Solution Set 1

1. Solve the characteristic equation, find the general solution of $u_x - u_y + yu = 0$

   Answer

   The characteristic equation is $dy/dx = -1$, so the characteristics are straight lines $y = -x + k$. Use the transformation

   $$\xi = x, \eta = x + y.$$

   The transformed differential equation is

   $$w_\xi + (\eta - \xi)w = 0.$$

   Multiply this equation by $e^{\eta\xi - \xi^2/2}$ to get

   $$\frac{d}{d\xi}\left(e^{\eta\xi - \xi^2/2} w\right) = 0.$$

   Then

   $$we^{\eta\xi - \xi^2/2} = g(\eta),$$

   so

   $$w(\xi, \eta) = g(\eta)e^{\xi^2/2}e^{-\eta\xi}.$$

   In terms of x and y,

   $$u(x, y) = g(x + y)e^{-xy - x^2/2}.$$

2. Solve the characteristic equation, find the general solution of $xu_x - yu_y + u = x$.

   Answer

   The characteristic equation is $dy/dx = -y/x$. The characteristics are hyperbolas $xy = k$. Let $\xi = x, \eta = xy$ to get

   $$\xi w_\xi + w = \xi,$$

   with the general solution

   $$w = \frac{1}{2}\xi + \frac{1}{\xi}g(\eta).$$

   Then

   $$u(x, y) = \frac{1}{2}x + \frac{1}{x}g(xy).$$

3. Use the method of characteristics to find a solution of the pde that passes through the given curve Γ.

$u_x - u_y = u^2$; $\quad \Gamma$ is given by $y = 2x - 1,\ u = 4$.

Answer

The characteristics are given by

$$x = t + A, y = -t + B, -\frac{1}{u} = t + C.$$

Suppose that a characteristic intersects Γ at $P : (s, 2s - 1, 4)$ when $t = 0$. Then

$$x = s = A, y = 2s - 1 = B, -\frac{1}{u} = t - \frac{1}{4} = C.$$

Eliminate s and t to obtain the explicit solution

$$u(x, t) = \frac{3}{y - 2x + \frac{7}{4}}.$$

4. Use the method of characteristics to find a solution of the pde that passes through the given curve Γ.

$xu_x + u_y = e^u$; $\quad \Gamma$ is given by $y = x - 1, u = 0$.

Answer

The characteristics are given by

$$x = Ae^t, y = t + B, -e^{-u} = t + C.$$

Suppose that a characteristic intersects Γ at $P : (s, s - 1, 0)$ when $t = 0$. Then

$$x = A = s, y = B = s - 1, C = -1.$$

Then

$$x = se^t, y = t + s - 1, -e^{-u} = t - 1.$$

Eliminate s and t to obtain the implicitly defined solution

$$x = \left(y + e^{-u}\right) e^{1-e^{-u}}.$$

5. Use the method of characteristics to find a solution of the pde that passes through the given curve Γ.

$uu_x + u_y = 0$; $\quad \Gamma$ is given by $y = 0, u = h(x)$.

Answer

The characteristics are given by the equations

$$dx/dt = u, \; dy/dt = 1, \text{ and } du/dt = 0.$$

Solutions are of the form

$$u = C(s), \; y = t + B(s), \text{ and } x = C(s)t + A(s).$$

At $t = 0$, $(x, y, u) = (A(s), B(s), C(s))$ and is on the curve $(s, 0, h(s))$. Therefore, $A(s) = s$, $B(s) = 0$, and $C(s) = h(s)$. The solution surface is parameterized as

$$(x, y, u) = (h(s)t + s, t, h(s)).$$

The implicitly defined solution is

$$u = h(x - uy).$$