Math 3043 Honors Real Analysis

Final Examination, Fall 2015

8:30 - 11:30, December 10, 2013

Instructions: This is an open book exam. You can use the textbook "Principles of Mathematical Analysis" by Walter Rudin and the Lecture Notes, but you cannot use any other materials, including the solutions of the exercises.

1. (25 Marks) Show that a bounded set A is Lebesgue measurable if and only if for every $\varepsilon > 0$, there are open sets G_1 and G_2 such that $A \subset G_1$, $A^c \subset G_2$ and $\mu(G_1 \cap G_2) < \varepsilon$.

2. (25 Marks) Show that if f is continuous on $[a, b]$, then $\{x \in [a, b] \mid f(x) > c\}$ is measurable for any c. This implies that continuous functions are measurable.

3. (25 Marks) Suppose $\{f_n\}$ and $\{g_n\}$ are two sequences of measurable functions on the measure space (X, Σ, μ) such that

$$|f_n(x)| \leq g_n(x), \qquad \text{for any } x \in X.$$

Assume that for any $x \in X$,

$$\lim_{n\to\infty} f_n(x) = f(x), \qquad \lim_{n\to\infty} g_n(x) = g(x).$$

Prove that if

$$\lim_{n\to\infty} \int_X g_n \, d\mu = \int_X g \, d\mu < \infty,$$

then

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

Hint: Apply Fatou's Lemma.

SOLUTION

1. (25 Marks) Show that a bounded set A is Lebesgue measurable if and only if for every $\varepsilon > 0$, there are open sets G_1 and G_2 such that $A \subset G_1$, $A^c \subset G_2$ and $\mu(G_1 \cap G_2) < \varepsilon$.

   Solution Let $\varepsilon > 0$ be given.

   "$\Leftarrow$": Suppose that there are open sets G_1 and G_2 such that $A \subset G_1$, $A^c \subset G_2$ and $\mu(G_1 \cap G_2) < \varepsilon$. Denote $K = G_2^c$. Then K is closed. It is obvious that
   $$A^c \subset G_2 \iff K = G_2^c \subset A.$$
   Since $A \subset G_1$ and G_1 is open, we have
   $$\mu^*(G_1 - A) \leq \mu(G_1 - K) = \mu(G_1 \cap K^c) = \mu(G_1 \cap G_2) < \varepsilon.$$
   By Exercise 3, A is Lebsegue measurable.

   "$\Rightarrow$": Suppose A is Lebesgue measurable. By Exercise 3, there are open set U with $A \subset U$ and closed set K with $K \subset A$, such that
   $$\mu(U - A) < \frac{\varepsilon}{2}, \qquad \mu(A - K) < \frac{\varepsilon}{2}.$$
   Let $G_1 = U$ and $G_2 = K^c$. Then G_1 and G_2 are open such that $A \subset G_1$, $A^c \subset G_2$ and
   $$\begin{aligned}\mu(G_1 \cap G_2) &= \mu(U \cap K^c) = \mu(U - K)\\ &= \mu((U - A) \cup (A - K))\\ &= \mu(U - A) + \mu(A - K) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

2. (25 Marks) Show that if f is continuous on $[a, b]$, then $\{x \in [a, b] \mid f(x) > c\}$ is measurable for any c. This implies that continuous functions are measurable.

   Solution In fact, if $x_0 \in A$, since f is continuous, there is $\delta(x_0) > 0$, such that whenever $x \in B(x_0; \delta(x_0)) \cap [a, b]$,
   $$f(x) > c.$$
   This means that
   $$B(x_0; \delta(x_0)) \cap [a, b] \subset A.$$

Thus, A is an open set and

$$\begin{aligned} A = \{x \in [a,b] \,\big|\, f(x) > c\} &= \bigcup_{x_0 \in \{x \in [a,b] \,\big|\, f(x) > c\}} \left(B(x_0; \delta(x_0)) \cap [a,b]\right) \\ &= \left(\bigcup_{x_0 \in \{x \in [a,b] \,\big|\, f(x) > c\}} B(x_0; \delta(x_0)\right) \cap [a,b]. \end{aligned}$$

This expression implies that A is Lebesgue measurable.

3. (25 Marks) Suppose $\{f_n\}$ and $\{g_n\}$ are two sequences of measurable functions on the measure space (X, Σ, μ) such that

$$|f_n(x)| \le g_n(x), \qquad \text{for any } x \in X.$$

Assume that for any $x \in X$,

$$\lim_{n\to\infty} f_n(x) = f(x), \qquad \lim_{n\to\infty} g_n(x) = g(x).$$

Prove that if

$$\lim_{n\to\infty} \int_X g_n \, d\mu = \int_X g \, d\mu < \infty,$$

then

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

Hint: Apply Fatou's Lemma.

Solution Since f_n and g_n are measurable, and

$$\lim_{n\to\infty} f_n(x) = f(x), \qquad \lim_{n\to\infty} g_n(x) = g(x),$$

we know that f, g, $g_n - f_n$, $g_n + f_n$ are all measurable. By the hypothesis that $|f_n| \le g_n$ on X, we know that $-g_n \le f_n \le g_n$, so that $g_n - f_n$ and $g_n + f_n$ are nonnegative on X. Because $\displaystyle\int_X g \, d\mu < \infty$, we have

$$\begin{aligned} 0 \le \int_X \inf(g_n - f_n) \, d\mu &\le \int_X (g_n - f_n) \, d\mu \\ &\le \int_X g_n \, d\mu + \int_X |f_n| \, d\mu \\ &\le 2 \int_X g_n \, d\mu < \infty \end{aligned}$$

for sufficiently large n, so that $\displaystyle\int_X \inf(g_n - f_n)\,d\mu$ is finite for sufficiently large n. Similarly, we know that $\displaystyle\int_X \inf(g_n + f_n)\,d\mu$ is finite for sufficiently large n.

By Fatou's Lemma, we have

$$\begin{aligned}
\int_X (g-f)\,d\mu &= \int_X \lim_{n\to\infty}(g_n - f_n)\,d\mu = \int_X \varliminf_{n\to\infty}(g_n - f_n)\,d\mu \\
&\leq \varliminf_{n\to\infty}\int_X (g_n - f_n)\,d\mu = \varliminf_{n\to\infty}\left(\int_X g_n - \int_X f_n\,d\mu\right) \\
&= \int_X g\,d\mu - \varlimsup_{n\to\infty}\int_X f_n\,d\mu, \\
\int_X (g+f)\,d\mu &= \int_X \lim_{n\to\infty}(g_n + f_n)\,d\mu = \int_X \varliminf_{n\to\infty}(g_n + f_n)\,d\mu \\
&\leq \varliminf_{n\to\infty}\int_X (g_n + f_n)\,d\mu = \varliminf_{n\to\infty}\left(\int_X g_n + \int_X f_n\,d\mu\right) \\
&= \int_X g\,d\mu + \varliminf_{n\to\infty}\int_X f_n\,d\mu.
\end{aligned}$$

These give

$$\int_X f\,d\mu \geq \varlimsup_{n\to\infty}\int_X f_n\,d\mu \geq \varliminf_{n\to\infty}\int_X f_n\,d\mu \geq \int_X f\,d\mu,$$

so that

$$\lim_{n\to\infty}\int_X f_n\,d\mu = \int_X f\,d\mu.$$