Math 3043 Honors Real Analysis

Final Examination, Spring 2013

12:30 - 15:30, May 20, 2013

Instructions: This is an open book exam. You can use the textbook "Principles of Mathematical Analysis" by Walter Rudin and the Lecture Notes, but you cannot use any other materials, including the solutions of the exercises.

1. Let X be a set and Σ_1 and Σ_2 are two σ-algebras of X. Show that
   (a) $\Sigma_1 \cap \Sigma_2$ is a σ-algebra of X.
   (b) $\Sigma_1 \cup \Sigma_2$ is not necessarily a σ-algebra of X.

2. (25 Marks) Let $([a,b], \sigma, \mu)$ be the Lebesgue measure space on $[a,b]$. Suppose $\{E_k\}$ be a sequence of Lebesgue measurable set in $[a,b]$ such that $\mu(E_k) = b - a$ for all $k \geq 1$. Show that $\mu(\bigcap_{k=1}^{\infty} E_k) = b - a$.

3. Let f be a measurable (not necessarily bounded) function on $X \subset \mathbb{R}$, with $\mu(X) < \infty$. Show that if f^2 is Lebesgue integrable on X, then

$$\sum_{k=0}^{\infty} k\mu(X_k) < \infty,$$

   where

$$X_k = \{x \in X \colon |f(x)| > k\}.$$

SOLUTION

1. Let X be a set and Σ_1 and Σ_2 are two σ-algebras of X. Show that

   (a) $\Sigma_1 \cap \Sigma_2$ is a σ-algebra of X.

   (b) $\Sigma_1 \cup \Sigma_2$ is not necessarily a σ-algebra of X.

   Solution

   (a) We verify the three axioms of σ-algebra.

   (σ_1) Since $X \in \Sigma_1$ and $X \in \Sigma_2$,

   $$X \in \Sigma_1 \cap \Sigma_2.$$

   (σ_2) Suppose $A, B \in \Sigma_1 \cap \Sigma_2$. Since $A, B \in \Sigma_1$, and Σ_1 is a σ-algebra, we know that $A - B \in \Sigma_1$. Similarly, $A - B \in \Sigma_2$. Thus, $A - B \in \Sigma_1 \cap \Sigma_2$.

   (σ_3) Suppose $\{A_i\}$ is a countable collection of sets in $\Sigma_1 \cap \Sigma_2$. Since $\{A_i\} \subset \Sigma_1$, and Σ_1 is a σ-algebra, we know that $\bigcup_{i=1}^{\infty} A_i \in \Sigma_1$. Similarly, $\bigcup_{i=1}^{\infty} A_i \in \Sigma_2$. Thus, $\bigcup_{i=1}^{\infty} A_i \in \Sigma_1 \cap \Sigma_2$.

   Hence, $\Sigma_1 \cap \Sigma_2$ is a σ-algebra.

   (b) Consider $X = \{0, 1, 2\}$, and

   $$\Sigma_1 = \{\emptyset, \{1\}, \{0, 2\}, X\},$$
   $$\Sigma_2 = \{\emptyset, \{2\}, \{0, 1\}, X\}.$$

   It is easy to show that both Σ_1 and Σ_2 are σ-algebras of X. However,

   $$\Sigma_1 \cup \Sigma_2 = \{\emptyset, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, X\}$$

   is not a σ-algebra, since $\{1\}, \{2\} \in \Sigma_1 \cup \Sigma_2$, but

   $$\{1\} \cup \{2\} = \{1, 2\} \not\in \Sigma_1 \cup \Sigma_2.$$

2. (25 Marks) Let $([a, b], \sigma, \mu)$ be the Lebesgue measure space on $[a, b]$. Suppose $\{E_k\}$ be a sequence of Lebesgue measurable set in $[a, b]$ such that $\mu(E_k) = b - a$ for all $k \geq 1$. Show that $\mu(\bigcap_{k=1}^{\infty} E_k) = b - a$.

Solution By the countable sub-additivity,

$$
\begin{aligned}
0 = \mu\left(\bigcup_{k=1}^{\infty}([a,b]-E_k)\right) &\le \textstyle\sum_{k=1}^{\infty}\mu([a,b]-E_k) \\
&= \sum_{k=1}^{\infty}[\mu([a,b])-\mu(E_k)] = \sum_{k=1}^{\infty}[(b-a)-(b-a)] = 0,
\end{aligned}
$$

so that

$$\mu\left(\bigcup_{k=1}^{\infty}([a,b]-E_k)\right) = 0.$$

Thus,

$$
\begin{aligned}
\mu\left(\bigcap_{k=1}^{\infty}E_k\right) &= \mu\left([a,b]-\left([a,b]-\bigcap_{k=1}^{\infty}E_k\right)\right) \\
&= \mu\left([a,b]-\bigcup_{k=1}^{\infty}([a,b]-E_k)\right) \\
&= \mu([a,b]) - \mu\left(\bigcup_{k=1}^{\infty}([a,b]-E_k)\right) \\
&= (b-a)-0 = b-a.
\end{aligned}
$$

3. Let f be a measurable (not necessarily bounded) function on $X \subset \mathbb{R}$, with $\mu(X) < \infty$. Show that if f^2 is Lebesgue integrable on X, then

$$\sum_{k=0}^{\infty} k\mu(X_k) < \infty,$$

where

$$X_k = \{x \in X\colon |f(x)| > k\}.$$

Solution Since on $X_k - X_{k+1}$,

$$k^2 < f^2(x) \le (k+1)^2,$$

we have

$$k^2[\mu(X_k)-\mu(X_{k+1})] \le \int_{X_k - X_{k+1}} f^2 d\mu \le (k+1)^2[\mu(X_k)-\mu(X_{k+1})]$$

Since f^2 is Lebesgue integrable on X, the left inequality implies

$$\sum_{k=0}^{N} k^2[\mu(X_k)-\mu(X_{k+1})] \le \sum_{k=0}^{N}\int_{X_k-X_{k+1}} f^2 d\mu = \int_{X_0-X_{N+1}} f^2 d\mu \le \int_X f^2 d\mu.$$

By shifting the indices, we have

$$\begin{aligned}
\sum_{k=0}^{N} k^2[\mu(X_k) - \mu(X_{k+1})] &= \sum_{k=0}^{N} k^2\mu(X_k) - \sum_{k=0}^{N} k^2\mu(X_{k+1}) \\
&= \sum_{k=1}^{N} k^2\mu(X_k) - \sum_{k=0}^{N} k^2\mu(X_{k+1}) \\
&= \sum_{i=0}^{N-1} (i+1)^2\mu(X_{i+1}) - \sum_{k=0}^{N} k^2\mu(X_{k+1}) \\
&\geq \sum_{k=0}^{N-1} (k+1)^2\mu(X_{k+1}) - \sum_{k=0}^{N-1} k^2\mu(X_{k+1}) \\
&= \sum_{k=0}^{N-1} (2k+1)\mu(X_{k+1}).
\end{aligned}$$

Thus,

$$\sum_{k=}^{N-1} k\mu(X_k) \leq \sum_{k=0}^{N-1} (2k+1)\mu(X_{k+1}) \leq \int_X f^2 d\mu.$$

Hence

$$\sum_{k=0}^{\infty} k\mu(X_k) < \infty.$$