Math 3043 Honors Real Analysis

Midterm Test, Fall 2015

19:30 - 21:30, November 4, 2015

Instructions: This is an open book exam. You can use the textbook "Principles of Mathematical Analysis" by Walter Rudin, but you cannot use any other materials, including the solutions of the exercises.

1. (20 Marks) Suppose f is a continuous on $[-\pi, \pi]$, $f(-\pi) = f(\pi)$, and

$$\int_{-\pi}^{\pi} f(x) \sin nx \, dx = 0$$

for all natural numbers n. Prove that f is an even function.

2. (20 Marks) Suppose that the function f satisfies the following conditions:

(a) $-\infty < a \le f(x) \le b < \infty$;

(b) $|f(x) - f(y)| \le L|x - y|$ for all $x, y \in [a, b]$, where L is constant satisfying $0 \le L < 1$.

Show that for any $x_0 \in [a, b]$, the sequence $\{x_n\}$, generated by the recursive formula

$$x_{n+1} = \tfrac{1}{2}\left[x_n + f(x_n)\right], \qquad n = 0, 1, 2, \ldots,$$

converges to the unique fixed point of f in $[a, b]$.

3. (20 Marks) Consider the equation

$$x^2 + y + e^{x^2+y} = 1.$$

(1) Show that the equation defines a unique continuous function $y = y(x)$ such that $y(0) = 0$ in a neighborhood of the point $(0, 0)$.

(2) Show that $y(x)$ is differentiable in a neighborhood of $x = 0$.

(3) Show that $y = y(x)$ has a local maximum at $x = 0$.

(4) Does the equation define a single-valued function $x = x(y)$ such that $x(0) = 0$? Explain.

SOLUTIONS

1. (20 Marks) Suppose f is a continuous on $[-\pi, \pi]$, $f(-\pi) = f(\pi)$, and

$$\int_{-\pi}^{\pi} f(x)\, \sin nx\, dx = 0$$

for all natural numbers n. Prove that f is an even function.

Solution For the given function f, the function $g(x) = f(x) - f(-x)$ is an odd continuous function, so that

$$\int_{-\pi}^{\pi} g(x)\, \cos nx\, dx.$$

By the hypothesis, we have

$$\begin{aligned}\int_{-\pi}^{\pi} g(x)\, \sin nx\, dx &= \int_{-\pi}^{\pi} f(x)\, \sin nx\, dx + \int_{-\pi}^{\pi} f(-x)\, \sin nx\, dx \\ &= 0 + 0 = 0.\end{aligned}$$

Thus, the function g has zero Fourier expansion, so that $\sigma_N(g; x) \equiv 0$ on $[-\pi, \pi]$, where

$$\sigma_N(g; x) = \frac{s_0 + s_1 + \cdots + s_N}{N+1},$$

and $s_N(g; x)$ is the Nth partial sum of the Fourier series of g. By Exercise 15, we know that $\sigma_N(g; x) \to g(x)$ uniformly on $[-\pi, \pi]$. By the hypothesis, $\sigma_N(f; x) \equiv 0$. Consequently,

$$g(x) = \lim_{N\to\infty} \sigma_N(g; x) = 0,$$

that is, $f(x) = f(-x)$. Therefore, f is an even function.

2. (20 Marks) Suppose that the function f satisfies the following conditions:

(a) $-\infty < a \leq f(x) \leq b < \infty$;

(b) $|f(x) - f(y)| \leq L|x - y|$ for all $x, y \in [a, b]$, where L is constant satisfying $0 \leq L < 1$.

Show that for any $x_0 \in [a, b]$, the sequence $\{x_n\}$, generated by the recursive formula

$$x_{n+1} = \tfrac{1}{2}\left[x_n + f(x_n)\right], \qquad n = 0, 1, 2, \ldots,$$

converges to the unique fixed point of f in $[a, b]$.

Solution We can show by induction that $x_n \in [a, b]$ for all integer n. Indeed, we know that $x_0 \in [a, b]$. Suppose $x_k \in [a, b]$ for some integer k. Then, by (a), $a \leq f(x_k) \leq b$. Thus,

$$x_{k+1} = \tfrac{1}{2}\left[x_n + f(x_n)\right] \geq \tfrac{1}{2}\left(a + a\right) = a,$$

and

$$x_{k+1} = \tfrac{1}{2}\left[x_n + f(x_n)\right] \le \tfrac{1}{2}\left(b + b\right) = b,$$

so that $x_{k+1} \in [a, b]$. Hence, $x_n \in [a, b]$ for all integer n.

Now, we show that $\{x_n\}$ is a Cauchy sequence. In fact, by (b),

$$\begin{aligned} |x_2 - x_1| &= \tfrac{1}{2}\left|[x_1 + f(x_1)] - [x_0 + f(x_0)]\right| \\ &\le \tfrac{1}{2}\left(|x_1 - x_0| + |f(x_1) - f(x_0)|\right) \\ &\le \tfrac{1}{2}\left(|x_1 - x_0| + L|x_1 - x_0|\right) = \left(\tfrac{1+L}{2}\right)|x_1 - x_0|. \end{aligned}$$

In general,

$$\begin{aligned} |x_{n+1} - x_n| &= \tfrac{1}{2}\left|[x_n + f(x_n)] - [x_{n-1} + f(x_{n-1})]\right| \\ &\le \tfrac{1}{2}\left(|x_n - x_{n-1}| + |f(x_n) - f(x_{n-1})|\right) \\ &\le \tfrac{1}{2}\left(|x_n - x_{n-1}| + L|x_n - x_{n-1}|\right) = \left(\tfrac{1+L}{2}\right)|x_n - x_{n-1}| \\ &\le \left(\tfrac{1+L}{2}\right)^2 |x_{n-1} - x_{n-2}| \le \cdots \le \left(\tfrac{1+L}{2}\right)^n |x_1 - x_0|. \end{aligned}$$

Thus, for all integers n and p,

$$\begin{aligned} |x_{n+p} - x_n| &= |x_{n+p} - x_{n+p-1}| + |x_{n+p-1} - x_{n+p-2}| + \cdots + |x_{n+1} - x_n| \\ &\le \left(\tfrac{1+L}{2}\right)^{n+p-1}|x_1 - x_0| + \left(\tfrac{1+L}{2}\right)^{n+p-2}|x_1 - x_0| + \cdots + \left(\tfrac{1+L}{2}\right)^n |x_1 - x_0| \\ &= \left(\tfrac{1+L}{2}\right)^n \cdot \frac{1 - \left(\frac{1+L}{2}\right)^p}{1 - \frac{1+L}{2}}|x_1 - x_0| \\ &< \left(\tfrac{1+L}{2}\right)^n \cdot \frac{1}{1 - \frac{1+L}{2}}|x_1 - x_0| = \left(\tfrac{1+L}{2}\right)^n \cdot \frac{2}{1 - L}|x_1 - x_0|. \end{aligned}$$

Since $0 \le L < 1$, we have $0 < \frac{1+L}{2} < 1$, so that $\lim\limits_{n\to 0}\left(\frac{1+L}{2}\right)^n = 0$. Hence, we see that $\{x_n\}$ is a Cauchy sequence.

Denote $\xi = \lim\limits_{n\to 0} x_n$. From (b), we know that f is continuous. By taking $n \to \infty$ on both sides of the recursive formula, we get $\xi = \frac{1}{2}\left[\xi + f(\xi)\right]$, which gives $f(\xi) = \xi$.

The uniquesness is a consequence of the condition (b).

3. (20 Marks) Consider the equation

$$x^2 + y + e^{x^2+y} = 1.$$

(1) Show that the equation defines a unique continuous function $y = y(x)$ such that $y(0) = 0$ in a neighborhood of the point $(0, 0)$.

(2) Show that $y(x)$ is differentiable in a neighborhood of $x = 0$.

(3) Show that $y = y(x)$ has a local maximum at $x = 0$.

(4) Does the equation define a single-valued function $x = x(y)$ such that $x(0) = 0$? Explain.

Solution

(1) Clearly, the elementary function $F(x, y) = x^2 + y + e^{x^2+y} - 1$ is smooth and maps $\mathbb{R}^2$ into $\mathbb{R}$. Since $F(0, 0) = 0$ and

$$\frac{\partial F}{\partial y}(0, 0) = 1 + 1 = 2 \neq 0,$$

we know that F satisfies the hypotheses of the Implicit Function Theorem. Thus, the equation $F(x, y) = 0$ defines a unique continuous function $y = y(x)$ such that $y(0) = 0$ in a neighborhood of the point $(0, 0)$.

(2) Since $F_x(x, y) = 2x + 2xe^{x^2+y}$ and $F_y(x, y) = 1 + e^{x^2+y}$ are continuous in a neighborhood of $(0, 0)$ and $F_y(0, 0) = 2 \neq 0$, we know that the derivative of $y = y(x)$ exists, and

$$y'(x) = -\frac{F_x(x, y)}{F_y(x, y)} = -\frac{2x + 2xe^{x^2+y}}{1 + e^{x^2+y}}.$$

(3) From (2), we see that $y'(0) = 0$ and the derivative of $y(x)$ changes from positive to negative when x varies from negative to positive near $x = 0$. By the First Derivative Test, $y = y(x)$ has a local maximum at $x = 0$.

(4) From (3), we see that $y = y(x)$ has a local maximum at $x = 0$. Thus, for small negative values of y, the equation $F(x, y) = 0$ will give at least two corresponding values of x in any small neighborhood of 0. Hence, the equation $F(x, y)$ does not define a single-valued function $x = x(y)$ such that $x(0) = 0$.