GREEN'S IDENTITIES AND GREEN'S FUNCTIONS

Green's first identity

First, recall the following theorem.

Theorem:

> (Divergence Theorem) Let $\mathcal{D}$ be a bounded solid region with a piecewise C^1 boundary surface $\partial\mathcal{D}$. Let $\mathbf{n}$ be the unit outward normal vector on $\partial\mathcal{D}$. Let $\mathbf{f}$ be any C^1 vector field on $\overline{\mathcal{D}} = \mathcal{D} \cup \partial\mathcal{D}$. Then
>
> $$\iiint_{\mathcal{D}} \vec{\nabla} \cdot \mathbf{f}\, dV = \iint_{\partial\mathcal{D}} \mathbf{f} \cdot \mathbf{n}\, dS$$
>
> where dV is the volume element in $\mathcal{D}$ and dS is the surface element on $\partial\mathcal{D}$.

By integrating the identity

$$\vec{\nabla} \cdot (v\vec{\nabla}u) = \vec{\nabla}v \cdot \vec{\nabla}u + v\ \Delta u$$

over $\mathcal{D}$ and applying the divergence theorem, one gets

$$\iint_{\partial\mathcal{D}} v\frac{\partial u}{\partial n} dS = \iiint_{\mathcal{D}} \vec{\nabla}v \cdot \vec{\nabla}u\, dV + \iiint_{\mathcal{D}} v\Delta u\, dV$$

where $\partial u/\partial n = \mathbf{n} \cdot \vec{\nabla}u$ is the directional derivative in the outward normal direction. This is *Green's first identity*.

Constraint on Neumann problems

Let $v = 1$. Green's first identity becomes

$$\iint_{\partial\mathcal{D}} \frac{\partial u}{\partial n} dS = \iiint_{\mathcal{D}} \Delta u \, dV. \qquad (*)$$

Consider the Neumann problem in any domain $\mathcal{D}$:

$$\Delta u = f(\mathbf{x}) \quad \text{in } \mathcal{D}$$

$$\frac{\partial u}{\partial n} = h(\mathbf{x}) \quad \text{on } \partial\mathcal{D}.$$

Applying the above identity, one gets

$$\iint_{\partial\mathcal{D}} h \, dS = \iiint_{\mathcal{D}} f \, dV.$$

Therefore h and f cannot be freely selected, else the problem may have no solution.

Mean value property

In three dimensions, the average value of any harmonic function over any sphere equals its value at the center. Applying the identity $(*)$ to a sphere with radius a in spherical coordinates, one gets

$$\int_0^{2\pi} \int_0^{\pi} u_r(a, \theta, \phi) a^2 \sin\theta d\theta d\phi = 0$$

$$= a^2 \int_0^{2\pi} \int_0^{\pi} u_r(a, \theta, \phi) \sin\theta d\theta d\phi$$

This gives

$$\frac{d}{dr}\left[\frac{1}{4\pi r^2}\int_0^{2\pi}\int_0^{\pi} u(r,\theta,\phi)r^2\sin\theta d\theta d\phi\right]$$

$$= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} u_r(r,\theta,\phi)\sin\theta d\theta d\phi = 0$$

Thus the mean value inside the rectangular bracket is independent of r and

$$\frac{1}{4\pi r^2}\int_0^{2\pi}\int_0^{\pi} u(r,\theta,\phi)r^2\sin\theta d\theta d\phi = u(\mathbf{0}).$$

Maximum principle

If $\mathcal{D}$ is a connected solid region, a non-constant harmonic function in $\mathcal{D}$ cannot take its maximum value inside $\mathcal{D}$, but only on $\partial\mathcal{D}$.

This can be shown in the same way as the two dimensional case.

Uniqueness of Dirichlet's problem

By substituting $v = u$, a harmonic function, in Green's first identity, one gets

$$\iint\limits_{\partial\mathcal{D}} u\frac{\partial u}{\partial n}dS = \iiint\limits_{\mathcal{D}} |\vec{\nabla}u|^2 dV.$$

Let u_1 and u_2 be two solutions of the same Dirichlet problem, then their difference $u = u_1 - u_2$ is a harmonic function satisfying the zero boundary Dirichlet problem, then $\iiint_{\mathcal{D}} |\vec{\nabla}u|^2 dV$ $= 0 \Rightarrow |\vec{\nabla}u|^2 = 0 \Rightarrow u = \text{constant} = 0$.

For Neumann problem, the solution is unique up to a constant on $\mathcal{D}$.

<u>Dirichlet's principle</u>

Among all the C^1 functions $w(\mathbf{x})$ in $\mathcal{D}$ that satisfy the Dirichlet boundary condition $w = h(\mathbf{x})$ on $\partial\mathcal{D}$, the harmonic function u that satisfies the boundary condition minimize the energy:

$$E[w] = \frac{1}{2} \iiint_{\mathcal{D}} |\vec{\nabla} w|^2 d\mathbf{x}.$$

Proof:

Let $w = u + v$ where both w and u satisfy the Dirichlet boundary condition, but u is a harmonic function. Then v is a function that has zero value on $\partial\mathcal{D}$.

$$E[w] = \frac{1}{2} \iiint_{\mathcal{D}} |\vec{\nabla}(u + v)|^2 d\mathbf{x}.$$

$$= E[u] + \iiint_{\mathcal{D}} \vec{\nabla} u \cdot \vec{\nabla} v \, d\mathbf{x} + E[v]$$

As $v = 0$ on $\partial\mathcal{D}$ and $\Delta u = 0$ in $\mathcal{D}$, Green's first identity gives $\iiint_{\mathcal{D}} \vec{\nabla} u \cdot \vec{\nabla} v \, d\mathbf{x} = 0$. Therefore,

$$E[w] = E[u] + E[v] \geq E[u].$$

This means that the energy is smallest when $w = u$.

Green's second identity

Switch u and v in Green's first identity, then subtract it from the original form of the identity. The result is

$$\iiint_{\mathcal{D}} (u\Delta v - v\Delta u) dV = \iint_{\partial\mathcal{D}} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) dS.$$

This is *Green's second identity.* It is valid for any pair of function u and v.

Special boundary conditions can be imposed on the functions to make the right hand side of these identity zero, so that $\iiint_{\mathcal{D}} u\Delta v = \iiint_{\mathcal{D}} v\Delta u$.

Definition:

A boundary condtion is called *symmetric* for the operator Δ on $\mathcal{D}$ if $\displaystyle\iint_{\partial\mathcal{D}} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) dS = 0$ for all pairs of functions u, v that satisfy the boundary condition.

Dirichlet, Neumann, and Robin BCs are symmetric.

<u>Representation formula</u>

Let $K(\mathbf{x}, \mathbf{x}_0) \equiv -1/(4\pi|\mathbf{x} - \mathbf{x}_0|)$. Any harmonic function u in an open solid region $\mathcal{D}$ can be express as an integral over the boundary $\partial\mathcal{D}$ as

$$u(\mathbf{x}_0) = \iint\limits_{\partial\mathcal{D}} \left[u(\mathbf{x}) \frac{\partial}{\partial n} K(\mathbf{x}, \mathbf{x}_0) - K(\mathbf{x}, \mathbf{x}_0) \frac{\partial u}{\partial n} \right] dS$$

where $\mathbf{x}_0 \in \mathcal{D}$ and $\mathbf{x}$ is on $\partial\mathcal{D}$.

Proof:

First, note that $\Delta K(\mathbf{x}, \mathbf{x}_0) = 0$ except at $\mathbf{x} = \mathbf{x}_0$. Therefore, both u and K are harmonic in the domain $\mathcal{D}' = \mathcal{D} - B_\epsilon(\mathbf{x}_0)$ where $B_\epsilon(\mathbf{x}_0)$ a ball in $\mathcal{D}$, centered at $\mathbf{x}_0$, and with radius $\epsilon > 0$. Applying Green's second identity to u and K in $\mathcal{D}'$, one gets

$$\iint\limits_{\partial(\mathcal{D} - B_\epsilon)} \left[u(\mathbf{x}) \frac{\partial}{\partial n} K(\mathbf{x}, \mathbf{x}_0) - K(\mathbf{x}, \mathbf{x}_0) \frac{\partial u}{\partial n} \right] dS = 0.$$

With $\mathbf{x}_0$ put at the origin of a spherical coordinate system, the unit normal for the directional derivative at the surface of B_ϵ is pointing towards the origin. Flipping the unit normal outward introduces a minus sign in front of the flux integral over ∂B_ϵ. The equation can be rewritten as:

$$\begin{aligned}
&\iint\limits_{\partial\mathcal{D}} \left[u(\mathbf{x}) \frac{\partial}{\partial n} K(\mathbf{x}, \mathbf{x}_0) - K(\mathbf{x}, \mathbf{x}_0) \frac{\partial u}{\partial n} \right] dS \\
&\quad = \frac{1}{4\pi} \iint\limits_{\partial B_\epsilon} \left[u(\mathbf{x}) \frac{\partial}{\partial r} \left(\frac{-1}{r} \right) - \left(\frac{-1}{r} \right) \frac{\partial u}{\partial r} \right] dS \\
&\quad = \frac{1}{4\pi} \iint\limits_{r=\epsilon} \left[u + \epsilon \frac{\partial u}{\partial r} \right] \sin\theta d\theta d\phi \to u(0) \quad \text{as } \epsilon \to 0^+.
\end{aligned}$$

The formula is thus obtained.

This formula is not useful for finding a solution. For a Dirichlet problem, u is uniquely determined by its value on $\partial\mathcal{D}$. There is no freedom in choosing $\partial u/\partial n$. However, this formula is a step towards Green's function, the use of which eliminates the $\partial u/\partial n$ term.

Green's Function

It is possible to derive a formula that expresses a harmonic function u in terms of its value on $\partial\mathcal{D}$ only.

Definition:

> Let $\mathbf{x}_0$ be an interior point of $\mathcal{D}$. The *Green's function* $G(\mathbf{x}, \mathbf{x}_0)$ for the operator Δ and the domain $\mathcal{D}$ is a function defined for $\mathbf{x} \in \mathcal{D}$ such that:
>
> (i) Let $K(\mathbf{x}, \mathbf{x}_0) = -1/(4\pi|\mathbf{x}-\mathbf{x}_0|)$. The function $H(\mathbf{x}) \equiv G(\mathbf{x}, \mathbf{x}_0) - K(\mathbf{x}, \mathbf{x}_0)$ has continuous second derivatives and is harmonic in $\mathcal{D}$ (including the point $\mathbf{x}_0$).
>
> (ii) $G(\mathbf{x}, \mathbf{x}_0) = 0$ for $\mathbf{x} \in \partial\mathcal{D}$.

Requirement (i) implies that as a function of $\mathbf{x}$, $G(\mathbf{x}, \mathbf{x}_0)$ possesses continuous second derivatives and $\Delta G = 0$ in $\mathcal{D}$, except at the point $\mathbf{x} = \mathbf{x}_0$. Requirement (ii) implies that the value of H on $\partial\mathcal{D}$ is given by $H(\mathbf{x}) = -K(\mathbf{x}, \mathbf{x}_0)$ where $\mathbf{x}$ is on $\partial\mathcal{D}$. *If the solution of a Dirichlet problem with arbitrary boundary value (described by a continuous function on $\partial\mathcal{D}$) exists*, then H exists (and so does G). However, this important *existence* theorem is not to be proven here.

Theorem:

If $G(\mathbf{x}, \mathbf{x}_0)$ is the Green's function, then the solution of the Dirichlet problem is given by the formula

$$u(\mathbf{x}_0) = \iint_{\partial \mathcal{D}} u(\mathbf{x}) \frac{\partial G(\mathbf{x}, \mathbf{x}_0)}{\partial n} \, dS.$$

Proof:

Recall that the representation formula is

$$u(\mathbf{x}_0) = \iint_{\partial \mathcal{D}} \left(u \frac{\partial K}{\partial n} - K \frac{\partial u}{\partial n} \right) ds.$$

The result of applying Green's second identity to the pair of harmonic functions u and H is

$$\iint_{\partial \mathcal{D}} \left(u \frac{\partial H}{\partial n} - H \frac{\partial u}{\partial n} \right) ds = 0.$$

Adding the two equations, the result becomes

$$u(\mathbf{x}_0) = \iint_{\partial \mathcal{D}} \left(u \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n} \right) ds = \iint_{\partial \mathcal{D}} u \frac{\partial G}{\partial n} ds.$$

It is the formula needed.

Principle of reciprocity

The Green's function $G(\mathbf{x}, \mathbf{x}_0)$ of a region $\mathcal{D}$ is *symmetric*, i.e.

$$G(\mathbf{x}, \mathbf{x}_0) = G(\mathbf{x}_0, \mathbf{x}) \qquad \text{for } \mathbf{x} \neq \mathbf{x}_0.$$

This relation ensures the C^2 and harmonic properties of G as a function of $\mathbf{x}_0$ (as long as $\mathbf{x}_0 \neq \mathbf{x}$).

Proof:

Apply Green's second identity to the pair of functions $u(\mathbf{x}) \equiv G(\mathbf{x}, \mathbf{a})$, $v(\mathbf{x}) \equiv G(\mathbf{x}, \mathbf{b})$ in the region $\mathcal{D}' = \mathcal{D} - B_\epsilon(\mathbf{a}) - B_\epsilon(\mathbf{b})$ in which u, v are harmonic. The result is

$$\iiint_{\mathcal{D}'} (u\Delta v - v\Delta u)dV$$

$$= \iint_{\partial\mathcal{D}} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) dS - I_\epsilon(\mathbf{a}) - I_\epsilon(\mathbf{b})$$

where

$$I_\epsilon(\mathbf{a}) = \iint_{\partial B_\epsilon(\mathbf{a})} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) dS$$

and

$$I_\epsilon(\mathbf{b}) = \iint_{\partial B_\epsilon(\mathbf{b})} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) dS.$$

Note that the unit normals for the directional derivatives in these two flux integrals are outward from the centers of the balls. Since Δu, $\Delta v = 0$ in $\mathcal{D} - B_\epsilon(\mathbf{a}) - B_\epsilon(\mathbf{b})$ and u, v vanish on $\partial\mathcal{D}$

$$I_\epsilon(\mathbf{a}) + I_\epsilon(\mathbf{b}) = 0 \quad \text{for any } \epsilon > 0.$$

To evaluate $I_\epsilon(\mathbf{a})$, put $\mathbf{a}$ at the origin of a spherical coordinate system, so that $|\mathbf{x} - \mathbf{a}| = r$ and $\dfrac{\partial}{\partial n} = \dfrac{\partial}{\partial r}$. Then, take the limit

$$\lim_{\epsilon\to 0^+} I_\epsilon(\mathbf{a}) =$$

$$\lim_{\epsilon\to 0^+} \iint\limits_{r=\epsilon} \left[\left(-\frac{1}{4\pi r}+H\right)\frac{\partial v}{\partial r} - v\,\frac{\partial}{\partial r}\left(-\frac{1}{4\pi r}+H\right)\right] r^2 \sin\theta d\theta d\phi$$

$$= -\lim_{\epsilon\to 0^+} \iint\limits_{r=\epsilon} v\,\frac{1}{4\pi r^2}\, r^2 \sin\theta d\theta d\phi = -v(\mathbf{a}).$$

Similarly, $\lim_{\epsilon\to 0^+} I_\epsilon(\mathbf{a}) = u(\mathbf{b})$.

Therefore, $-v(\mathbf{a}) + u(\mathbf{b}) = 0. \quad \Rightarrow\; G(\mathbf{a},\mathbf{b}) = G(\mathbf{b},\mathbf{a})$.

The Green's function can also be applied to solve the Poisson equation.

Theorem:

The solution of the problem

$$\Delta u = f \;\text{ in } \mathcal{D} \qquad u = h \;\text{ on } \partial\mathcal{D}$$

is given by

$$u(\mathbf{x}_0) = \iint\limits_{\partial\mathcal{D}} u(\mathbf{x})\frac{\partial G(\mathbf{x},\mathbf{x}_0)}{\partial n}\,dS + \iiint\limits_{\mathcal{D}} f(\mathbf{x})G(\mathbf{x},\mathbf{x}_0)\,dV.$$

Green's function in special geometries

Even though the proof of the existence for Green's function in a general region is difficult, Green's functions can be found explicitly (therefore shown to exist) for certain special cases.

The half-space

For the half-space with $z > 0$, the Green's function is

$$G(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{4\pi|\mathbf{x} - \mathbf{x}_0|} + \frac{1}{4\pi|\mathbf{x} - \mathbf{x}_0^*|}.$$

where $\mathbf{x}_0^* = (x_0, y_0, -z_0)$ for $\mathbf{x}_0 = (x_0, y_0, z_0)$.

As $\mathbf{n}$ is pointing downward at the boundary,

$$\partial G/\partial n = -\partial G/\partial z|_{z=0} \; = \frac{1}{2\pi}\frac{z_0}{|\mathbf{x} - \mathbf{x}_0|^3}.$$

Therefore,

$$u(\mathbf{x}_0) = \frac{z_0}{2\pi} \iint_{\partial D} \frac{h(\mathbf{x})}{|\mathbf{x} - \mathbf{x}_0|^3}\, dS.$$

The Ball

The Green's function for the ball $\mathcal{D} = \{|\mathbf{x}| < a\}$ is

$$G(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{4\pi|\mathbf{x} - \mathbf{x}_0|} + \frac{q*}{4\pi|\mathbf{x} - \mathbf{x}_0^*|}.$$

where $q^* = a/|\mathbf{x}_0|$ (> 1) and $\mathbf{x}_0^* = (a/|\mathbf{x}_0|)^2\mathbf{x}_0 = q^{*2}\,\mathbf{x}_0$.

For any $\mathbf{x}$ on the surface of the ball,

$$|\mathbf{x}-\mathbf{x}_0| = \left|\frac{|\mathbf{x}_0|}{a}\mathbf{x}-\frac{a}{|\mathbf{x}_0|}\mathbf{x}_0\right| = \frac{|\mathbf{x}_0|}{a}|\mathbf{x}-\mathbf{x}_0^*| = (q^*)^{-1}|\mathbf{x}-\mathbf{x}_0^*|.$$

Therefore, $G(\mathbf{x},\mathbf{x}_0) = 0$.

Furthermore, as

$$\vec{\nabla}G = \frac{\mathbf{x}-\mathbf{x}_0}{4\pi|\mathbf{x}-\mathbf{x}_0|^3} - q^*\frac{\mathbf{x}-\mathbf{x}_0^*}{4\pi|\mathbf{x}-\mathbf{x}_0^*|^3} = \frac{1-(q^*)^{-2}}{4\pi}\frac{\mathbf{x}}{|\mathbf{x}-\mathbf{x}_0|^3},$$

$$\frac{\partial G}{\partial n} = \frac{\mathbf{x}}{a}\cdot\vec{\nabla}G = \frac{a^2-|\mathbf{x}_0|^2}{4\pi a|\mathbf{x}-\mathbf{x}_0|^3}.$$

Thus,

$$u(\mathbf{x}_0) = \frac{a^2-|\mathbf{x}_0|^2}{4\pi a}\iint\limits_{|\mathbf{x}|=a}\frac{h(\mathbf{x})}{|\mathbf{x}-\mathbf{x}_0|^3}\,dS.$$

—— Problem Set 10 ——