lambda(U) = \sum_{i=1}^{\infty} (b_i - a_i).$$

10.2 **Theorem** Suppose U and $\{V_i\}$ are bounded open sets in \mathbb{R} , and $U \subset \cup V_i$. Then

$$\lambda(U) \leq \sum \lambda(V_i).$$

In particular, $U \subset V$ implies $\lambda(U) \leq \lambda(V)$.

Proof If we expressing each V_i as a union of disjoint segments, we only need to show that $U \subset \cup (c_j, d_j)$ implies $\lambda(U) \leq \sum (d_j - c_j)$.

In fact, since U is open, we write $U = \sqcup (a_i, b_i)$. For given $\epsilon > 0$, the compact set $K = [a_1 + \epsilon, b_1 - \epsilon] \cup \dots \cup [a_n + \epsilon, b_n - \epsilon]$ is covered by the open segments $\{(c_j, d_j)\}$. Hence there exists a finite cover:

$$K \subset (c_{j_1}, b_{j_1}) \cup \dots \cup (c_{j_k}, b_{j_k}).$$

Thus,

$$\sum_{i=1}^n (b_i - a_i) - 2n\epsilon = \sum_{i=1}^n \lambda(a_i + \epsilon, b_i - \epsilon) \leq \sum_{l=1}^k (d_{j_l} - c_{j_l}) \leq \sum (d_j - c_j).$$

Since ϵ and n are arbitrary, we obtain

$$\lambda(U) = \sum (b_i - a_i) \leq \sum (d_j - c_j) = \sum \lambda(V_j).$$

10.3 **Theorem** Suppose U and V are bounded open sets in \mathbb{R} . Then

$$\lambda(U \cup V) = \lambda(U) + \lambda(V) - \lambda(U \cap V).$$

Proof Let $U = \sqcup(a_i, b_i)$, $V = \sqcup(c_j, d_j)$. Put $U_n = (a_1, b_1) \sqcup \cdots \sqcup (a_n, b_n)$ and $V_n = (c_1, d_1) \sqcup \cdots \sqcup (c_n, d_n)$. Since U and V are bounded sets, by Theorem 10.2, both $\sum(b_i - a_i) < +\infty$ and $\sum(d_j - c_j) < +\infty$. Hence, for given $\epsilon > 0$, there is N , such that $n \geq N$ implies

$$0 \leq \lambda(U) - \lambda(U_n) = \sum_{i>n} (b_i - a_i) < \epsilon, \quad 0 \leq \lambda(V) - \lambda(V_n) = \sum_{j>n} (d_j - c_j) < \epsilon.$$

Since

$$\begin{aligned} U_n \cup V_n &\subset U \cup V = (U_n \cup V_n) \cup (U - U_n) \cup (V - V_n), \\ U_n \cap V_n &\subset U \cap V \subset (U_n \cap V_n) \cup (U - U_n) \cup (V - V_n), \end{aligned}$$

we have

$$\begin{aligned} \lambda(U_n \cup V_n) &\leq \lambda(U \cup V) \\ &\leq \lambda(U_n \cup V_n) + \lambda(U - U_n) + \lambda(V - V_n) \\ &< \lambda(U_n \cup V_n) + 2\epsilon, \\ \lambda(U_n \cap V_n) &\leq \lambda(U \cap V) \\ &\leq \lambda(U_n \cap V_n) + \lambda(U - U_n) + \lambda(V - V_n) \\ &< \lambda(U_n \cap V_n) + 2\epsilon. \end{aligned}$$

It follows that

$$\begin{aligned} \lim_{n \rightarrow \infty} \lambda(U_n) &= \lambda(U), & \lim_{n \rightarrow \infty} \lambda(V_n) &= \lambda(V), \\ \lim_{n \rightarrow \infty} \lambda(U_n \cup V_n) &= \lambda(U \cup V), & \lim_{n \rightarrow \infty} \lambda(U_n \cap V_n) &= \lambda(U \cap V). \end{aligned}$$

Since U_n and V_n are finite unions of segments, it is easy to show

$$\lambda(U_n \cup V_n) = \lambda(U_n) + \lambda(V_n) - \lambda(U_n \cap V_n).$$

By taking $n \rightarrow \infty$, we get the equality in the theorem. ■

10.4 **Length of Closed Set:** Let K be a bounded closed set, and U is a bounded open set containing K . Define the length of K to be:

$$\lambda(K) = \lambda(U) - \lambda(U - K).$$

- $\lambda(K)$ is well-defined for any U . In fact, let V be a bounded open set containing K . For any open set S , with $K \subset S \subset U$, by Theorem 10.3, we have

$$\begin{aligned} \lambda(U) &= \lambda((U - K) \cup S) \\ &= \lambda(U - K) + \lambda(S) - \lambda((U - K) \cap S) \\ &= \lambda(U - K) + \lambda(S) - \lambda(S - K). \end{aligned}$$

Hence, for $S = U \cap V$,

$$\lambda(V) = \lambda(V - K) + \lambda(U \cap V) - \lambda(U \cap V - K).$$

This identity and Theorem 10.3 imply

$$\begin{aligned} \lambda(U) &= \lambda((U - K) \cup (U \cap V)) \\ &= \lambda(U - K) + \lambda(U \cap V) - \lambda((U - K) \cap (U \cap V)) \\ &= \lambda(U - K) + \lambda(U \cap V) - \lambda((U \cap V) - K) \\ &= \lambda(U - K) + \lambda(V) - \lambda(V - K), \end{aligned}$$

or

$$\lambda(U) - \lambda(U - K) = \lambda(V) - \lambda(V - K), \quad \text{for any } U, V \text{ containing } K.$$

10.5 **Lebesgue Outer Measure, Lebesgue Inner Measure, Lebesgue Measurable:** For any bounded set A in \mathbb{R} , the Lebesgue outer measure of A is defined as

$$\mu^*(A) = \inf\{\lambda(U) : U \supset A, U \text{ open}\};$$

the greatest lower bound, the infimum

the Lebesgue inner measure of A is

$$\mu_*(A) = \sup\{\lambda(K) : K \subset A, K \text{ closed}\}.$$

If $\mu^*(A) = \mu_*(A)$, then A is said to be Lebesgue measurable. The common value is the Lebesgue measure of A , and we denote it as $\mu(A)$.

- The definition immediately implies that, if A is Lebesgue measurable, then for any $\epsilon > 0$, then there exist open set U and closed set K , such that $K \subset A \subset U$ and $\lambda(U - K) < \epsilon$.
- If A is Lebesgue measurable, then for any open set U and closed set K , with $K \subset A \subset U$, the measure $\mu(A)$ is the only number satisfying $\lambda(K) \leq \mu(A) \leq \lambda(U)$.

10.6 **Theorem** The Lebesgue outer and Lebesgue inner measures have the following properties:

- (a) Non-Negativity: $0 \leq \mu_*(A) \leq \mu^*(A)$.
- (b) Monotonicity: $A \subset B$ implies $\mu_*(A) \leq \mu_*(B)$, $\mu^*(A) \leq \mu^*(B)$.
- (c) Countable Sub-Additivity: $\mu^*(\cup A_i) \leq \sum \mu^*(A_i)$.

Proof (a) and (b) are clear from the definition. (c) follows from Theorem 10.2. ■

- **Corollary:** If $\mu^*(A) = 0$, then A is Lebesgue measurable with $\mu(A) = 0$. Any subset of a set of measure zero is also a set of measure zero.
- **Corollary:** If A is a (bounded) countable set, then A is Lebesgue measurable, and $\mu(A) = 0$.

Proof If $A = \{a\}$ contains only one point, then, for any $\epsilon > 0$, $A \subset (a - \epsilon/2, a + \epsilon/2)$. It follows that

$$\mu^*(A) \leq \lambda(a - \epsilon/2, a + \epsilon/2) = \epsilon.$$

Since ϵ is arbitrary, we know that $\mu^*(A) = 0$, which implies A is Lebesgue measurable, and $\mu(A) = 0$, by the Corollary of Theorem 10.6.

If A is a bounded countable set, write $A = \cup\{a_i\}$. Then, by the countable sub-additivity,

$$\mu^*(A) \leq \sum \mu^*(\{a_i\}) = 0,$$

we know that A is Lebesgue measurable, and $\mu(A) = 0$. ■

- We have seen in 2.44 that Cantor set is uncountable, and the measure is also zero.

10.7 **Theorem** Any finite segment $\langle a, b \rangle$ “open”, “closed”, “half-open and half-closed”) is Lebesgue measurable, with the usual length $b - a$ as its measure. Any bounded open subset U is Lebesgue measurable, with $\mu(U) = \lambda(U)$.

Proof Let $\epsilon > 0$ be given. The set $A = \langle a, b \rangle$ is measurable, since

$$K = [a + \epsilon/4, b - \epsilon/4] \subset A \subset U = (a - \epsilon/4, b + \epsilon/4),$$

and $\lambda(U - K) \leq 2 \cdot \epsilon/2 = \epsilon$. The measure $\mu(A) = b - a$, since, by Theorem 10.6,

$$b - a - \epsilon/2 = \lambda(K) \leq \mu_*(A) \leq \mu^*(A) \leq \lambda(U) = b - a + \epsilon/2.$$

Let $A = \sqcup(a_i, b_i)$ be a bounded open set. Since $\sum(b_i - a_i)$ converges, there is N such that

$$\sum_{i>N} (b_i - a_i) < \epsilon/2.$$

Let

$$K_n = [a_1 + \epsilon/(4N), b_1 - \epsilon/(4N)] \cup \dots \cup [a_n + \epsilon/(4N), b_n - \epsilon/(4N)].$$

It is clear that, for any n , $K_n \subset A \subset U = A$. For any fixed n , with $n \geq N$, we have

$$\lambda(U - K_n) \leq \sum_{i>N} (b_i - a_i) + N \cdot 2\epsilon/(4N) < \epsilon.$$

Hence, A is Lebesgue measurable. The measure $\mu(A) = \sum(b_i - a_i) = \lambda(U)$, since,

$$\begin{aligned} \sum(b_i - a_i) - \epsilon &< \sum(b_i - a_i) - \sum_{i>N} (b_i - a_i) - \epsilon/2 \\ &\leq \sum(b_i - a_i) - \sum_{i>n} (b_i - a_i) - \epsilon/2, \quad \text{if } n \geq N \\ &= \lambda(K_n) \leq \mu_*(A) \leq \mu^*(A) \leq \lambda(U) = \sum(b_i - a_i). \end{aligned}$$

10.8 **Theorem** If A and B are disjoint bounded sets, then

$$\mu_*(A \sqcup B) \leq \mu_*(A) + \mu^*(B) \leq \mu^*(A \sqcup B).$$

Proof Let $\epsilon > 0$ be given. To prove the first inequality, by the definitions of the Lebesgue outer and lower measures, there are bounded closed set $K \subset A \cup B$ and bounded open set $U \supset B$, such that

$$\mu_*(A \sqcup B) - \epsilon < \lambda(K), \quad \lambda(U) < \mu^*(B) + \epsilon.$$

Since $K - U$ is closed and $K - U \subset A$, we know that $\lambda(K - U) \leq \mu_*(A)$. Hence, we have

$$\mu_*(A \sqcup B) - \epsilon < \lambda(K), \quad \lambda(K - U) + \lambda(U) < \mu_*(A) + \mu^*(B) + \epsilon,$$

Thus, since ϵ is arbitrary, the first inequality in the theorem holds if

$$\lambda(K) \leq \lambda(K - U) + \lambda(U).$$

In fact, for any bounded open set V , with $K \subset V$, since $V - (K - U) \subset (V - K) \cup U$, by Theorem 10.3,

$$\lambda(V - (K - U)) \leq \lambda((V - K) \cup U) = \lambda(V - K) + \lambda(U) - \lambda((V - K) \cap U) \leq \lambda(V - K) + \lambda(U),$$

which implies

$$\lambda(K) = \lambda(V) - \lambda(V - K) \leq \lambda(V) - \lambda(V - (K - U)) + \lambda(U) = \lambda(K - U) + \lambda(U).$$

To prove the second inequality of the theorem, we choose bounded closed set $K \subset A$ and bounded open set $U \supset A \cup B$, such that

$$\mu_*(A) - \epsilon < \lambda(K), \quad \lambda(U) < \mu^*(A \sqcup B) + \epsilon.$$

Since $U - K$ is open and $B \subset U - K$, we have $\mu^*(B) \leq \lambda(U - K)$. Thus, by the definition of $\lambda(K)$,

$$\mu_*(A) + \mu^*(B) - \epsilon < \lambda(K) + \lambda(U - K) = \lambda(U) < \mu^*(A \sqcup B) + \epsilon,$$

which implies the second inequality since ϵ is arbitrary. ■

10.9 **Theorem (Carathéodory Theorem)** A bounded set A in \mathbb{R} is Lebesgue measurable if and only if the Carathéodory condition

$$\mu^*(X) = \mu^*(X \cap A) + \mu^*(X - A)$$

holds for any bounded X . (In other words, a subset is Lebesgue measurable if and only if it and its complement can be used to “split” the outer measure of any subset.)

Proof If the Carathéodory condition holds for any bounded X , we choose X to be a bounded segment containing A . Then X is Lebesgue measurable and $X \cap A = A$. By Theorem 10.8, we have

$$\mu^*(A) + \mu^*(X - A) = \mu^*(X) = \mu_*(X) = \mu_*(A \sqcup (X - A)) \leq \mu_*(A) + \mu^*(X - A),$$

which implies $\mu^*(A) \leq \mu_*(A)$. Thus, A is Lebesgue measurable.

Conversely, suppose A is Lebesgue measurable. Let $\epsilon > 0$ be given. We first show that the Carathéodory condition holds for any bounded X , provided that $A \cap U$ is Lebesgue measurable for any bounded open set U . In fact, for any bounded X , there is a bounded open $U \supset X$, such that $\lambda(U) < \mu^*(X) + \epsilon$. Hence, by Theorem 10.8,

$$\begin{aligned} \mu^*(X) + \epsilon &> \lambda(U) = \mu^*(U) \\ &\geq \mu_*(U \cap A) + \mu^*(U - A) \\ &= \mu^*(U \cap A) + \mu^*(U - A) \\ &\geq \mu^*(X \cap A) + \mu^*(X - A), \end{aligned}$$

which implies $\mu^*(X) \geq \mu^*(X \cap A) + \mu^*(X - A)$, since ϵ is arbitrary. On the other hand, Theorem 10.6 implies $\mu^*(X) \leq \mu^*(X \cap A) + \mu^*(X - A)$. Thus, the Carathéodory condition holds in this case.

Next, we show that if A is Lebesgue measurable, and if U is a bounded open set, then $A \cap U$ is Lebesgue measurable. In fact, since U is open, U is Lebesgue measurable by Theorem 10.7. Since an intersection of two open sets is again an open set, so that it is Lebesgue measurable. Hence, the Carathéodory condition can be applied to U . This gives

$$\mu^*(A \cap U) + \mu^*(A - U) = \mu^*(A).$$

Since A is Lebesgue measurable, by Theorem 10.8, we have

$$\mu^*(A) = \mu_*(A) \leq \mu_*(A \cap U) + \mu^*(A - U).$$

Thus, $\mu^*(A \cap U) \leq \mu_*(A \cap U)$, which implies that $A \cap U$ is Lebesgue measurable.

These two steps indicate that the Carathéodory condition holds if A is Lebesgue measurable. ■

1.2 MEASURE SPACES

10.10 **Outer Measure, Measurable:** Suppose X is a set. An outer measure μ^* on X is a function defined for each subset $A \subset X$ so that

- (a) $\mu^*(\emptyset) = 0$.
- (b) Monotonicity: If $A \subset B$, then $\mu^*(A) \leq \mu^*(B)$,
- (c) Countable Sub-Additivity: $\mu^*(\cup A_i) \leq \sum \mu^*(A_i)$.

A subset A of X is said to be measurable if the Carathéodory condition

$$\mu^*(Y) = \mu^*(Y \cap A) + \mu^*(Y - A)$$

holds for any $Y \subset X$. In this case, we denote the outer measure of A as its measure and write $\mu(A) = \mu^*(A)$.

- If A is measurable, then $\mu(A) \geq 0$.
- By symmetry of the Carathéodory condition, A is measurable if and only if $X - A$ is measurable.

10.11 **Theorem** Suppose A , B , and $\{A_i\}$ are all measurable.

- (a) The countable union $\cup A_i$ is measurable.
- (b) The countable intersection $\cap A_i$ is measurable.
- (c) The difference $A - B$ is measurable.

Moreover, if $\{A_i\}$ are disjoint, then $\mu(\sqcup A_i) = \sum \mu(A_i)$.

Proof Let us first show that a finite union of measurable subsets is measurable. This Step 1 can be done by proving that if A and B are measurable, so is $A \cup B$.

Suppose Y is any subset of X . Since A is measurable, we have A splits Y .

$$\mu^*(Y) = \mu^*(Y \cap A) + \mu^*(Y - A).$$

Since B is measurable, we have B splits $Y - A$.

$$\mu^*(Y - A) = \mu^*((Y - A) \cap B) + \mu^*(Y - A - B)$$

Since A splits $Y \cap (A \cup B)$.

$$\begin{aligned} \mu^*(Y \cap (A \cup B)) &= \mu^*(Y \cap (A \cup B) \cap A) + \mu^*(Y \cap (A \cup B) - A) \\ &= \mu^*(Y \cap A) + \mu^*((Y - A) \cap B), \end{aligned}$$

combining these equations gives

$$\mu^*(Y) = \mu^*(Y \cap (A \cup B)) + \mu^*(Y - (A \cup B)),$$

which means $A \cup B$ is measurable.

If, in addition, A and B are disjoint, then

$$\mu(A \sqcup B) = \mu^*((A \sqcup B) \cap A) + \mu^*(A \sqcup B - A) = \mu(A) + \mu(B).$$

This implies the finite additivity:

$$\mu \left(\bigsqcup_{i=1}^k A_i \right) = \sum_{i=1}^k \mu(A_i).$$

Since $A \cap B = X - (X - A) \cup (X - B)$, hence, if A and B are measurable, so is $A \cap B$. Thus, a finite intersection of measurable subsets is measurable.

It follows that $A - B$ is measurable, since $A - B = A \cap (X - B)$.

Next, let us consider a countable union of disjoint subsets. Suppose $\{A_i\}$ are measurable and disjoint. Put $B_n = \bigsqcup_{i=1}^n A_i$, $B = \bigsqcup A_i = \cup B_n$. Then $\{B_n\}$ are measurable. We shall show that B is measurable. Indeed, by the finite additivity, we have

$$\sum_{i=1}^n \mu(A_i) = \mu(B_n) \leq \mu^*(B) \leq \mu(X).$$

This implies that the non-negative series $\sum \mu(A_i)$ converges. Hence, for any $\epsilon > 0$, there is N such that $\sum_{i>N} \mu(A_i) < \epsilon$. For any subset Y , since B_N is measurable,

$$\begin{aligned} \mu^*(Y \cap B) + \mu^*(Y - B) &= \mu^*(Y \cap B \cap B_N) + \mu^*(Y \cap B - B_N) + \mu^*(Y - B) \\ &\leq \mu^*(Y \cap B_N) + \mu^*(B - B_N) + \mu^*(Y - B_N) \\ &= \mu^*(Y) + \sum_{i>N} \mu(A_i) < \mu^*(Y) + \epsilon. \end{aligned}$$

This gives $\mu^*(Y \cap B) + \mu^*(Y - B) \leq \mu^*(Y)$. By the sub-additivity of the outer measure, it follows that B is measurable.

The countable sub-additivity gives $\mu(\bigsqcup A_i) \leq \sum \mu(A_i)$. On the other hand, the earlier inequality $\sum_{i=1}^n \mu(A_i) = \mu(B_n) \leq \mu(\bigsqcup A_i)$ implies $\sum \mu(A_i) \leq \mu(\bigsqcup A_i)$. Thus we have the countable additivity $\mu(\bigsqcup A_i) = \sum \mu(A_i)$.

Suppose $\{A_i\}$ are measurable, but not necessarily disjoint. Put $C_i = A_i - A_1 \cup \dots \cup A_{i-1}$. Then $\{C_i\}$ are measurable and disjoint. Hence $\cup A_i = \bigsqcup C_i$ is measurable.

From $\cap A_i = X - \cup(X - A_i)$, we know that the countable intersection of measurable subsets is measurable. ■

10.12 **σ -Algebra, Measure:** A collection Σ of subsets of X is a σ -algebra if

1. $X \in \Sigma$.
2. If $A, B \in \Sigma$, then $A - B \in \Sigma$.
3. If $A_i \in \Sigma$, then $\cup A_i \in \Sigma$.

A measure on a given σ -algebra Σ assigns a number $\mu(A)$ for each $A \in \Sigma$, such that the following properties holds:

1. Non-Negativity: $\mu(A) \geq 0$.

Step 2

Step 3

So, $\mu(\emptyset) = 0$.

2. Countable Additivity: If $A_i \in \Sigma$ and disjoint, then $\mu(\sqcup A_i) = \sum \mu(A_i)$.

- The σ -algebra is also closed under countable intersection, since

$$\cap A_i = X - \cup(X - A_i).$$

10.13 **Measure Space:** A measure space (X, Σ, μ) consists of a set X , a σ -algebra Σ on X , and a measure μ on Σ . The subsets in Σ are called measurable.

10.14 **Theorem** A measure satisfies the following properties

- (a) Monotonicity: If $A \subset B$, then $\mu(A) \leq \mu(B)$.
- (b) Countable Sub-Additivity: $\mu(\cup A_i) \leq \sum \mu(A_i)$.
- (c) Monotonic Limit: If $A_i \subset A_{i+1}$ for all i , then $\mu(\cup A_i) = \lim \mu(A_i)$. If $A_i \supset A_{i+1}$ for all i , then $\mu(\cap A_i) = \lim \mu(A_i)$.

Proof (a) Since $A \sqcup (B - A) = B$, by the additivity,

$$\mu(B) = \mu(A) + \mu(B - A) \geq \mu(A).$$

(b) Put $C_i = A_i - A_1 \cup \dots \cup A_{i-1}$. Then $\{C_i\}$ are measurable, disjoint, and $C_i \subset A_i$. By the countable additivity, and (a), we have

$$\mu(\cup A_i) = \mu(\sqcup C_i) = \sum \mu(C_i) \leq \sum \mu(A_i).$$

(c) For $A_i \subset A_{i+1}$, we have $\cup A_i = \sqcup (A_i - A_{i-1})$. By the countable additivity,

$$\mu(\cup A_i) = \mu(\sqcup (A_i - A_{i-1})) = \sum \mu(A_i - A_{i-1}).$$

Thus, the non-negative series $\sum \mu(A_i - A_{i-1})$ converges. Similarly, by the countable sub-additivity,

$$\mu(\cup A_i) - \mu(A_n) = \mu(\cup A_i - A_n) = \mu\left(\bigsqcup_{i>n} (A_i - A_{i-1})\right) = \sum_{i>n} \mu(A_i - A_{i-1}).$$

Since the convergence of $\sum \mu(A_i - A_{i-1})$ implies that $\lim_{n \rightarrow \infty} \sum_{i>n} \mu(A_i - A_{i-1}) = 0$, we obtain

$$\lim_{n \rightarrow \infty} \mu(A_n) = \mu(\cup A_i).$$

If $A_i \supset A_{i+1}$, then $X - A_i \subset X - A_{i+1}$. Hence, by what we just proved,

$$\lim \mu(X - A_n) = \mu(\cup(X - A_i)).$$

Hence,

$$\begin{aligned} \lim \mu(A_n) &= \lim [\mu(X) - \mu(X - A_n)] = \mu(X) - \mu(\cup(X - A_i)) \\ &= \mu(X - \cup(X - A_i)) = \mu(\cap A_i). \end{aligned}$$

1.3 EXAMPLES

10.15 **Cantor Set:** Recall the Cantor set P obtained by repeatedly deleting the open middle thirds of the closed interval $[0, 1]$, discussed in 2.44. The set removed altogether from the closed set $[0, 1]$ is a disjoint union of open intervals

$$U = \underbrace{\left(\frac{1}{3}, \frac{2}{3}\right)} \sqcup \underbrace{\left(\frac{1}{9}, \frac{2}{9}\right) \sqcup \left(\frac{7}{9}, \frac{8}{9}\right)} \sqcup \underbrace{\left(\frac{1}{27}, \frac{2}{27}\right) \sqcup \left(\frac{7}{27}, \frac{8}{27}\right) \sqcup \left(\frac{19}{27}, \frac{20}{27}\right) \sqcup \left(\frac{25}{27}, \frac{26}{27}\right)} \sqcup \cdots,$$

so that

$$P = [0, 1] - U.$$

We know that P is a closed set, so it is Lebesgue measurable. It is clear that

$$\lambda(U) = \frac{1}{3} + \frac{2}{3^2} + \frac{2^2}{3^3} + \cdots = \frac{1}{3} \cdot \frac{1}{1 - \frac{2}{3}} = 1.$$

By the additivity of Lebesgue measure,

$$\mu([0, 1]) = \mu(P \sqcup U) = \mu(P) + \mu(U) = \mu(P) + \lambda(U),$$

we obtain

$$\mu(P) = \mu([0, 1]) - \lambda(U) = 1 - 1 = 0.$$

It is known that P is an uncountable set. Hence an uncountable set may still have Lebesgue measure zero.

10.16 **Theorem** Any Lebesgue measurable subset of \mathbb{R} with positive measure contains a non-measurable subset.

Proof In \mathbb{R} , we define a relation $x \sim y$ of two points x and y , if $x - y \in \mathbb{Q}$. It is easy to check that this is an equivalence relation, by

- (a) $x \sim x$, since $x - x = 0 \in \mathbb{Q}$;
- (b) if $x \sim y$, then $y \sim x$, since $y - x = -(x - y) \in \mathbb{Q}$;
- (c) if $x \sim y$ and $y \sim z$, then $x \sim z$, since $x - z = (x - y) + (y - z) \in \mathbb{Q}$.

Hence, we can decompose the set \mathbb{R} into a disjoint union of equivalence classes. Each class contains the real numbers that are equivalent in terms of this relation. Thus, each class is in the form of $x + \mathbb{Q}$, where x is any member in the class. For each class, we choose one member in $(x + \mathbb{Q}) \cap [0, 1]$, so that we form a subset X of $[0, 1]$. Then, we have the decomposition

$$\mathbb{R} = \bigsqcup_{r \in \mathbb{Q}} (r + X).$$

In fact, for any $p \in \mathbb{R}$, there is $x \in X$ such that $p - x \in \mathbb{Q}$. This gives that $p = r + x \in r + X$ for some $r \in \mathbb{Q}$. For any two distinct rational numbers r_1 and r_2 , we must have $(r_1 + X) \cap (r_2 + X) = \emptyset$. Otherwise there are $x_1, x_2 \in X$ such that

$r_1 + x_1 = r_2 + x_2$, which implies $x_1 - x_2 = r_2 - r_1 \in \mathbb{Q}$. By the construction of X , the last equation holds only if $x_1 = x_2$, so that $r_1 = r_2$, a contradiction.

We claim that $\mu_*(X) = 0$. In fact, for any closed subset $K \subset X$, if $r_1, \dots, r_n \in [0, 1] \cap \mathbb{Q}$, since $\{r_n + K\}$ are disjoint closed subsets in $[0, 2]$, we have

$$\begin{aligned} n\lambda(K) &= \lambda(r_1 + K) + \dots + \lambda(r_n + K) \\ &= \lambda((r_1 + K) \sqcup \dots \sqcup (r_n + K)) \\ &\leq \lambda([0, 2]) = 2. \end{aligned}$$

Since n is arbitrary, we have $\lambda(K) = 0$, which implies $\mu_*(X) = 0$.

Let A be a measurable subset with $\mu(A) > 0$. We claim that there is a rational number r such that $A \cap (r + X)$ is not measurable. In fact, if $A \cap (r + X)$ is measurable for each rational r , then, from

$$A = A \cap \bigsqcup_{r \in \mathbb{Q}} (r + X) = \bigsqcup_{r \in \mathbb{Q}} (A \cap (r + X)),$$

we have

$$\mu(A) = \sum_{r \in \mathbb{Q}} \mu(A \cap (r + X)),$$

by the countable additivity. Since $\mu(A \cap (r + X)) \leq \mu_*(r + X) = \mu_*(X) = 0$, we have $\mu(A) = 0$, a contradiction to the hypothesis $\mu(A) > 0$. It follows that there is a rational number r such that the subset $A \cap (r + X)$ of A is not measurable. ■

1.4 MEASURABLE FUNCTIONS

10.17 **Measurable Function:** Let f be a function defined on a measure space (X, Σ, μ) . The function f is said to be measurable if the set

$$\{x \in X : f(x) > a\} = f^{-1}(a, \infty)$$

measurable, for every real a .

in Σ .

10.18 **Theorem** The following statements are equivalent:

- (a) $\{x \in X : f(x) > a\}$ is measurable for every real a .
- (b) $\{x \in X : f(x) \geq a\}$ is measurable for every real a .
- (c) $\{x \in X : f(x) < a\}$ is measurable for every real a .
- (d) $\{x \in X : f(x) \leq a\}$ is measurable for every real a .

Proof The equivalences can be proved by the following relations:

$$\begin{aligned} (a) &\Rightarrow (b) \Rightarrow (c) \Rightarrow \\ (d) &\Rightarrow (a) \end{aligned}$$

$$\begin{aligned} \{x \in X : f(x) \geq a\} &= \bigcap_{n=1}^{\infty} \{x \in X : f(x) > a - 1/n\}, \\ \{x \in X : f(x) < a\} &= X - \{x \in X : f(x) \geq a\}, \\ \{x \in X : f(x) \leq a\} &= \bigcap_{n=1}^{\infty} \{x \in X : f(x) < a + 1/n\}, \\ \{x \in X : f(x) > a\} &= X - \{x \in X : f(x) \leq a\}. \end{aligned}$$

10.19 **Theorem** Let (X, Σ, μ) be a measure space. The measurable functions have the following properties:

- (a) If f and g are measurable, then $c_1f + c_2g$ is measurable for any real c_1, c_2 .
- (b) If f and g are measurable, then fg is measurable.
- (c) If f is measurable, and g is continuous, then $g \circ f$ is measurable. In particular, $|f|$ is measurable, and $1/f$ is measurable if $f(x) \neq 0$ for any $x \in X$.
- (d) If $\{f_n\}$ is a sequence of measurable functions, then

$$\sup f_n, \quad \inf f_n, \quad \overline{\lim} f_n, \quad \underline{\lim} f_n$$

are all measurable functions.

- (e) If $X = \cup X_i$, and $X_i \in \Sigma$, then f is measurable if and only if the restrictions $f|_{X_i}$ are measurable.

Proof (a) We only need to show that $f + g$ and cf are measurable, where c is any real number.

For any real a , $f(x) + g(x) > a$ if and only if $f(x) > r$ and $g(x) > a - r$ for some rational r . Hence,

$$\{x \in X : f(x) + g(x) > a\} = \bigcup_{r \in \mathbb{Q}} [\{x \in X : f(x) > r\} \cap \{x \in X : g(x) > a - r\}].$$

This implies $f + g$ is measurable if f and g are measurable.

For the scalar multiplication cf , if $c = 0$, then

$$\{x \in X : cf(x) > a\} = \begin{cases} \emptyset, & \text{if } a \geq 0, \\ X, & \text{if } a < 0. \end{cases}$$

If $c \neq 0$, then

$$\{x \in X : cf(x) > a\} = \begin{cases} \{x \in X : f(x) > a/c\}, & \text{if } c > 0, \\ \{x \in X : f(x) < a/c\} & \text{if } c < 0. \end{cases}$$

Hence cf is measurable.

(b) If we can show that f^2 is measurable if f is measurable, then, from $fg = \frac{1}{4}[(f+g)^2 - (f-g)^2]$, by (a), we know fg is measurable. Indeed, if $a < 0$, $\{x \in X : f^2(x) < a\} = \emptyset$. If $a \geq 0$, then

$$\{x \in X : f^2(x) < a\} = \{x \in X : f(x) < \sqrt{a}\} \cap \{x \in X : f(x) > -\sqrt{a}\}.$$

(c) Since $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$, to show that $g \circ f$ is measurable, we need to show that $f^{-1} \circ g^{-1}(a, \infty)$ is measurable for any real a . In fact, g is continuous, by Theorem 4.8, $g^{-1}(a, \infty)$ is open. This allows us to write $g^{-1}(a, \infty) = \sqcup(c_i, d_i)$. Hence,

$$\begin{aligned} \{x \in X : (g \circ f)(x) > a\} &= (g \circ f)^{-1}(a, \infty) \\ &= f^{-1} \circ g^{-1}(a, \infty) \\ &= f^{-1}(\sqcup(c_i, d_i)) \\ &= \cup f^{-1}(c_i, d_i) \\ &= \cup \{x \in X : c_i < f(x) < d_i\} \\ &= \cup [\{x \in X : f(x) < d_i\} - \{x \in X : f(x) \leq c_i\}]. \end{aligned}$$

Thus, $g \circ f$ is measurable.

In particular, since $g(y) = |y|$ is continuous, we know that $|f|$ is measurable if f is measurable. Similarly, the function $g(y) = 1/y$ is continuous for $y \neq 0$. Hence the function $1/f$ is measurable if $f(x) \neq 0$ for any $x \in X$.

(d) Put $g = \sup f_n$. By the definition, $g(x) = \sup_n \{f_n(x)\}$. Hence,

$$\begin{aligned} \{x \in X : g(x) > a\} &= \{x \in X : \text{there exist } n \text{ for which } f_n(x) > a\} \\ &= \bigcup_{n=1}^{\infty} \{x \in X : f_n(x) > a\}. \end{aligned}$$

Thus, $\sup f_n$ is measurable. It follows that $\inf f_n = -\sup(-f_n)$ is measurable. The functions $\overline{\lim} f_n$ and $\underline{\lim} f_n$ are measurable, since

$$\overline{\lim} f_n(x) = \inf_n (\sup \{f_n(x), f_{n+1}(x), f_{n+2}(x), \dots\}),$$

and

$$\underline{\lim} f_n(x) = \sup_n (\inf \{f_n(x), f_{n+1}(x), f_{n+2}(x), \dots\}).$$

(e) The equivalence is due to the following two identities:

$$f^{-1}(U) = \cup (f|_{X_i})^{-1}(U), \quad (f|_{X_i})^{-1}(U) = X_i \cap f^{-1}(U).$$

• **Corollary:**

(a) If f and g are measurable, then $\max(f, g)$ and $\min(f, g)$ are measurable. In particular,

$$f^+ = \max(f, 0), \quad f^- = -\min(f, 0),$$

are measurable.

(b) The limit of a convergent sequence of measurable functions is measurable.

10.20 **Characteristic Function, Simple Function:** For any subset $A \subset X$, put

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

χ_A is called the characteristic function of A .

For a finite collection of distinct numbers c_1, \dots, c_n , we define a simple function as

$$s(x) = \sum_{i=1}^n c_i \chi_{X_i}(x),$$

a finite linear combination of characteristic functions.

- A simple function s is measurable if and only if $\{X_i\}$ are measurable.
- If necessary, we can re-write the expression of a simple function s so that $\{X_i\}$ are disjoint. By adding $0 \cdot \chi_{X - \cup X_i}$, we may further assume that $\{X_i\}$ form a partition of X :

$$P: X = X_1 \sqcup \dots \sqcup X_n.$$

- Finite linear combinations of simple functions are simple. The maximum and minimum of finitely many simple functions are simple. The absolute value of simple functions are simple.

10.21 **Theorem** Let (X, Σ, μ) be a measure space. A measurable function with lower bound is the limit of an increasing sequence of simple functions. A bounded measurable function is the uniform convergent limit of an increasing sequence of simple functions.

Proof If f is a bounded measurable function, say $a < f(x) < b$ for $x \in X$, we consider a partition of $[a, b]$,

$$\Pi : a = c_0 < c_1 < \dots < c_k = b.$$

Put $X_i = \{x \in X : c_{i-1} < f(x) \leq c_i\}$, $1 \leq i \leq k$. The collection $\{X_i\}$ forms a partition of X : $X = \bigsqcup_{i=1}^k X_i$. Define a simple function s by

$$s(x) = \sum_{i=1}^k \left(\inf_{X_i} f \right) \chi_{X_i}(x).$$

Since $c_{i-1} \leq s(x) = \inf_{X_i} f \leq f(x) \leq c_i$ on X_i , we have

$$f(x) - \|\Pi\| \leq s(x) \leq f(x), \quad x \in X,$$

$$\begin{aligned} \text{If } s(x) &= \sum_{i=1}^n c_i \chi_{X_i}(x), \\ t(x) &= \sum_{i=1}^m d_i \chi_{Y_i}(x), \\ \text{then } \max(s(x), t(x)) &= \sum_{i,j} \max(c_i, d_j) \chi_{X_i \cap Y_j}(x) \end{aligned}$$

where $\|\Pi\| = \max_i(c_i - c_{i-1})$.

Hence, for each partition Π , we can construct a simple function s as above. Take a sequence of partitions $\{\Pi_n\}$ such that for each n , Π_n is a refinement of Π_{n-1} , and $\|\Pi_n\| \rightarrow 0$ as $n \rightarrow \infty$. We obtain the corresponding sequence of simple functions $\{s_n\}$. Obviously they are increasing. By

$$0 \leq f(x) - s_n(x) \leq \|\Pi_n\|, \quad x \in X,$$

we know that s_n converges to f uniformly on X .

If $f(x) > a$ for $x \in X$, then for any fixed integer n , we consider a partition Π of $[a, n]$,

$$\Pi : a = c_0 < c_1 < \cdots < c_k = n.$$

Put $X_i = \{x \in X : c_{i-1} < f(x) \leq c_i\}$, $1 \leq i \leq k$, and $\tilde{X} = \{x \in X : f(x) > n\}$. We have a partition of X : $X_1 \sqcup \cdots \sqcup X_k \sqcup \tilde{X}$, and define a simple function s by

$$s(x) = \sum_{i=1}^k \left(\inf_{X_i} f \right) \chi_{X_i}(x) + n \chi_{\tilde{X}}(x).$$

Similarly to the bounded case, we have

$$f(x) - \|\Pi\| \leq s(x) \leq f(x), \quad x \in \bigsqcup_{i=1}^k X_i = X - \tilde{X}.$$

For each n , we take a partition Π_n , such that

- (1) In $[a, n]$, the partition Π_n is a refinement of $\Pi_{n-1} \cup (n-1, n]$.
- (2) $\|\Pi_n\| \rightarrow 0$ as $n \rightarrow \infty$.

we can take $\|\Pi_n\| \leq (n-a)/n^2$.

Then, the corresponding sequence of simple functions $\{s_n\}$ is increasing. Since for any fixed $x \in X$, there is an integer n such that $a < f(x) \leq n$, we have

$$0 \leq f(x) - s_n(x) \leq \|\Pi_n\|,$$

we know that $s_n(x) \rightarrow f(x)$ as $n \rightarrow \infty$. ■

1.5 INTEGRATION

10.22 **Lebesgue Integral:** Let (X, σ, μ) be a measure space, with $\mu(X) < \infty$. A partition P of X is a finite collection of measurable subsets $\{X_i\}$, $1 \leq i \leq n$, such that

$$P: X = X_1 \sqcup X_2 \sqcup \cdots \sqcup X_n.$$

Suppose f is a bounded real function defined on X . On each subset X_i , put

$$M_i = \sup_{x \in X_i} f(x), \quad m_i = \inf_{x \in X_i} f(x).$$

Similarly to the upper and lower Riemann sums, we can form the following upper and lower Lebesgue sums:

$$U(P, f) = \sum_{i=1}^n M_i \mu(X_i), \quad L(P, f) = \sum_{i=1}^n m_i \mu(X_i).$$

Since f is bounded, there exist m and M such that for $x \in [a, b]$,

$$m \leq f(x) \leq M.$$

Hence, for each fixed partition P ,

$$m\mu(X) \leq L(P, f) \leq U(P, f) \leq M\mu(X),$$

so that the upper and lower Lebesgue sums $U(P, f)$ and $L(P, f)$ are bounded with respect to the partitions.

By taking the inf and the sup over all partitions P of X , we define the upper Lebesgue integral

$$\int_X^{\bar{}} f \, d\mu = U(f) = \inf_P U(P, f)$$

and the lower Lebesgue integral

$$\int_X^{\underline{}} f \, d\mu = L(f) = \sup_P L(P, f).$$

These two values, the upper Lebesgue integral and the lower Lebesgue integral, are well-defined and finite, for any bounded function f .

If the upper and lower integrals are equal, we denote the common value by

$$\int_X f \, d\mu,$$

and we say that f is Lebesgue-integrable on X , and write $f \in \mathcal{L}(X)$. Here $\mathcal{L}(X)$ denotes the set of Lebesgue-integrable functions on X .

- Suppose A is a measurable subset of X . If f is a bounded function on X such that the function $f\chi_A$ is Lebesgue-integrable on X . We define the Lebesgue integral of f on A to be the Lebesgue integral of $f\chi_A$ on X , and write

$$\int_A f \, d\mu = \int_X f\chi_A \, d\mu.$$

- For convenience, we define the oscillation of f on a set I by

$$\omega_I(f) = \sup_{x \in I} f(x) - \inf_{x \in I} f(x) = \sup\{|f(x) - f(y)| : x, y \in I\}.$$

It is clear that

$$U(P, f) - L(P, f) = \sum_i \omega_{X_i}(f) \mu(X_i).$$

10.23 **Theorem** Suppose f is a bounded function on a measure space (X, Σ, μ) with $\mu(X) < \infty$. Then $f \in \mathcal{L}(X)$ if and only if for any $\epsilon > 0$, there is a partition $P = \{X_i\}$ of X such that

$$U(P, f) - L(P, f) = \sum_i \omega_{X_i}(f) \mu(X_i) < \epsilon.$$

Proof Let $\epsilon > 0$ be given. Suppose there is a partition P of X such that $U(P, f) - L(P, f) < \epsilon$. Since

$$L(P, f) \leq \int_X f \, d\mu \leq \bar{\int}_X f \, d\mu \leq U(P, f),$$

we have

$$0 \leq \bar{\int}_X f \, d\mu - \int_X f \, d\mu < \epsilon.$$

Since ϵ is arbitrary, we have

$$\bar{\int}_X f \, d\mu - \int_X f \, d\mu = 0,$$

which implies $f \in \mathcal{L}$.

Conversely, suppose $f \in \mathcal{L}$. There are partitions $P_1 = \{X_i\}$ and $P_2 = \{Y_j\}$ of X , such that

$$\int_X f \, d\mu - L(P_1, f) < \epsilon/2,$$

and

$$U(P_2, f) - \bar{\int}_X f \, d\mu < \epsilon/2.$$

If we choose $P = \{Z_l\}$ as

$$Z_l = X_i \cap Y_j,$$

then each Z_l is measurable. It is also clear that, if $l_1 \neq l_2$, then $(i_1, j_1) \neq (i_2, j_2)$, so that at least one of $X_{i_1} \cap X_{i_2}$ and $Y_{j_1} \cap Y_{j_2}$ is empty. Hence,

$$Z_{l_1} \cap Z_{l_2} = (X_{i_1} \cap Y_{j_1}) \cap (X_{i_2} \cap Y_{j_2}) = (X_{i_1} \cap X_{i_2}) \cap (Y_{j_1} \cap Y_{j_2}) = \emptyset.$$

and

$$\cup_l Z_l = \cup_{i,j} (X_i \cap Y_j) = \cup_i X_i \cap (\cup_j Y_j) = \cup_i X_i \cap X = \cup_i X_i = X.$$

In other words, P is the common refinement of P_1 and P_2 , with each Z_l being a subset of some X_i and Y_j . By the definitions of the upper Lebesgue sum, we have

$$\begin{aligned} U(P, f) &= \sum_l \left(\sup_{x \in Z_l} f(x) \right) \mu(Z_l) = \sum_i \sum_j \left(\sup_{x \in X_i \cap Y_j} f(x) \right) \mu(X_i \cap Y_j) \\ &\leq \sum_i \sum_j \left(\sup_{x \in X_i} f(x) \right) \mu(X_i \cap Y_j) \\ &= \sum_i \left(\sup_{x \in X_i} f(x) \right) \sum_j \mu(X_i \cap Y_j) \\ &= \sum_i \left(\sup_{x \in X_i} f(x) \right) \mu(X_i) = U(P_1, f). \end{aligned}$$

Similarly, we have $L(P_2, f) \leq L(P, f)$. Hence, by combining the early inequalities, and $\int_X f d\mu = \int_X f d\mu = \int_X f d\mu$, we have

$$U(P, f) \leq U(P_2, f) < \int_X f d\mu + \epsilon/2 < L(P_1, f) + \epsilon \leq L(P, f) + \epsilon.$$

that is,

$$U(P, f) - L(P, f) < \epsilon.$$

- **Corollary** If f is a bounded function defined on $[a, b]$ such that f is Riemann integrable, then f is Lebesgue integrable and

$$\int_a^b f(x) dx = \int_{[a,b]} f d\mu.$$

10.24 **Almost Everywhere:** Suppose E is a subset of X . If a property holds for every $x \in E - A$, and if $\mu(A) = 0$, we say that the property holds almost everywhere on E .

- Let B be the set on which the property does not hold. It is clear that $B \subset A$. In general, the set B is not necessarily measurable, unless the measure space (X, Σ, μ) is *complete*.

10.25 **Theorem** Suppose f is a bounded function on a measure space (X, Σ, μ) with $\mu(X) < \infty$. Then $f \in \mathcal{L}(X)$ if and only if f is equal to a measurable function almost everywhere on X .

Proof Suppose $f = g$ on $X - A$, with g being a measurable function and $\mu(A) = 0$. Assume $m < f(x) < M$. Choose a partition Π of $[m, M]$:

$$\Pi: m = c_0 < c_1 < \cdots < c_k = M,$$

Here the Lebesgue measure is the one studied at the beginning of the Chapter.

so that $\|\Pi\| = \max_i(c_i - c_{i-1}) < \epsilon/\mu(X)$. Put

$$X_i = \{x \in X : c_{i-1} < f(x) \leq c_i\} \cap (X - A), \quad 1 \leq i \leq k.$$

Since $f = g$ on $X - A$, we know that $X_i = \{x \in X : c_{i-1} < g(x) \leq c_i\} \cap (X - A)$, so that each X_i is measurable. It is clear that

$$P: X_1 \sqcup \cdots \sqcup X_k \sqcup A = X$$

form a partition of X . Since

$$\omega_{X_i}(f) \leq c_i - c_{i-1} < \epsilon/\mu(X), \quad \mu(A) = 0,$$

we have

$$\begin{aligned} & U(P, f) - L(P, f) \\ & \leq \sum_{i=1}^k \omega_{X_i}(f)\mu(X_i) + \omega_A(f)\mu(A) \\ & < \frac{\epsilon}{\mu(X)} \sum_{i=1}^k \mu(X_i) + (M - m) \cdot 0 = \epsilon. \end{aligned}$$

By Theorem 10.21, we know that $f \in \mathcal{L}(X)$.

Conversely, suppose $f \in \mathcal{L}(X)$. By Theorem 10.23, for each positive integer n , there is a partition $P_n = \{X_i^{(n)}\}$ such that

$$U(P_n, f) - L(P_n, f) < \frac{1}{n}, \quad n = 1, 2, 3, \dots$$

Put

$$\phi_n(x) = \sum_i \left(\inf_{X_i^{(n)}} f \right) \chi_{X_i^{(n)}}(x), \quad \psi_n(x) = \sum_i \left(\sup_{X_i^{(n)}} f \right) \chi_{X_i^{(n)}}(x),$$

and

$$f_*(x) = \sup_n \phi_n(x), \quad f^*(x) = \inf_n \psi_n(x).$$

Since f is bounded on X , these functions are all well defined on X , with

$$f_*(x) \leq f(x) \leq f^*(x), \quad x \in X.$$

Since all $\{X_i^{(n)}\}$ are measurable, the simple functions $\{\phi_n\}$ and $\{\psi_n\}$ are measurable, so that f_* and f^* are measurable by Theorem 10.19.

We claim that for any $\delta > 0$, the measurable set

$$A_\delta = \{x \in X : f^*(x) - f_*(x) > \delta\}$$

is of zero measure. In fact, for any positive integer n , by the above construction, there is a partition P_n , with the corresponding two simple functions ϕ_n and ψ_n , such that

$$\frac{1}{n} > \sum_i \omega_{X_i^{(n)}}(f) \mu(X_i^{(n)}) \geq \sum_{\omega_{X_i^{(n)}}(f) > \delta} \omega_{X_i^{(n)}}(f) \mu(X_i^{(n)}) \geq \delta \mu(\cup_{\omega_{X_i^{(n)}}(f) > \delta} X_i^{(n)}).$$

By the inequality

$$f^*(x) - f_*(x) \leq \psi_n(x) - \phi_n(x),$$

and

$$\psi_n(x) - \phi_n(x) = \omega_{X_i^{(n)}} f(x), \quad x \in X_i^{(n)},$$

we have

$$A_\delta \subset \cup_{\omega_{X_i^{(n)}}(f) > \delta} X_i^{(n)},$$

which implies

$$\mu(A_\delta) \subset \mu(\cup_{\omega_{X_i^{(n)}}(f) > \delta} X_i^{(n)}) \leq \frac{1}{n\delta}.$$

Since n is arbitrary, we conclude that $\mu(A_\delta) = 0$.

Since

$$A_0 = \{x: f^*(x) \neq f_*(x)\} = \bigcup_{n=1}^{\infty} A_{1/n},$$

by the countable sub-additivity, we have

$$\mu(A_0) \leq \sum_{n=1}^{\infty} \mu(A_{1/n}) = 0.$$

It is clear that $f(x) = f^*(x) = f_*(x)$ on $X - A_0$. Hence f is equal to a measurable function almost everywhere on X . ■

10.26 **Theorem** Let (X, Σ, μ) be a measure space. The following properties of the Lebesgue integral hold:

(a) If A is a measurable set, and if $f \in \mathcal{L}(X)$, then f is Lebesgue integrable on A .

(b) If A is measurable, then $\int_X \chi_A d\mu = \mu(A)$;

(c) If $f, g \in \mathcal{L}(X)$, and if a, b are constants, then

$$\int_X (af + bg) d\mu = a \int_X f d\mu + b \int_X g d\mu;$$

(d) If $f, g \in \mathcal{L}(X)$, and if $f \geq g$ on X , then

$$\int_X f d\mu \geq \int_X g d\mu,$$

and the equality holds if and only if $f = g$ almost everywhere;

(e) If $f \in \mathcal{L}(X)$, and if A, B are measurable sets, then

$$\int_{A \cup B} f \, d\mu = \int_A f \, d\mu + \int_B f \, d\mu - \int_{A \cap B} f \, d\mu.$$

Proof (a) If A is measurable, then χ_A is a measurable function. If $f \in \mathcal{L}(X)$, by Theorem 10.25, f is equal to a measurable function g almost everywhere on X . It follows from Theorem 10.19(b) that $g\chi_A$ is measurable. Hence $f\chi_A$ is Lebesgue integrable since it is equal to $g\chi_A$ almost everywhere on X .

(b) If A is measurable, then for the partition P_0 of X :

$$P_0 : A \sqcup (X - A),$$

we have $L(P_0, \chi_A) = U(P_0, \chi_A) = 1 \cdot \mu(A) + 0 \cdot \mu(X - A) = \mu(A)$. Thus

$$\mu(A) \leq \sup_P L(P, \chi_A) = \int_X \chi_A \, d\mu \leq \int_X \chi_A \, d\mu = \inf_P U(P, \chi_A) \leq \mu(A),$$

i.e., $\int_X \chi_A \, d\mu = \mu(A)$.

We omit the proofs of (c) and the inequality in (d), since they are similar to that of Theorem 6.12. We will prove the second part of (d) at the end of this proof.

(e) The identity

$$\chi_{A \cup B} = \chi_A + \chi_B - \chi_{A \cap B}$$

gives the desired identity.

(d) For the second part of (d), we only need to prove that if $f \geq 0$ is Lebesgue integrable, and $\int_X f \, d\mu = 0$, then $f = 0$ almost everywhere.

Let $\epsilon > 0$ be given. The hypotheses imply there is a partition P of X , $P = \{X_i\}$, such that

$$0 \leq L(P, f) \leq \sum f(x_i^*)\mu(X_i) \leq U(P, f) < \epsilon^2,$$

for any choice of $x_i^* \in X_i$. Put

$$Y_\epsilon = \sqcup \{X_i : \sup_{x \in X_i} f(x) > 2\epsilon\}.$$

Then, on each X_i on which $\sup_{x \in X_i} f(x) > 2\epsilon$, we can choose $x_i^* \in X_i$ such that $f(x_i^*) \geq \epsilon$. It follows that

$$\begin{aligned} \epsilon^2 &\geq U(P, f) \geq \sum f(x_i^*)\mu(X_i) \\ &\geq \sum_{\sup_{X_i} f > 2\epsilon} f(x_i^*)\mu(X_i) \geq \sum_{\sup_{X_i} f > 2\epsilon} \epsilon\mu(X_i) = \epsilon \cdot \mu(Y_\epsilon), \end{aligned}$$

so that $\mu(Y_\epsilon) < \epsilon$.

Take any positive sequence $\{\epsilon_n\}$ such that $\sum \epsilon_n$ converges. It is clear that $0 \leq f(x) \leq 2\epsilon_n$ for all $n \geq k$ and $x \in X - \cup_{n \geq k} Y_{\epsilon_n}$. Since $\lim_{n \rightarrow \infty} \epsilon_n = 0$, we know that $f = 0$ on $X - \cup_{n \geq k} Y_{\epsilon_n}$ for every k . Hence $f = 0$ on

$$\cup_k (X - \cup_{n \geq k} Y_{\epsilon_n}) = X - \cap_k (\cup_{n \geq k} Y_{\epsilon_n}).$$

Put $Y = \cap_k (\cup_{n \geq k} Y_{\epsilon_n})$. If we can show that $\mu(Y) = 0$, then $f = 0$ is almost everywhere. Indeed, by the monotonicity and the countable sub-additivity, we have

$$\mu(Y) \leq \mu(\cup_{n \geq k} Y_{\epsilon_n}) \leq \sum_{n \geq k} \mu(Y_{\epsilon_n}) \leq \sum_{n \geq k} \epsilon_n \rightarrow 0,$$

as $k \rightarrow \infty$, which implies $\mu(Y) = 0$. ■

1.6 EXTENSION OF INTEGRATION

10.27 Unbounded Functions on Extended Measure Space: Let (X, Σ, μ) is an extended measure space, with $\mu(X)$ possibly being infinite. For any function f on X , define

$$f_{[a,b]}(x) = \begin{cases} f(x), & \text{if } a \leq f(x) \leq b, \\ b, & \text{if } f(x) > b, \\ a, & \text{if } f(x) < a. \end{cases}$$

- If f is an almost measurable function on an extended measure space (X, Σ, μ) , then $f_{[a,b]}$ is almost measurable, since

$$\{x \in X : f_{[a,b]}(x) > c\} = \begin{cases} \{x \in X : f(x) > c\}, & \text{if } a \leq c \leq b, \\ \emptyset, & \text{if } c < a \text{ or } b < c. \end{cases}$$

- Let f be an almost measurable function on an extended measure space (X, Σ, μ) , with $\mu(X)$ possibly being infinite. Suppose I is a fixed finite number. We say that f is *Lebesgue integrable* on X , with $\int_X f \mu = I$, if for any $\epsilon > 0$, there are $N > 0$ and $Y \in \Sigma$ with $\mu(Y) < \infty$, such that

$$\left| \int_A f_{[a,b]} d\mu - I \right| < \epsilon$$

whenever $A \in \Sigma$, $A \supset Y$, $\mu(A) < \infty$, $a < -N$, and $b > N$.

- The assumption that f is almost measurable implies that the function $f_{[a,b]}$ is almost measurable, by Theorem 10.25, so that the integral $\int_A f_{[a,b]} d\mu$ is well-defined. Hence, the integrability means the *convergence* of the integral as $a \rightarrow -\infty$, $b \rightarrow +\infty$, and

" $A \rightarrow X$ " with $\mu(A) < \infty$. In particular, if f is bounded and $\mu(X) < +\infty$, then, by Theorem 10.25, the integrability of f on X is equivalent to almost measurable of f on X .

10.28 **Non-Negative Functions:** For any function f on X ,

$$f^+ = \max(f, 0) \geq 0, \quad f^- = -\min(f, 0) \geq 0,$$

then

$$\begin{aligned} f &= f^+ - f^-, \\ |f| &= f^+ + f^-. \end{aligned}$$

If $a < 0$ and $b > 0$, then

$$f_{[a,b]} = f_{[0,b]} + f_{[a,0]} = f_{[0,b]}^+ - f_{[0,-a]}^-,$$

so that

$$\int_A f_{[a,b]} d\mu = \int_A f_{[0,b]}^+ d\mu - \int_A f_{[0,-a]}^- d\mu.$$

10.29 **Theorem** An almost measurable function f on a measure space (X, Σ, μ) is Lebesgue integrable if and only if $\int_X f d\mu$ is bounded, i.e., there is $M > 0$ such that

$$\left| \int_A f_{[a,b]} d\mu \right| \leq M,$$

for any $a < b$ and $A \in \Sigma$ with finite $\mu(A)$. Moreover, the following are equivalent:

- (a) f is Lebesgue integrable;
- (b) $|f|$ is Lebesgue integrable;
- (c) f^+ and f^- are Lebesgue integrable.

Proof Let $\epsilon > 0$ be given. For the convenience, we use $I(f)$ to denote that f is Lebesgue integrable on X , and $B(f)$ that $\int_X f d\mu$ is bounded.

(1) We show that for a non-negative function f , $I(f)$ is equivalent to $B(f)$. In fact, if f is Lebesgue integrable, there are $N > 0$ and $Y \in \Sigma$ with $\mu(Y) < \infty$, such that

$$\left| \int_A f_{[a,b]} d\mu - I \right| < 1$$

whenever $A \in \Sigma$, $A \supset Y$, $\mu(A) < \infty$, $a < -N$, and $b > N$. Thus, for any $a < b$ and $A \in \Sigma$ with finite $\mu(A)$, we have

$$\begin{aligned} \left| \int_A f_{[a,b]} d\mu \right| &\leq \left| \int_{A \cup Y} f_{[-|a|-N-1, |b|+N+1]} d\mu \right| \\ &\leq \left| \int_{A \cup Y} f_{[-|a|-N-1, |b|+N+1]} d\mu - I \right| + |I| \leq 1 + |I|, \end{aligned}$$

so that $\int_X f \, d\mu$ is bounded. Conversely, if for a non-negative function f , $\int_X f \, d\mu$ is bounded, we can put

$$I = \sup_{a < b, A} \int_A f_{[a,b]} \, d\mu < \infty,$$

where the supremum is taken for all $a < b$ and $A \in \Sigma$ with finite $\mu(A)$. Hence, for any $\epsilon > 0$, there are $a_0 < b_0$, $A_0 \in \Sigma$ with finite $\mu(A_0)$ such that

$$0 \leq I - \int_{A_0} f_{[a_0, b_0]} \, d\mu < \epsilon.$$

Put $N = \max\{|a_0|, |b_0|\}$. Then, since f is non-negative, for any $A \in \Sigma$, $A \supset A_0$, $\mu(A) < \infty$, $a < -N$, and $b > N$, we have

$$0 \leq I - \int_A f_{[a,b]} \, d\mu \leq I - \int_{A_0} f_{[a_0, b_0]} \, d\mu < \epsilon,$$

which implies that $\int_X f \, d\mu = I$.

(2) We show that $I(f)$ is equivalent to $I(f^+)$ and $I(f^-)$. In fact, if f^+ and f^- are Lebesgue integrable, then, by

$$f = f^+ - f^-,$$

we know that f is Lebesgue integrable. Conversely, if f is Lebesgue integrable, then, there are $N > 0$ and $Y \in \Sigma$ with $\mu(Y) < \infty$, such that

$$\left| \int_A f_{[a,b]} \, d\mu - I \right| < \epsilon$$

whenever $A \in \Sigma$, $A \supset Y$, $\mu(A) < \infty$, $a < -N$, and $b > N$. It follows that whenever $A \in \Sigma$, $A \supset Y$, $\mu(A) < \infty$, $a_1, a_2 < -N$, and $b_1, b_2 > N$, we have

$$\left| \int_A f_{[a_1, b_1]} \, d\mu - \int_A f_{[a_2, b_2]} \, d\mu \right| \leq \left| \int_A f_{[a_1, b_1]} \, d\mu - I \right| + \left| \int_A f_{[a_2, b_2]} \, d\mu - I \right| < 2\epsilon.$$

Since

$$f_{[a,b]} = f_{[0,b]}^+ - f_{[0,-a]}^-,$$

if we take $a_1 = a_2$, the last inequality gives

$$\left| \int_A f_{[0, b_1]}^+ \, d\mu - \int_A f_{[0, b_2]}^+ \, d\mu \right| < 2\epsilon.$$

This implies that f^+ is Lebesgue integrable. Similarly, we can also show that f^- is Lebesgue integrable.

(3) We show that if $B(f)$ is equivalent to $B(f^+)$ and $B(f^-)$. In fact, it is easy to see that if $\int_X f^+ d\mu$ and $\int_X f^- d\mu$ are bounded, then $\int_X f d\mu$ is bounded, since $f = f^+ - f^-$. Conversely, if $\int_X f d\mu$ is bounded, then, there is $M > 0$ such that

$$\left| \int_A f_{[a,b]} d\mu \right| \leq M,$$

for any $a < b$ and $A \in \Sigma$ with $\mu(A)$ finite. By putting $a = 0$ and $b > 0$, and $a < 0$ and $b = 0$, respectively, we have the boundness of $\int_X f^+ d\mu$ and $\int_X f^- d\mu$.

Now we ready to prove that $I(f)$ is equivalent to $B(f)$. This is obvious from the following equivalences:

$$\begin{aligned} I(f) &\iff I(f^+) \text{ and } I(f^-) \text{ by part (2)} \\ &\iff B(f^+) \text{ and } B(f^-) \text{ by part (1)} \\ &\iff B(f) \text{ by part (3)}. \end{aligned}$$

For the integrability of $|f|$, we can similarly, as part (2), prove that $I(|f|)$ is equivalent $I(f^+)$ and $I(f^-)$, so that $I(|f|)$ is equivalent to $I(f)$. ■

10.30 **Theorem (Comparison Test)** If f is almost measurable, g is Lebesgue integrable, and $|f| \leq g$, then f is Lebesgue integrable.

Proof By Theorem 10.29, the integral $\int_X g d\mu$ is bounded. The inequality implies that the integral $\int_X f d\mu$ is bounded, so that f is Lebesgue integrable, again by Theorem 10.29. ■

1.7 CONVERGENCE THEOREMS

10.31 **Theorem** Suppose f is a non-negative Lebesgue integrable function on a measure space (X, Σ, μ) . Then there is an increasing sequence of measurable simple functions $\{\phi_n\}$, such that $\phi_n \leq f$, $\lim_{n \rightarrow \infty} \phi_n = f$ almost everywhere, and

$$\int_X f d\mu = \lim_{n \rightarrow \infty} \int_X \phi_n d\mu.$$

The same conclusion holds even when $\int_X f d\mu = +\infty$. The proof is omitted.

Proof If f is a bounded function on a measure space (X, Σ, μ) with $\mu(X) < \infty$, in the proof of Theorem 10.25, we know that

$$\int_X f d\mu = \sup_{\substack{\phi \leq f \\ \phi \text{ simple}}} \int_X \phi d\mu.$$

It follows that there is a sequence of measurable simple functions $\{\psi_n\}$, such that $\psi_n \leq f$ and $\int_X \psi_n d\mu = \int_X f d\mu$. The same result is also true when f is unbounded, or $\mu(X) = +\infty$, since

$$\begin{aligned} \int_X f d\mu &= \sup_{\substack{\mu(A) < +\infty \\ b > 0}} \int_A f_{[0,b]} d\mu \\ &= \sup_{\substack{\mu(A) < +\infty \\ b > 0}} \sup_{\substack{\phi \leq f_{[0,b]} \\ \phi \text{ simple}}} \int_A \phi d\mu \\ &= \sup_{\substack{\mu(A) < +\infty \\ \phi \leq f_{[0,b]} \text{ for some } b > 0 \\ \phi \text{ simple}}} \int_A \phi d\mu = \sup_{\substack{\phi \leq f \\ \phi \text{ simple}}} \int_X \phi d\mu. \end{aligned}$$

Put $\phi_n = \max\{\psi_1, \dots, \psi_n\}$. Then $\{\phi_n\}$ is an increasing sequence of measurable simple functions, satisfying $\phi_n \leq f$ and $\lim_{n \rightarrow \infty} \int_X \phi_n d\mu = \int_X f d\mu$.

To see that $\lim_{n \rightarrow \infty} \phi_n = f$ almost everywhere, we notice that

$$\int_X f d\mu = \lim_{n \rightarrow \infty} \int_X \phi_n d\mu \leq \int_X \lim_{n \rightarrow \infty} \phi_n d\mu \leq \int_X f d\mu.$$

It follows from Theorem 10.26(d) that $\lim_{n \rightarrow \infty} \phi_n = f$ almost everywhere. ■

10.32 **Theorem** (**Monotone Convergence Theorem**) Suppose $\{f_n\}$ is an increasing sequence of almost measurable functions on a measure space (X, Σ, μ) . Assume that $\int_X f_1 d\mu$ is finite. Then

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu = \int_X \lim_{n \rightarrow \infty} f_n d\mu.$$

Proof Put $g_n = f_n - f_1$. Then $\{g_n\}$ is an increasing sequence of non-negative measurable functions. Let $g = \lim_{n \rightarrow \infty} g_n$. Then g is measurable and $g_n \leq g$, so that

$$\lim_{n \rightarrow \infty} \int_X g_n d\mu \leq \int_X g d\mu.$$

To prove the last inequality in the reverse direction,

$$\int_X g d\mu \leq \lim_{n \rightarrow \infty} \int_X g_n d\mu,$$

by Theorem 10.31, we only need to show that

$$\int_X \phi d\mu \leq \lim_{n \rightarrow \infty} \int_X g_n d\mu,$$

for any measurable simple function ϕ satisfying $\phi \leq g$. In fact, for any fixed $0 < \alpha < 1$, the set

$$X_n = \{x: g_n(x) \geq \alpha\phi(x)\}$$

is measurable, and $X_n \leq X_{n+1}$. For any $x \in X$, if $\phi(x) = 0$, then $g_n(x) \geq \alpha\phi(x)$; if $\phi(x) > 0$, then $\lim_{n \rightarrow \infty} g_n(x) \geq \phi(x) > \alpha\phi(x)$, so that $g_n(x) > \alpha\phi(x)$ for some n .

It follows that $X = \cup X_n$. For any measurable set A with finite $\mu(A)$, by Theorem 10.14(c), we have

$$\int_X \chi_A d\mu = \mu(A) = \lim_{n \rightarrow \infty} \mu(A \cap X_n) = \lim_{n \rightarrow \infty} \int_{X_n} \chi_A d\mu.$$

Hence, by taking finite linear combinations of χ_A for such A , we have

$$\int_X \alpha\phi d\mu = \lim_{n \rightarrow \infty} \int_{X_n} \alpha\phi d\mu \leq \lim_{n \rightarrow \infty} \int_{X_n} g_n d\mu \leq \lim_{n \rightarrow \infty} \int_X g_n d\mu.$$

We let $\alpha \rightarrow 1^-$ and have

$$\int_X \phi d\mu \leq \lim_{n \rightarrow \infty} \int_X g_n d\mu.$$

Finally, since $\int_X f_1 d\mu$ is finite, by adding it to the both sides of the equality

$$\lim_{n \rightarrow \infty} \int_X g_n d\mu = \int_X \lim_{n \rightarrow \infty} g_n d\mu,$$

we conclude the equality in the theorem for $\{f_n\}$. ■

10.33 **Theorem** (**Fatou's Lemma**) Suppose $\{f_n\}$ is a sequence of almost measurable functions, such that $\int_X \inf f_n d\mu$ is finite. Then

$$\int_X \underline{\lim}_{n \rightarrow \infty} f_n d\mu \leq \underline{\lim}_{n \rightarrow \infty} \int_X f_n d\mu.$$

Similarly, if $\int_X \sup f_n d\mu$ is finite. Then

$$\overline{\lim}_{n \rightarrow \infty} \int_X f_n d\mu \leq \int_X \overline{\lim}_{n \rightarrow \infty} f_n d\mu.$$

Proof We only need to prove the first inequality, since the second one can be obtain from the first one by considering $\{-f_n\}$.

Put $g_n = \inf\{f_n, f_{n+1}, f_{n+2}, \dots\}$. Then $\{g_n\}$ is an increasing sequence of measurable functions, with $\int_X g_1 d\mu$ being finite. It is clear that $\lim_{n \rightarrow \infty} g_n = \underline{\lim}_{n \rightarrow \infty} f_n$. By the Monotone Convergence Theorem, we have

$$\int_X \underline{\lim}_{n \rightarrow \infty} f_n d\mu = \int_X \lim_{n \rightarrow \infty} g_n d\mu = \lim_{n \rightarrow \infty} \int_X g_n d\mu.$$

On the other hand, since $g_n \leq f_n$, we have

$$\lim_{n \rightarrow \infty} \int_X g_n \, d\mu = \underline{\lim}_{n \rightarrow \infty} \int_X g_n \, d\mu \leq \underline{\lim}_{n \rightarrow \infty} \int_X f_n \, d\mu.$$

Combining these yields the inequality of the theorem. ■

10.34 **Example** Let $(\mathbb{R}, \Sigma, \mu)$ be Borel σ -algebra, equipped with the usual measure. In other words, a subset $A \in \Sigma$ if it can be formed from open sets through the operations of countable union, countable intersection, and complement, while $\mu(a, b) = b - a$.

Consider the sequence $\{f_n\}$, where

$$f(x) = \begin{cases} \frac{1}{n}, & \text{if } x \in [0, n], \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that $\{f_n\}$ is a sequence of measurable functions in the measure space $(\mathbb{R}, \Sigma, \mu)$, and $\lim_{n \rightarrow \infty} f_n = 0$. The strict inequality in Fatou's Lemma holds:

$$\int_X \underline{\lim}_{n \rightarrow \infty} f_n \, d\mu = 0 < 1 = \underline{\lim}_{n \rightarrow \infty} \int_X f_n \, d\mu.$$

10.35 **Theorem (Dominated Convergence Theorem)** Suppose $\{f_n\}$ is a sequence of almost measurable functions, and $\lim_{n \rightarrow \infty} f_n = f$ almost everywhere. Assume that $|f_n| \leq g$ almost everywhere, with $\int_X g \, d\mu < \infty$. Then

$$\lim_{n \rightarrow \infty} \int_X f_n \, d\mu = \int_X \lim_{n \rightarrow \infty} f_n \, d\mu = \int_X f \, d\mu.$$

Proof The inequality $|f_n| \leq g$ implies $|\inf f_n| \leq g$ and $|\sup f_n| \leq g$. By the Comparison Test (Theorem 10.30), we know that the integrability of g implies that both $\inf f_n$ and $\sup f_n$ are integrable. By Fatou's Lemma,

$$\int_X \underline{\lim}_{n \rightarrow \infty} f_n \, d\mu \leq \underline{\lim}_{n \rightarrow \infty} \int_X f_n \, d\mu \leq \overline{\lim}_{n \rightarrow \infty} \int_X f_n \, d\mu \leq \int_X \overline{\lim}_{n \rightarrow \infty} f_n \, d\mu.$$

Since $\underline{\lim}_{n \rightarrow \infty} f_n = \lim_{n \rightarrow \infty} f_n = \overline{\lim}_{n \rightarrow \infty} f_n$, we conclude the theorem. ■

10.35 **Theorem (Lebesgue Theorem)** A bounded function on $[a, b]$ is Riemann integrable if and only if it is continuous almost everywhere.

Proof Suppose f is Riemann integrable. For each $x \in [a, b]$, we define the oscillation of f as

$$\omega(x) = \lim_{\epsilon \rightarrow 0^+} \omega_{(x-\epsilon, x+\epsilon)} f.$$

It is clear that f is continuous at x if and only if $\omega(x) = 0$. Denote A to be the collection of points at which f is not continuous. Then

$$A = \{x: \omega(x) > 0\} = \cup A_{1/n},$$

where $A_\delta = \{x: \omega(x) > \delta\}$. If we can prove that $\mu(A_\delta) = 0$ for any $\delta > 0$, then, by the countable sub-additivity, we conclude that f is continuous almost everywhere.

Indeed, for any $\epsilon > 0$, there is a partition P of $[a, b]$ by intervals,

$$P: a = x_0 < x_1 < \cdots < x_n = b,$$

such that

$$\sum \omega_{[x_{i-1}, x_i]}(f) \Delta x_i < \epsilon \delta.$$

It is known that if $\omega(x) > \delta$ for some $x \in (x_{i-1}, x_i)$, then $\omega_{[x_{i-1}, x_i]}(f) \geq \omega(x) > \delta$. Thus, if we put

$$\Omega_\delta = \bigsqcup_{\omega_{[x_{i-1}, x_i]}(f) > \delta} (x_{i-1}, x_i),$$

then

$$A_\delta \subset \Omega_\delta \sqcup \{x_0, x_1, \cdots, x_n\}.$$

On the other hand, since

$$\epsilon \delta > \sum \omega_{[x_{i-1}, x_i]}(f) \Delta x_i \geq \sum_{\omega_{[x_{i-1}, x_i]}(f) > \delta} \omega_{[x_{i-1}, x_i]}(f) \Delta x_i \geq \delta \mu(\Omega_\delta),$$

we have

$$\mu(\Omega_\delta) < \epsilon,$$

so that $\mu(A_\delta) < \epsilon$. By the arbitrariness of ϵ , we have $\mu(A_\delta) = 0$.

Conversely, suppose f is continuous almost everywhere. Let $\{P_n\}$ be a sequence of partitions of $[a, b]$ by intervals, such that P_{n+1} refines P_n and $\|P_n\| \rightarrow 0$. For each P_n , denote

$$\phi_n = \sum \left(\inf_{[x_{i-1}, x_i]} f \right) \chi_{(x_{i-1}, x_i]}, \quad \psi_n = \sum \left(\sup_{[x_{i-1}, x_i]} f \right) \chi_{(x_{i-1}, x_i]}.$$

Since P_{n+1} refines P_n , we know that ϕ_n is decreasing and ψ_n increasing. For a point x at which f is continuous, since $\|P_n\| \rightarrow 0$, we know that

$$\lim_{n \rightarrow \infty} \phi_n(x) = \lim_{n \rightarrow \infty} \psi_n(x) = f(x).$$

The boundedness of f implies that ϕ_1 and ψ_1 are both Lebesgue integrable, since we can take P_1 to be the whole interval $[a, b]$. By the Monotone Convergence Theorem, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{[a, b]} \phi_n \, d\mu &= \int_{[a, b]} \lim_{n \rightarrow \infty} \phi_n \, d\mu = \int_{[a, b]} f \, d\mu \\ &= \int_{[a, b]} \lim_{n \rightarrow \infty} \psi_n \, d\mu = \lim_{n \rightarrow \infty} \int_{[a, b]} \psi_n \, d\mu. \end{aligned}$$

On the other hand, since

$$\begin{aligned} \int_{[a,b]} \psi_n d\mu - \int_{[a,b]} \phi_n d\mu &= \sum \left(\sup_{[x_{i-1}, x_i]} f - \inf_{[x_{i-1}, x_i]} f \right) \Delta x_i \\ &= \sum \omega_{[x_{i-1}, x_i]}(f) \Delta x_i, \end{aligned}$$

by Theorem 6.6, the function f is Riemann integrable on $[a, b]$. ■

Exercises

- 1 If K_1 and K_2 are bounded disjoint closed subsets of \mathbb{R} , then $\lambda(K_1 \cup K_2) = \lambda(K_1) + \lambda(K_2)$.
- 2 If two sets A and B differ by a subset of Lebesgue measure zero, then A is Lebesgue measurable if and only if B is Lebesgue measurable.
- 3 Suppose A is a bounded subset of \mathbb{R} . Prove that the following statements are all equivalent:
 - (a) A is Lebesgue measurable.
 - (b) For any $\epsilon > 0$, there is a bounded open set U containing A , such that $\mu^*(U - A) < \epsilon$.
 - (c) For any $\epsilon > 0$, there is a closed set K contained in A , such that $\mu^*(A - K) < \epsilon$.
- 4 Suppose A is a bounded subset of \mathbb{R} . Prove that A is Lebesgue measurable if and only if for any $\epsilon > 0$, there is a finite union U of open intervals, such that $\mu^*((U - A) \cup (A - U)) < \epsilon$.
- 5 Let f be a real Lipschitz function on \mathbb{R} , that is, there is a constant $L > 0$ such that, for any $x, y \in \mathbb{R}$,

$$|f(x) - f(y)| \leq L|x - y|.$$

Prove that if A is Lebesgue measurable, then $f(A)$ is Lebesgue measurable.

- 6 Let \mathcal{C} be a collection of some subsets in X such that $X, \emptyset \in \mathcal{C}$, and λ be a non-negative-valued function on \mathcal{C} satisfying $\lambda(\emptyset) = 0$. Prove that

$$\mu^*(A) = \inf \left\{ \sum \lambda(C_i) : A \subset \cup C_i, C_i \in \mathcal{C} \right\}$$

is an outer measure, where the union $\cup C_i$ is countable.

- 7 Prove that if an outer measure is invariant under some invertible transformation, then the induced measure is also invariant under the transformation.

- 8 Let μ be a measure on the σ -algebra of Lebesgue measurable subsets. Prove that if $\mu(a, b) = b - a$ for any open interval (a, b) , then μ is the Lebesgue measure. Then use this prove that the Lebesgue measure is translation invariant

$$\mu(A + a) = \mu(A),$$

and satisfies the dialation property

$$\mu(cA) = |c|\mu(A).$$

- 9 Let Σ be a σ -algebra on X that contains infinitely many subsets.
- Suppose $A \in \Sigma$ and A contains infinitely many subsets in Σ . Prove that there is a subset $B \subset A$, such that $B \in \Sigma$, $B \neq A$ and B contains infinitely many subsets in Σ .
 - Prove that there is a strictly decreasing sequence of subsets in Σ . This implies that $X = \sqcup_{i=1}^{\infty} A_i$ for some nonempty and disjoint $A_i \in \Sigma$.
 - Prove that there are uncountably infinitely many subsets in Σ .
- 10 A measure space (X, Σ, μ) is complete if $A \in \Sigma$ with $\mu(A) = 0$, and if $B \subset A$, then $B \in \Sigma$. Prove that in a complete measure space, a subset $A \subset X$ is measurable if and only if for any $\epsilon > 0$, there are measurable subsets B and C , such that $B \subset A \subset C$ and $\mu(C - B) < \epsilon$.
- 11 Suppose (X, Σ, μ) is a measure space with finite $\mu(X)$. Egoroff Theorem says that if f_n are measurable and $\lim_{n \rightarrow \infty} f_n = f$, then for any $\epsilon > 0$, there is $A \in \Sigma$, such that $\mu(X - A) < \epsilon$ and f_n converges uniformly on A .

Define

$$X_{n,\epsilon} = \{x : |f_k(x) - f(x)| < \epsilon \text{ for all } k \geq n\}.$$

Prove Egoroff Theorem in the following steps.

- Show that $\lim_{n \rightarrow \infty} f_n = f$ on X if and only if $X = \cup_n X_{n,\epsilon}$ for any $\epsilon > 0$.
 - Show that $\lim_{n \rightarrow \infty} f_n = f$ uniformly if and only if for any $\epsilon > 0$, there is n , such that $X = X_{n,\epsilon}$.
 - Show that $X_{n,\epsilon}$ is increasing as $n \rightarrow \infty$ and is decreasing as $\epsilon \rightarrow 0^+$.
 - Prove that for any $\epsilon > 0$ and $\delta > 0$, there is n , such that $\mu(X - X_{n,\epsilon}) < \delta$.
 - Prove that for any $\epsilon > 0$, there is a sequence n_k , such that $A = \cap_k X_{n_k, \frac{1}{k}}$ satisfies $\mu(X - A) < \epsilon$, and $\lim_{n \rightarrow \infty} f_n = f$ uniformly on A .
- 12 Suppose A is a bounded Lebesgue measurable subset of \mathbb{R} , and f is a real bounded measurable function defined on A . Then, for any $\epsilon > 0$, there is a compact set $K \subset A$ with $\mu(A - K) < \epsilon$ such that the restriction of f to K is continuous.

This is a special case of Lusin's Theorem. The general statement allows A to be unbounded but $\mu(A) < \infty$, and f to be bounded almost everywhere on A .

- 13 Suppose f is a bounded measurable function on a measure space (X, Σ, μ) with finite $\mu(X)$. Suppose the values of f lies in $(a, b]$. For any partition $\Pi: a = c_0 < c_1 < \dots < c_n = b$, choose $c_i^* \in [c_{i-1}, c_i]$ and define the sum

$$\hat{S}(\Pi, f) = \sum c_i^* \mu(f^{-1}(c_{i-1}, c_i]).$$

Prove that

$$\int_X f \, d\mu = \lim_{\|\Pi\| \rightarrow 0} \hat{S}(\Pi, f) = \int_a^b x \, d\mu(f^{-1}[a, x]).$$

The right side is Riemann-Stieljes integral, and the increasing function $\alpha(x) = \mu(f^{-1}(a, x])$ is the *distribution* of f .

- 14 Suppose f is an integrable function on \mathbb{R} . Prove that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} f(x) \cos nx \, dx = \lim_{n \rightarrow \infty} \int_{\mathbb{R}} f(x) \sin nx \, dx = 0.$$

More generally, if g is a bounded integrable periodic function of period T , then

$$\lim_{t \rightarrow \infty} \int_{\mathbb{R}} f(x) g(tx) \, dx = \frac{1}{T} \int_{[0, T]} g(x) \, dx \cdot \int_{\mathbb{R}} f(x) \, dx.$$

This special case is the so-called Riemann-Lebesgue Lemma.