Math4424 Homework 4

8.1. Determine the population principal components Y_1 and Y_2 for the covariance matrix

$$\mathbf{\Sigma} = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}.$$

Also, calculate the proportion of the total population variance explained by the first principal component. ■

8.3. Let

$$\mathbf{\Sigma} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Determine the principal components Y_1, Y_2 and Y_3. What can you say about the eigenvectors (and principal components) associated with eigenvalues that are not distinct? ■

8.4. Find the principal components and the proportion of the total population variance explained by each when the covariance matrix is

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma^2 & \sigma^2\rho & 0 \\ \sigma^2\rho & \sigma^2 & \sigma^2\rho \\ 0 & \sigma^2\rho & \sigma^2 \end{bmatrix}, \qquad -\frac{1}{\sqrt{2}} < \rho < \frac{1}{\sqrt{2}}.$$ ■

8.5. (a). Find the eigenvalues of the correlation matrix

$$\boldsymbol{\rho} = \begin{bmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{bmatrix},$$

Are your results consistent with (8-16) and (8-17)?

(b). Verify the eigenvalue-eigenvector pairs for the $p \times p$ matrix $\boldsymbol{\rho}$ given in (8-15).

8.6. Data on x_1 = sales and x_2 = profits for the 10 largest companies in the world were listed in Exercise 1.4 of Chapter 1. From Example 4.12,

$$\bar{\mathbf{x}} = \begin{bmatrix} 155.60 \\ 14.70 \end{bmatrix}, \qquad \mathbf{S} = \begin{bmatrix} 7476.45 & 303.62 \\ 303.62 & 26.19 \end{bmatrix}.$$

(a). Determine the sample principal components and their variances for these data. (You may need the quadratic formula to solve for the eigenvalues of $\mathbf{S}$).

(b). Find the proportion of the total sample variance explained by $\hat{y}_1$.

(c). Sketch the constant density ellipse $(\mathbf{x}-\bar{\mathbf{x}})'\mathbf{S}^{-1}(\mathbf{x}-\bar{\mathbf{x}}) = 1.4$, and indicate the principal components $\hat{y}_1$ and $\hat{y}_2$ on your graph.

(d). Compute the correlation coefficients $r_{\hat{y}_1, x_k}, k = 1, 2$. What interpretation, if any, can you give to the first principal component? ■