ROTATING CONVECTION AND THE SOLAR DIFFERENTIAL ROTATION

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Abstract. We discuss the implication of a numerical experiment on rotating convection and its relevance to the construction of a model for the solar differential rotation.

1. A Brief Overview

Space borne experiments have now established that the total solar irradiance varies over a full solar cycle (Willson and Hudson, 1991, Hickey et al., 1988). It is therefore clear that the solar output is linked to the dynamo process. The solar dynamo, differential rotation, and convection are closely related (Parker, 1955). Since the dynamo arises from the differential rotation and convection (through the ω and α processes), explaining the differential rotation and sorting out its relation to convection is then a prerequisite for understanding the magnetic cycle and the associated variations in solar irradiance.

Global numerical models of solar differential rotation (Gilman, 1977, Glatzmaier, 1984, Gilman and Miller, 1986) are successful in obtaining the solar angular velocity distribution at the surface. In the interior, the distributions more or less obey the well-known Taylor-Proudman theorem, so that the isorotation surfaces are cylindrical and parallel to the rotation axis. However, results of helioseismology, now confirmed by many groups (Duvall et al., 1986, Brown and Morrow, 1987, Rhodes et al., 1987, Libbrecht, 1989), show that inside the convection zone, the isorotation surfaces tend to align radially, and that beneath a thin shear layer the stable radiative region rotates more or less uniformly.

This contradiction between the observational and numerical results cannot be easily reconciled and raises the question: What is missing from the numerical models? In an earlier paper, Chan and Serizawa (1991) argued that the Taylor-Proudman theorem would not hold if the buoyance force is as important as the Coriolis force. The buoyance force depends on the distribution of the entropy [or equivalently, the superadiabatic gradient $\delta\nabla = (\partial \ln T / \partial \ln p) - (\partial \ln T / \partial \ln p)_{adiabatic}$], and if the entropy distribution is wrong, the buoyance force would be in error. To obtain the correct $\delta\nabla$ profile, it is necessary to perform the simulation for a period comparable to the thermal relaxation time of the layer, about 10^5 years for the solar convection zone. But the numerical calculations were usually run only for a period on the order of ten years. Thus there is a problem with direct numerical simulations. To make progress, semi-analytical approaches are therefore more practical. The idea here is to bypass the thermal relaxation calculation by making

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an assumption about the distribution of $\delta\nabla$. Since the computational load for each model is light, it is possible to construct many models and search for conditions under which the correct angular velocity distribution develops.

Semi-analytical models of solar differential rotation appeared well before the development of numerical models. One class of these models, the 'anisotropic viscosity' model, is based on the argument that the turbulent exchange of momentum along the vertical direction (radial) should be quantitatively different from that in the horizontal direction (Wasiutynski 1946). The anisotropic Reynolds stresses in the momentum equation then generate a differential rotation in the large scale. This concept was first applied in calculations by Kippenhahn (1963) and Kohler (1970). Elaborations and extensions of this approach have later been made, for example, by Durney and Spruit (1979), Rudiger (1980, 1989), and Schmidt (1982). Another class of models, the 'latitude-dependent heat transport' model, argues that the interaction of rotation with convection leads to a differential heat transport between the equator and the poles. The resulting source term in the energy equation thus sets up a meridional flow which in turn generates a differential rotation. The idea was originally proposed by Weiss (1965) and developments have been made, for example, by Durney and Roxburgh (1971), Belvedere and Paterno (1977), and Pidatella et al. (1986). Both classes of models aimed at satisfying the solar surface constraints by adjusting the heat transport parameter, the viscosity, and the Prandtl number. With the exception of a few recent specialized versions (e.g. Tuominen and Rudiger, 1989, Brandenburg et al., 1992), most of the above described semi-analytical models cannot reproduce the solar internal angular velocity distribution inferred from helioseismology.

In these semi-analytical models, the formulation is based on perturbation expansions of the 'inverse Rossby number' ($\sim \Omega$ L/ V where Ω is the mean rotation rate, L and V are the characteristic length and velocity respectively; we shall call it the Coriolis number). The expansions diverge when the Coriolis number is larger than one, which is approximately the case in the solar convection zone, and therefore strictly speaking, the formulation is not applicable. For the same reason, such theories cannot be applied to explain the extreme differential rotation observed on Jupiter and Saturn where the angular velocities form alternately positive and negative latitudinal bands.

Conventional semi-analytical theories emphasize the action of the Coriolis force on the sub-global scales. But actually the effects of the Coriolis force are maximized in the global scale as the Coriolis number increases with the length scale. We therefore proposed a theory which has close ties with conventional semi-analytical theories but emphasizes the global scale interaction of convection and rotation (Chan et al., 1987). This model interprets the differential rotation as the zonal (azimuthal) wind component of a global, axisymmetric mode of convection under the influence of rotation. In conventional theories, the linearized, coupled fluid equations describing the differential rotation are directly driven by latitude-dependent terms in the momentum and/or energy equations, and in most cases, the amount of driving required is excessively large. In our case, instead, we seek an appropriate resonant response to such driving. The response being resonant, the driver can be arbitrarily small and its exact form becomes unimportant. The response is in fact a self-excited convective mode. The pattern of such a mode depends on the balance of the Coriolis force and the buoyance force and therefore the distribution of $\delta\nabla$ plays a crucial role. This theory can be applied to explain the

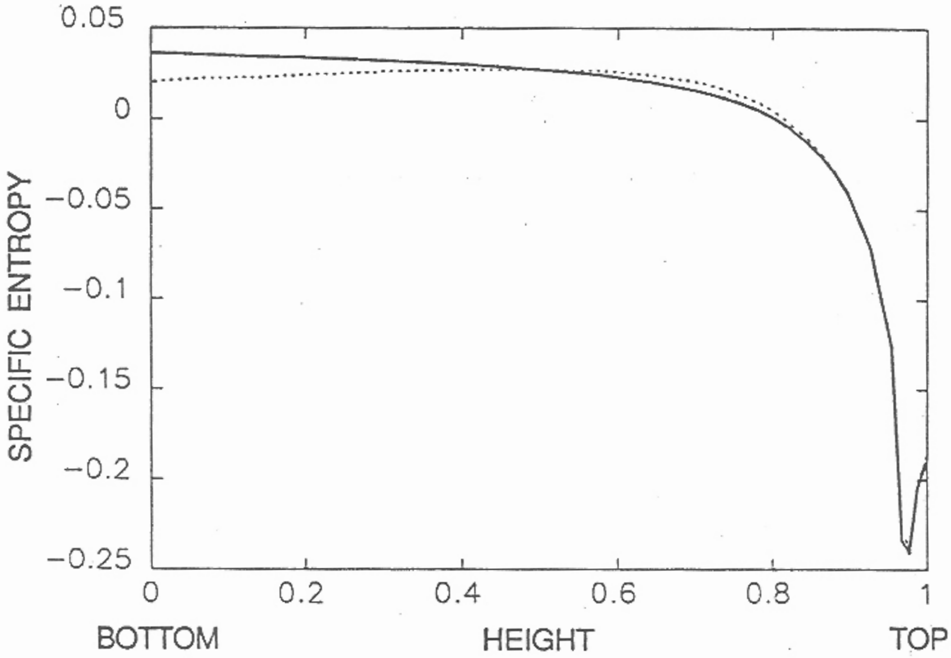


Fig. 1. The effect of rotation on the entropy distribution of a convection zone.

alternating wind bands of Jupiter and Saturn (Mayr et al., 1984; 1991). Both Boussinesq (Chan et al., 1987) and compressible flow (Chan and Mayr, 1991) models have been constructed to describe the solar differential rotation. The compressible model can describe the observed internal solar angular distribution rather well.

Recently, using 3D numerical experiments, Chan and Gigas (1991) found that $\delta\nabla$ can turn slightly negative (subadiabatic) in the lower region of a deep convection zone. Supposing that to be the case, we have tried to construct solar differential rotation models with background structures that followed such behavior. In all the cases, however, the differential rotations could not be made to penetrate significantly into the subadiabatic lower convection zone, in contradiction with observations. A question then arose: Since the Coriolis number in the lower solar convection zone is close to 1, can the rotation alter the sign of $\delta\nabla$ in this region? To answer this question, we performed a numerical experiment on rotating deep convection. The results were striking. The rotation turned the lower convective region from slightly subadiabatic to significantly superadiabatic. Details of this numerical experiment are given in Section 2. The following conclusions are reached: (i) Estimates of $\delta\nabla$ based on the standard mixing length theory are incorrect in the lower part of a deep convection zone. (ii) Strong rotation plays an important role in determining $\delta\nabla$. At the moment, we do not have a theory of $\delta\nabla$ for rotating convection. But we can now assume that it is positive in the whole convection zone and take its distribution as a free parameter in our model. We made a systematic study on the parametric combinations that produce solar-like differential rotation and found a number of possible solutions. The results of this study are summarized in Section 3.

2. A Numerical Experiment of Rotating Convection

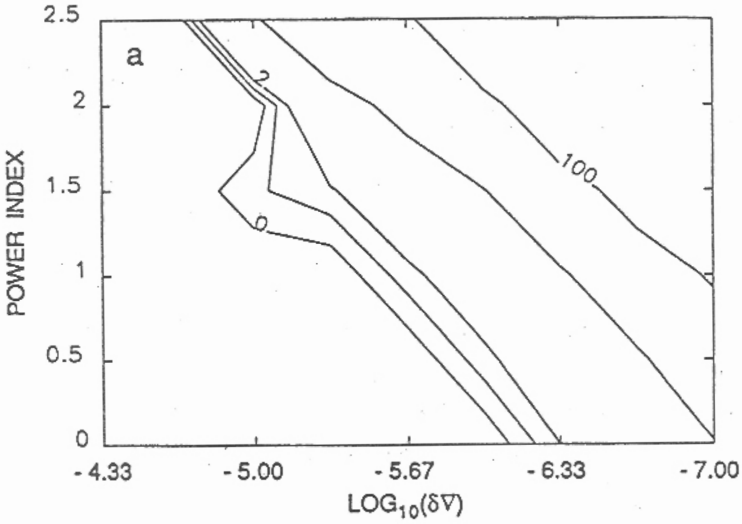
Using the numerical code described in Chan and Sofia (1986), we performed a numerical experiment with the following specifications: (i) The fluid is rotating, with the axis of rotation at an angle 45° from the vertical direction. The magnitude of the angular velocity is 0.5 $[(p_t/\rho_t)^{1/2}$ / depth] where the subscript 't' denotes values of variables at the upper boundary. The Rossby number is about 1. (ii) The depth of the convection zone contains 4.5 pressure scale heights. To have better control on the conditions near the top of the convection zone, a stable radiative layer with 1 pressure scale height is attached at the top. (iii) The domain of computation is a 3D rectangular box with an aspect ratio (horizontal width / depth) of 1.5. (iv) The grid consists of 35x35x39 points. (v) The side boundaries are periodic; the upper and lower boundaries are impenetrable and stress-free. (vi) There is a constant input flux from the lower boundary (= 0.25 $[p_t(p_t/\rho_t)^{1/2}]$); the temperature at the upper boundary is fixed.

The effect of rotation on the entropy distribution is illustrated in Fig. 1. The solid curve shows the distribution of the specific entropy for the rotating case; the dashed curve shows the distribution for an identical case without rotation. To overcome the impediment from rotation, the layer adjusts its structure to make $\delta\nabla$ positive and large enough so that convection prevails.

3. Generation of 'Sun-Like' Differential Rotation

Having found that $\delta\nabla$ could be positive in the whole convection zone, we returned to the modeling of the solar differential rotation. Using a linearized, axisymmetric version of a recently developed spectral code that solves the compressible fluid equations (Chan et al. 1993), we studied convection zone models with assumed distributions of $\delta\nabla$ and looked for situations which can produce sun-like differential rotation. The common characteristics of the models are: (i) The domain of computation spans between 0.575 and 0.934 (or 0.99) solar radii. It contains a major portion of the convection zone and the upper part of the stable radiative layer below. (ii) The upper boundary is stress-free but the lower boundary is slip-free (the implication is that magnetic field plays a role in locking up the flow). (iii) The number of vertical grid levels is 52 (or 101). For studying the subcritical responses (see later discussion), the degree of spherical harmonics used is 10. For the supercritical cases which require many iterations for finding the resonances, the degree is 4 (or 6). (iv) The gas can be conveniently approximated as an ideal gas with a ratio of specific heat 5/3. (v) The background distributions of temperature, density, and pressure are held fixed. (vi) Inside the convection zone, the superadiabatic gradient is assumed to vary as $\delta\nabla_t(\rho_t/\rho)^n$ where ρ is the density. $\delta\nabla_t$ and the power index n are free parameters. The eddy diffusivity κ is assumed to satisfy $f = \kappa \rho T \delta\nabla / H$ where f is the total energy flux and H is the pressure scale height. The eddy Prandtl number Pr, another free parameter, is assume to be independent of depth in the convection zone. (vii) In the stable layer, the radiative conductivity K is taken to satisfy $f = K\, dT/dr$. The viscosity there is arbitrarily set at 10% of the value at the bottom of the convection zone (exact value unessential).

To drive the linearized system,we introduce a perturbation of the form $\varepsilon f P_2(\cos\theta)$ in the energy equation, near the bottom of the convection zone (the results are insensitive to the exact location). ε is an amplitude factor, f is the solar flux, P_2 is the normalized



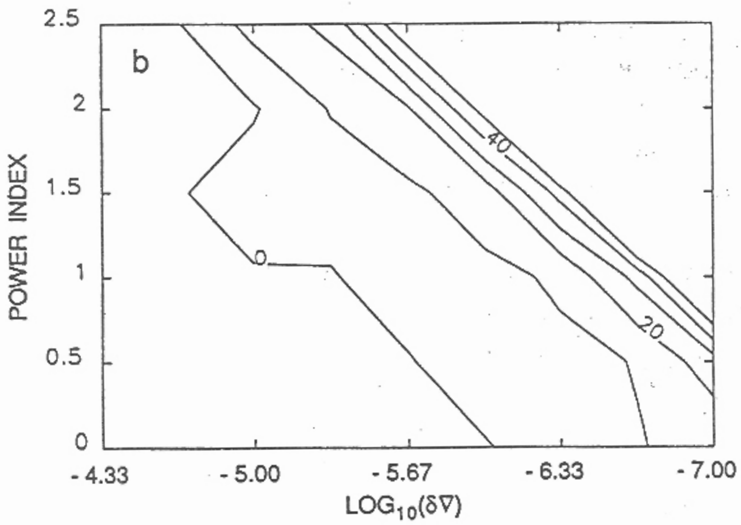


Fig. 2. a. Distribution of the amplitude factor of perturbation in the ($\delta\nabla_t$, n) plane.
b. Distribution of the maximal surface meridional velocity.

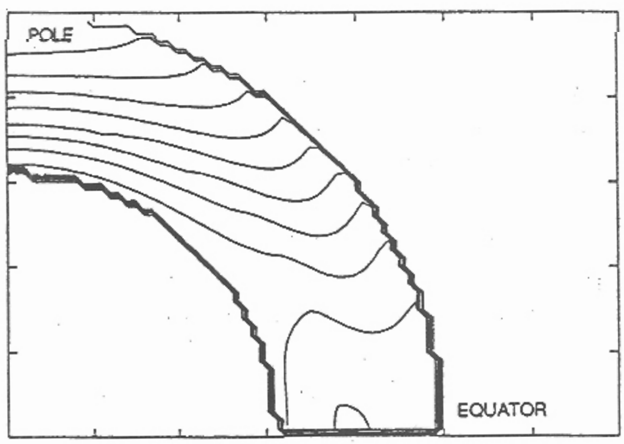


Fig. 3. The angular velocity distribution for a subcritical response with n = 1, Pr = 1, and $\delta\nabla_t = 2.x10^{-6}$.

Legendre function of the second degree, and θ is the co-latitude. The physical rationale and the formulation of this procedure are similar to those adopted by the 'latitudinal-dependent heat transport' model. Under these conditions, there exists a large parameter region (see Fig. 2 for the Pr = 1 case; cases with Pr = 0.1 and 10 have similar patterns) which can produce radially oriented isorotation contours in the convection zone (see Fig. 3), similar to those observed. This region however is characterized by large viscosities and diffusivities (small $\delta\nabla_t$ since f is fixed), corresponding to subcritical effective Rayleigh numbers (no self-excited modes). Over a large portion of this region (towards the right), the pattern of differential rotation as depicted in Fig. 3 is quite robust; it is insensitive to $\delta\nabla_t$, n, and Pr, but it relies on the form of the driver $P_2(\cos\theta)$. The magnitude of the driving needed to produce the correct solar amplitude (Fig. 2a) and the size of the surface meridional velocity (Fig. 2b) quickly increase with smaller $\delta\nabla_t$ and larger diffusivity. The lines labeled by 0 in Fig. 2a,b approximately delineate the boundary which separates the super- and sub- critical regions. Beyond a thin strip on the right of this critical boundary, the perturbation factor required is unreasonably large ($\varepsilon > 1$; latitudinal variation in flux > f). In the region $1 > \varepsilon > 0$, the angular velocity distributions often show irregularities; sun-like distributions appear only when ε is close to 1; the required size of flux perturbation is at best marginally acceptable.

Resonant responses to the energy driver occur on the left side of the critical boundary at discrete locations of $\delta\nabla_t$. These are unstable eigenmodes described by the linearized fluid equations and do not always produce sun-like differential rotation. The angular velocity distributions depend on $\delta\nabla_t$, n, and Pr.

A number of solutions with sun-like differential rotation have been found for a variety of combinations of parameters. Table 1 provides some examples (N is the degree of the harmonic expansion). The values of $\delta\nabla_t$ at which resonances occur are quite insensitive to the location of the upper boundary of the model and the number of spherical harmonics included, and they are close to the critical boundary. They also approximately satisfy an inverse proportion relationship with Pr as derived in Chan et al. (1987; eqn. 21). Since they are self-excited, the resonant modes do not depend on the amplitude and form of the excitation driver. Fig. 4 shows a surface plot of the angular velocity distribution for one such case. The amplitude of the mode is found by matching the surface angular velocity distribution to that observed on the sun. Though the patterns of the zonal flows are simple and similar, these modes possess complicated, multi-cellular meridional flows. The magnitudes and patterns of the meridional circulations are highly dependent on the free parameters. In the case presented by Fig. 4, the maximal meridional velocity at the surface is 28.6 m/s.

In the solar convection zone, the axisymmetric convective modes discussed here are not the first to be excited. Many other modes (especially the small scale ones, e.g., granule scale) grow more readily and at much faster rates. But in the 'mean-field' picture (as identified by Tuominen et al., 1993) adopted by the semi-analytical models and inherited here, these nonlinearly interacting and fully developed sub-global scale or non-axisymmetric modes are treated as part of the background turbulence and their actions are lumped into the mean-field transport coefficients (e.g. eddy diffusion). The lowest degree axisymmetric mode may thus represent the 'first' growing mode in this idealization, and

TABLE I

Examples of $\delta\nabla_t$ at which resonances with 'sun-like' differential rotation occur.

Pr =	10	5	3
r_t = 0.934; N = 4	9.57×10^{-6}	1.76×10^{-5}	2.79×10^{-5}
r_t = 0.990; N = 4	1.13×10^{-5}	$(1.75\times10^{-5})^{*}$	2.38×10^{-5}
r_t = 0.934; N = 6	9.7×10^{-6}	1.72×10^{-5}	2.62×10^{-5}

* a local maximum in response, not a resonance

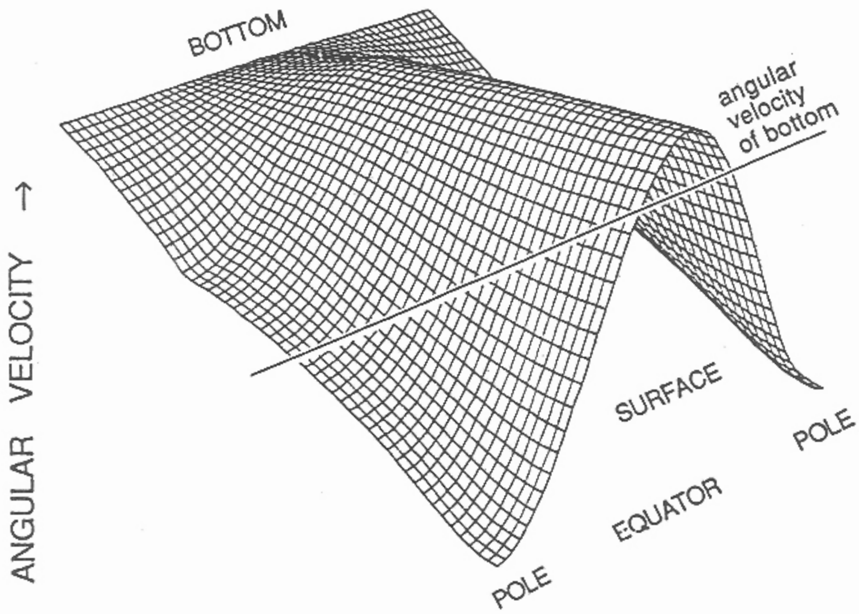


Fig. 4. Angular velocity distribution for the convective mode with Pr = 10, r_t = 0.990, N = 4, and $\delta\nabla_t = 1.13\times10^{-5}$.

the linear mode may then approximately describe the 'mildly nonlinear' situation. Whether the mean-field simplification is valid and whether a resonant mode of the linearized mean-field equations can indeed mimic the global scale circulation are questions that need to be addressed by future numerical experiments.

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References

Belvedere, G., and Paterno, L.: 1977, *Solar Phys.* **54**, 289.

Brandenburg, A., Moss, D., Tuominen, I.: 1992, *Astron. Astrophys.* **265**, 328.

Brown, T. M., and Morrow, C. A.: 1987, *Ap. J. Lett.* **314**, L21.

Chan, K. L., and Gigas, D.: 1991, *Astrophys. J.* **389**, L87.

Chan, K. L., and Mayr, H. G.: 1991, in *The Sun and Cool Stars: Activities, Magnetism, and Dynamos*, eds. I. Tuominen, D. Moss, and G. Rudiger, Springer-Verlag, Berlin, 178.

Chan, K. L., and Serizawa, K.: 1991, in *The Sun and Cool Stars: Activities, Magnetism, and Dynamos*, ed. I. Tuominen, D. Moss, and G. Rudiger, Springer-Verlag, Berlin, 15.

Chan, K. L., and Sofia, S.: 1986, *Astrophys. J.* **307**, 222.

Chan, K. L., Sofia, and Mayr, H. G.: 1987, in *The Internal Solar Angular Velocity*, eds. B. R. Durney and S. Sofia, D. Reidel, Dordrecht, 347.

Chan, K. L., Mayr, H. G., Mengel, J. G., and Harris, I.: 1993, *J. Comput. Phys.*, in press.

Durney, B., and Roxburgh, I. W.: 1971, *Solar Phys.*, **16**, 3.

Durney, B., and Spruit, H. C.: 1979, *Astrophys. J.* **243**, 1067.

Duvall, T. L., Harvey, J. W., and Pomerantz, M. A.: 1986, *Nature* **321**, 500.

Glatzmaier, G. A.: 1984, *J. Comput. Phys.* **55**, 461.

Gilman, P. A.: 1977, *Geophys. Astrophys. Fluid Dyn.* **8**, 93.

Gilman, P. A., and Miller, J.: 1986, *Astrophys. J. Suppl.* **61**, 585.

Hickey, J. R., Alton, B. M., Kyle, H. L., and Major, E. R.: 1988, *Adv. Space Res.* **8**, (7) 5.

Kippenhahn, R.: 1963, *Astrophys. J.* **137**, 664.

Kohler, H.: 1970, *Solar Phys.* **13**, 3.

Libbrecht, K. G.: 1989, *Astrophys. J.* **336**, 1092.

Mayr, H. G., Harris, I., and Chan, K. L.: 1984, *Earth, Moon, and Planets* **30**, 245.

Mayr, H. G., Chan, K. L., Harris, I., and Schatten, K.: 1991, *Astrophys. J.* **367**, 361.

Parker, E. N.: 1955, *Astrophys. J.* **122**, 293.

Pidatella, R. M., Stix, M., Belvedere, G., and Paterno, L.: 1986, *Astron. Astrophys.* **156**, 22.

Rhodes, E. J., Cacciani, A., Woodard, M., Tomczyk, S., Korennik, S., Ulrich, R. K.: 1987, in The Internal Solar Angular Velocity, eds. B. Durney and S. Sofia, D. Reidel, Dordrecht, 75.

Rudiger, G.: 1980, *Geophys. Astroshys. Fluid Dyn.* **16**, 239.

_____ 1989, *Differential Rotation and Stellar Convection: Sun and Solar Type Stars*, Gordon and Breach, New York.

Schmidt, W.: 1982, *Geophys. Astrophys. Fluid Dyn.* **21**, 27.

Tuominen, I., and Rudiger, G.: 1989, *Astron. Astrophys.* **217**, 217.

Tuominen, I., Brandenburg, A., Moss, D., and Rieutord, M. 1993, *Astron. Astrophys.*, in press.

Wasiutynski, J.: 1946, *Astrophys. Norvegica* **4**.

Weiss, N. O.: 1965, *Observatory* **85**, 37.

Willson, R. C., and Hudson, H. S.: 1991, *Nature* **351**, 42.