Math 3043 Honors Real Analysis

Midterm Test, Spring 2013

19:00 - 22:00, April 12, 2013

Instructions: This is an open book exam. You can use the textbook "Principles of Mathematical Analysis" by Walter Rudin, but you cannot use any other materials, including the solutions of the exercises.

1. (25 Marks) Suppose f is a continuous on $[0, 2\pi]$, and $f(0) = f(2\pi)$. Show that if the Fourier expansion of f is zero, then $f \equiv 0$ on $[0, 2\pi]$. What about if $f(0) \neq f(2\pi)$?

2. (25 Marks) Suppose $F\colon \mathcal{R} = \mathbb{R} \times [a, b] \to \mathbb{R}$ is continuous, and differentiable with respect to the first variable. Assume that

$$0 < m \leq \frac{\partial}{\partial x} F(x, t) \leq M$$

for all $(x, t) \in \mathcal{R}$.

(a) Prove that the mapping $\mathcal{F}$ defined by

$$\mathcal{F}(\phi)(t) = \phi(t) - \frac{1}{M} F(\phi(t), t)$$

is a contraction from $C[a, b]$ to $C[a, b]$.

(b) Show that for every continuous function ψ on $[a, b]$, there is a unique continuous function ϕ on $[a, b]$, such that

$$F(\phi(t), t) = \psi(t).$$

3. (50 Marks)

(a) Show that

$$\lim_{n\to\infty} \left[\frac{2}{\pi} + \frac{4}{\pi} \sum_{k=1}^{n} \frac{(-1)^{n+1}}{4k^2 - 1} \cos 2kx \right] = |\cos x|,$$

uniformly on $\mathbb{R}$.

(b) Use Part (a) to prove that for any Riemann integrable function f on $[a, b]$,

$$\lim_{\lambda \to \infty} \int_a^b f(x) |\cos \lambda x| \, dx = \frac{2}{\pi} \int_a^b f(x) \, dx.$$

(c) Give a generalization of Part (b). Justify your generalization.

SOLUTIONS

1. (25 Marks) Suppose f is a continuous on $[0, 2\pi]$, and $f(0) = f(2\pi)$. Show that if the Fourier expansion of f is zero, then $f \equiv 0$ on $[0, 2\pi]$. What about if $f(0) \neq f(2\pi)$?

Solution By Exercise 15, we know that $\sigma_N(f; x) \to f(x)$ uniformly on $[0, 2\pi]$. By the hypothesis, $\sigma_N(f; x) \equiv 0$. Hence,

$$\int_0^{2\pi} |f(x)|^2 \, dx = \int_0^{2\pi} \lim_{N\to\infty} |\sigma_N(f; x) - f(x)|^2 \, dx = 0.$$

Since f is continuous, this implies that $f \equiv 0$ on $[0, 2\pi]$.

If $f(0) \neq f(2\pi)$, consider the continuous function $F(x) = f(x) \sin x$. We need to check that all Fourier coefficients of F are zero. In fact,

$$\begin{aligned}
a_n &= \frac{1}{\pi} \int_0^{2\pi} F(x) \cos nx \, dx = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin x \, \cos nx \, dx, \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(x) \left[\sin(n+1)x - \sin(n-1)x\right] \, dx = 0, \quad n \geq 0, \\
b_n &= \frac{1}{\pi} \int_0^{2\pi} F(x) \sin nx \, dx = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin x \, \sin nx \, dx, \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(x) \left[\cos(n-1)x - \cos(n+1)x\right] \, dx = 0.
\end{aligned}$$

Hence, we know that $f(x) \sin x = 0$ for all $x \in [0, 2\pi]$. This implies that $f(x) = 0$ for all $x \in [0, 2\pi]$ except at $x = 0, \pi$, and 2π. By the continuity of f, we also have $f(x) = 0$ at these three points. So, $f \equiv 0$ on $[0, 2\pi]$.

2. (25 Marks) Suppose $F \colon \mathcal{R} = \mathbb{R} \times [a, b] \to \mathbb{R}$ is continuous, and differentiable with respect to the first variable. Assume that

$$0 < m \leq \frac{\partial}{\partial x} F(x, t) \leq M$$

for all $(x, t) \in \mathcal{R}$.

(a) Prove that the mapping $\mathcal{F}$ defined by

$$\mathcal{F}(\phi)(t) = \phi(t) - \frac{1}{M} F(\phi(t), t)$$

is a contraction from $C[a, b]$ to $C[a, b]$.

(b) Show that for every continuous function ψ on $[a, b]$, there is a unique continuous function ϕ on $[a, b]$, such that

$$F(\phi(t), t) = \psi(t).$$

Solution

(a) Since F is continuous, it is clear that $\mathcal{F}$ is from $C[a,b]$ to $C[a,b]$. For any $\phi, \psi \in C[a,b]$,

$$\begin{aligned}
|\mathcal{F}(\phi)(t) - \mathcal{F}(\psi)(t)| &= \left|[\phi(t) - \frac{1}{M}F(\phi(t),t)] - [\psi(t) - \frac{1}{M}F(\psi(t),t)]\right| \\
&= \left|[\phi(t) - \psi(t)]\left(1 - \frac{1}{M}F_x(c,t)\right)\right| \\
&\le |\phi(t) - \psi(t)| \cdot \left(1 - \frac{m}{M}\right).
\end{aligned}$$

Put $\alpha = 1 - \dfrac{m}{M}$. It is easy to see that $0 \le \alpha < 1$. Thus, we have

$$\|\mathcal{F}(\phi) - \mathcal{F}(\psi)\| \le \alpha\|\phi - \psi\|.$$

Hence, $\mathcal{F}$ is a contraction from $C[a,b]$ to $C[a,b]$.

(b) Similar to Part (a), the mapping $\mathcal{G}$ defined by

$$\mathcal{G}(\phi)(t) = \phi(t) - \frac{1}{M}[F(\phi(t),t) - \psi(t)]$$

is a contraction from $C[a,b]$ to $C[a,b]$. By Theorem 7.15, $C[a,b]$ is a complete metric space. Hence, $\mathcal{G}$ is a unique fixed point:

$$\phi(t) - \frac{1}{M}[F(\phi(t),t) - \psi(t)] = \phi(t),$$

which is equivalent to that the $F(\phi(t),t) = \psi(t)$ has a unique solution in $C[a,b]$.

3. (50 Marks)

(a) Show that

$$\lim_{n\to\infty}\left[\frac{2}{\pi} + \frac{4}{\pi}\sum_{k=1}^{n}\frac{(-1)^{n+1}}{4k^2-1}\cos 2kx\right] = |\cos x|,$$

uniformly on $\mathbb{R}$.

(b) Use Part (a) to prove that for any Riemann integrable function f on $[a,b]$,

$$\lim_{\lambda\to\infty}\int_a^b f(x)|\cos\lambda x|\,dx = \frac{2}{\pi}\int_a^b f(x)\,dx.$$

(c) Give a generalization of Part (b). Justify your generalization.

Solution

(a) We expand the function $|\cos x|$ as a Fourier series on $[-\pi, \pi]$. The coefficients are

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} |\cos x|\, dx = \frac{4}{\pi}\int_0^{\pi/2} \cos x\, dx = \frac{4}{\pi}, \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} |\cos x|\, \cos nx\, dx = \frac{2}{\pi}\int_0^{\pi} |\cos x|\, \cos nx\, dx \\
&= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos x\, \cos nx\, dx - \int_{\pi/2}^{\pi} \cos x\, \cos nx\, dx\right] \\
&= \frac{1}{\pi}\left[\int_0^{\pi/2} (\cos(n+1)x + \cos(n-1)x)\, dx\right. \\
&\qquad \left. - \int_{\pi/2}^{\pi} (\cos(n+1)x + \cos(n-1)x)\, dx\right] \\
&= \frac{2}{\pi}\left[\frac{\sin(n+1)\frac{\pi}{2}}{n+1} + \frac{\sin(n-1)\frac{\pi}{2}}{n-1}\right] \\
&= \begin{cases} \dfrac{4}{\pi}\dfrac{(-1)^{k+1}}{4k^2-1}, & \text{if } n = 2k, \\ 0, & \text{if } n = 2k+1. \end{cases}
\end{aligned}$$

Since $|\cos x|$ is a continuous, 2π-periodic, and piecewise differentiable, we know that on $\mathbb{R}$,

$$|\cos x| = \frac{2}{\pi} + \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{4k^2-1} \cos 2kx,$$

and the convergence of the series is uniform. Since f is bounded, we have a uniformly convergent expansion:

$$f(x)|\cos \lambda x| = \frac{2}{\pi} f(x) + \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^{k+1} f(x)}{4k^2-1} \cos 2\lambda kx.$$

Hence, for any finite interval $[a, b]$,

$$\int_a^b f(x)|\cos \lambda x|\, dx = \frac{2}{\pi}\int_a^b f(x)\, dx + \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{4k^2-1} \int_a^b f(x) \cos 2\lambda kx\, dx.$$

The series on the right, as a function of λ, is uniformly convergent on $\mathbb{R}$, since

$$\left| \frac{(-1)^{k+1}}{4k^2-1} \int_a^b f(x) \cos 2\lambda kx\, dx \right| \le \frac{1}{4k^2-1} \int_a^b |f(x)|\, dx.$$

Thus,

$$\lim_{\lambda\to\infty}\int_a^b f(x)|\cos\lambda x|\,dx = \frac{2}{\pi}\int_a^b f(x)\,dx + \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{4k^2-1}\lim_{\lambda\to\infty}\int_a^b f(x)\cos 2\lambda kx\,dx$$
$$= \frac{2}{\pi}\int_a^b f(x)\,dx,$$

by the Riemann-Lebesgue Lemma.

(c) A generalization of the result in Part (b) can be stated as follows.

Suppose g is a continuous, 2π-periodic, and piecewise continuously differentiable. Then for any Riemann integrable function f on $[a,b]$,

$$\lim_{\lambda\to\infty}\int_a^b f(x)g(\lambda x)\,dx = \frac{1}{2\pi}\int_a^b f(x)\,dx\cdot\int_0^{2\pi} g(x)\,dx$$

In fact, the function g can be expanded as a uniformly convergent Fourier series,

$$g(x) = a_0 + \sum_{k=1}^{\infty}\left[a_k\cos kx + b_k\sin kx\right],$$

where $a_0 = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} g(x)\,dx$. Since f is bounded, the function $f(x)g(\lambda x)$ has a uniformly convergent expansion

$$f(x)g(\lambda x) = a_0 f(x) + \sum_{k=1}^{\infty}\left[a_k f(x)\cos\lambda kx + b_k f(x)\sin\lambda kx\right].$$

Thus, for any finite interval $[a,b]$,

$$\int_a^b f(x)g(\lambda x)\,dx = a_0\int_a^b f(x)\,dx + \sum_{k=1}^{\infty}\left[a_k\int_a^b f(x)\cos\lambda kx\,dx + b_k\int_a^b f(x)\sin kx\,dx\right]$$

Since

$$\left|\int_a^b f(x)\cos\lambda kx\,dx\right| \le \int_a^b |f(x)|\,dx, \qquad \left|\int_a^b f(x)\sin\lambda kx\,dx\right| \le \int_a^b |f(x)|\,dx,$$

and since $\sum(|a_k|+|b_k|)<\infty$, so that the last infinite series, as a function of λ, converges uniformly on $\mathbb{R}$. Thus,

$$\lim_{\lambda\to\infty}\int_a^b f(x)g(\lambda x)\,dx$$
$$= a_0\int_a^b f(x)\,dx + \sum_{k=1}^{\infty}\left[a_k\lim_{\lambda\to\infty}\int_a^b f(x)\cos\lambda kx\,dx + b_k\lim_{\lambda\to\infty}\int_a^b f(x)\sin\lambda kx\,dx\right]$$
$$= a_0\int_a^b f(x)\,dx = \frac{1}{2\pi}\int_a^b f(x)\,dx\cdot\int_0^{2\pi} g(x)\,dx,$$

by the Riemann-Lebesgue Lemma.