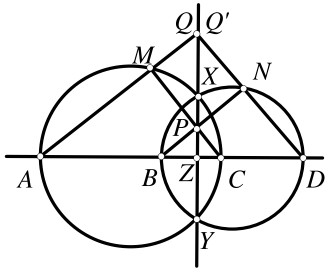
1. Coordinate Geometry

When we do a geometry problem, we should first look at the given facts and the conclusion. If all these involve intersection points, midpoints, feet of perpendiculars, parallel lines, then there is a good chance we can solve the problem by coordinate geometry. However, if they involve two or more circles, angle bisectors and areas of triangles, then sometimes it is still possible to solve the problem by choosing a good place to put the origin and the x-axis. Below we will give some examples. *It is important to stay away from messy computations!*

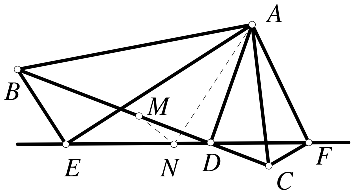
Example 1. (*1995 IMO*) Let A, B, C and D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at the points X and Y. The line XY meets BC at the point Z. Let P be a point on the line XY different from Z. The line CP intersects the circle with diameter AC at the points C and M, and the line BP intersects the circle with diameter BD at the points B and N. Prove that the lines AM, DN, and XY are concurrent.



(*Remarks*. Quite obvious we should set the origin at Z. Although the figure is not symmetric with respect to line XY, there are pairs such as M, N and A, D and B, C that are symmetric in *roles*! So we work on the left half of the figure, the computations will be similar for the right half.)

Solution. (*Due to Mok Tze Tao, 1995 Hong Kong Team Member*) Set the origin at Z and the x-axis on line AD. Let the coordinates of the circumcenters of triangles AMC and BND be $(x_1, 0)$ and $(x_2, 0)$, and the circumradii be r_1 and r_2, respectively. Then the coordinates of A and C are $(x_1 - r_1, 0)$ and $(x_1 + r_1, 0)$, respectively. Let the coordinates of P be $(0, y_0)$. Since $AM \perp CP$ and the slope of CP is $-y_0/(x_1 + r_1)$, so the equation of AM is $(x_1 + r_1)x - y_0 y = x_1^2 - r_1^2$. Let Q be the intersection of AM with XY, then Q has coordinates $(0, (r_1^2 - x_1^2)/y_0)$. Similarly, let Q' be the intersection of DN with XY, then Q' has coordinates $(0, (r_2^2 - x_2^2)/y_0)$. Since $r_1^2 - x_1^2 = ZX^2 = r_2^2 - x_2^2$, so $Q = Q'$.

Example 2. (*1998 APMO*) Let ABC be a triangle and D the foot of the altitude from A. Let E and F be on a line passing through D such that AE is perpendicular to BE, AF is perpendicular to CF, and E and F are different from D. Let M and N be the midpoints of the line segments BC and EF, respectively. Prove that AN is perpendicular to NM.

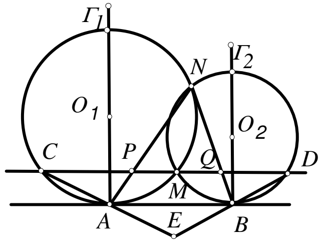


(*Remarks*. We can set the origin at D and the x-axis on line BC. Then computing the coordinates of E and F will be a bit messy. A better choice is to set the line through D, E, F horizontal.)

Solution. (*Due to Cheung Pok Man, 1998 Hong Kong Team Member*) Set the origin at A and the x-axis parallel to line EF. Let the coordinates of D, E, F be (d, b), (e, b), (f, b), respectively. The case $b=0$ leads to $D=E$, which is not allowed.

So we may assume $b \neq 0$. Since $BE \perp AE$ and the slope of AE is b/e, so the equation of line BE works out to be $ex + by = e^2 + b^2$. Similarly, the equations of lines CF and BC are $fx + by = f^2 + b^2$ and $dx + by = d^2 + b^2$, respectively. Solving the equations for BE and BC, we find B has coordinates $(d+e, b-(de/b))$. Similarly, C has coordinates $(d+f, b-(df/b))$. Then M has coordinates $(d+(e+f)/2, b-(de+df)/(2b))$ and N has coordinates $((e+f)/2, b)$. So the slope of AN is $2b/(e+f)$ and the slope of MN is $-(e+f)/(2b)$. Therefore, $AN \perp MN$.

Example 3. (*2000 IMO*) Two circles Γ_1 and Γ_2 intersect at M and N. Let ℓ be the common tangent to Γ_1 and Γ_2 so that M is closer to ℓ than N is. Let ℓ touch Γ_1 at A and Γ_2 at B. Let the line through M parallel to ℓ meet the circle Γ_1 again at C and the circle Γ_2 again at D. Lines CA and DB meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP=EQ$.



(*Remarks.* Here if we set the x-axis on the line through the centers of the circles, then the equation of the line AB will be complicated. So it is better to have line AB on the x-axis.)

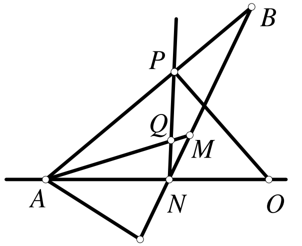
Solution. Set the origin at A and the x-axis on line AB. Let B, M have coordinates $(b,0)$, (s,t), respectively. Let the centers O_1, O_2 of Γ_1, Γ_2 be at $(0, r_1)$, (b, r_2), respectively. Then C, D have coordinates $(-s,t)$, $(2b-s,t)$, respectively. Since AB, CD are parallel, $CD=2b=2AB$ implies A, B are midpoints of CE, DE, respectively. So E is at $(s, -t)$. We see $EM \perp CD$.

To get $EP=EQ$, it is now left to show M is the midpoint of segment PQ. Since $O_1 O_2 \perp MN$ and the slope of $O_1 O_2$ is $(r_2-r_1)/b$, the equation of line MN is $bx+(r_2-r_1)y = bs+(r_2-r_1)t$. (This line should pass through the midpoint of segment AB.) Since $O_2M=r_2$ and $O_1M=r_1$, we get

$$(b-s)^2+(r_2-t)^2=r_2^2 \quad \text{and} \quad s^2+(r_1-t)^2=r_1^2\,.$$

Subtracting these equations, we get $b^2/2=bs+(r_2-r_1)t$, which implies $(b/2, 0)$ is on line MN. Since PQ, AB are parallel and line MN intersects AB at its midpoint, M must be the midpoint of segment PQ. Together with $EM \perp PQ$, we get $EP=EQ$.

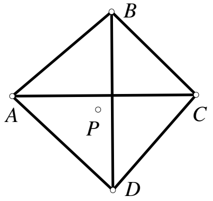
Example 4. (*2000 APMO*) Let ABC be a triangle. Let M and N be the points in which the median and the angle bisector, respectively, at A meet the side BC. Let Q and P be the points in which the perpendicular at N to NA meets MA and BA, respectively, and O the point in which the perpendicular at P to BA meets AN produced. Prove that QO is perpendicular to BC.



(*Remarks*. Here the equation of the angle bisector is a bit tricky to obtain unless it is the x-axis. In that case, the two sides of the angle is symmetric with respect to the x-axis.)

Solution. *(Due to Wong Chun Wai, 2000 Hong Kong Team Member)* Set the origin at N and the x-axis on line NO. Let the equation of line AB be $y = ax+b$, then the equation of lines AC and PO are $y=-ax-b$ and $y=(-1/a)x+b$, respectively. Let the equation of BC be $y=cx$. Then B has coordinates $(b/(c-a),\ bc/(c-a))$, C has coordinates $(-b/(c+a),\ -bc(c+a))$, M has coordinates $(ab/(c^2-a^2),\ abc/(c^2-a^2))$, A has coordinates $(-b/a,\ 0)$, O has coordinates $(ab,\ 0)$ and Q has coordinates $(0,\ ab/c)$. Then BC has slope c and QO has slope $-1/c$. Therefore, $QO \perp BC$.

Example 5. (1998 IMO) In the convex quadrilateral $ABCD$, the diagonals AC and BD are perpendicular and the opposite sides AB and DC are not parallel. Suppose that the point P, where the perpendicular bisectors of AB and DC meet, is inside $ABCD$. Prove that $ABCD$ is a cyclic quadrilateral if and only if the triangles ABP and CDP have equal areas.



(*Remarks*. The area of a triangle can be computed by taking the half length of the cross product. A natural candidate for the origin is P and having the diagonals parallel to the axes will be helpful.)

Solution. (*Due to Leung Wing Chung, 1998 Hong Kong Team Member*) Set the origin at P and the x-axis parallel to line AC. Then the equations of lines AC and BD are $y=p$ and $x=q$, respectively. Let $AP=BP=r$ and $CP=DP=s$. Then the coordinates of A,B,C,D are $(-\sqrt{r^2-p^2},p)$, $(q,\sqrt{r^2-q^2})$, $(\sqrt{s^2-p^2},p)$, $(q,-\sqrt{s^2-q^2})$, respectively. Using the determinant formula for finding the area of a triangle, we see that the areas of triangles ABP and CDP are equal if and only if

$$\frac{1}{2}\begin{vmatrix} -\sqrt{r^2-p^2} & p \\ q & \sqrt{r^2-q^2} \end{vmatrix} = \frac{1}{2}\begin{vmatrix} \sqrt{s^2-p^2} & p \\ q & -\sqrt{s^2-q^2} \end{vmatrix},$$

which after cancelling $\frac{1}{2}$ on both sides is equivalent to

$$-\sqrt{r^2-p^2}\sqrt{r^2-q^2}-pq=-\sqrt{s^2-p^2}\sqrt{s^2-q^2}-pq.$$

Since $f(x)=-\sqrt{x^2-p^2}\sqrt{x^2-q^2}-pq$ is strictly decreasing when $x \geq |p|$ and $|q|$, equality of areas hold if and only if $r=s$, which is equivalent to A, B, C, D concyclic (since P being on the perpendicular bisectors of AB, CD is the only possible place for the center).

After seeing these examples, we would like to remind the readers that there are pure geometric proofs to each of the problems. For examples (1) and (3), there are proofs that only take a few lines. We encourage the readers to discover these simple proofs.

Although in the opinions of many people, a pure geometric proof is better and more beautiful than a coordinate geometric proof, we should point out that sometimes the coordinate geometric proofs may be preferred when there are many cases. For example (2), the different possible orderings of the points *D, E, F* on the line can all happen as some pictures will show. The coordinate geometric proofs above cover all cases.

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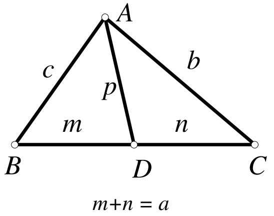
2. Stewart's Theorem and Ceva's Theorem

Notations: For ΔABC, its area will be denoted by $[ABC]$ or S_{ABC}. As usual, let a be the length of side BC, b be the length of side CA and c be the length of side AB.



Stewart's Theorem. Let D be a point on side BC. Let p, m and n be the lengths of line segments AD, BD and CD respectively. Then

$$b^2m + c^2n = a(p^2+mn).$$

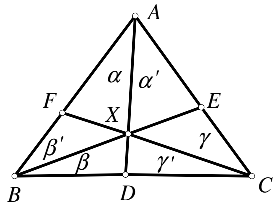
Proof. Since $\angle ADB + \angle ADC = 180^\circ$, so $\cos\angle ADB + \cos\angle ADC = 0$.The formula follows from the cosine law as $\cos\angle ADB = (m^2+p^2-c^2)/(2mp)$ and $\cos\angle ADC = (n^2+p^2-b^2)/(2np)$.

Formulas. (1) If AD is the median to side BC, then $m = a/2 = n$. Let m_a denote the length of AD. Stewart's theorem yields $m_a = \frac{1}{2}\sqrt{2b^2+2c^2-a^2}$. This formula is sometimes refered to as <u>*Apollonius' formula*</u>. Note we have the interesting formula

$$4(m_a^2+m_b^2+m_c^2) = 3(a^2+b^2+c^2).$$

(2) If AD is the angle bisector of $\angle BAC$, then $m/n=c/b$ and $m+n=a$ imply $m=ca/(b+c)$ and $n=ba/(b+c)$. Let t_a denote the length of AD. Stewart's theorem yields $t_a = \sqrt{bc\left(1-\frac{a^2}{(b+c)^2}\right)}$.

Next we come to an important theorem known as Ceva's theorem. It explains why medians or altitudes or angle bisectors of a triangle are concurrent.



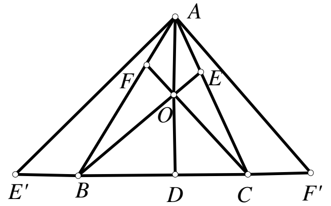
Ceva's Theorem. For ΔABC, let D be on line BC, E be on line CA and F be on line AB. If lines AD, BE and CF are concurrent, then

$$\frac{AF}{FB}\times\frac{BD}{DC}\times\frac{CE}{EA} = 1.$$

(*Remarks*. By the sine law, $AF/\sin\gamma = AC/\sin\angle AFC$ and $FB/\sin\gamma' = CB/\sin\angle CFB$ Since $\sin\angle AFC = \sin\angle CFB$, we get $AF/FB = (AC\sin\gamma)/(CB\sin\gamma')$. Using similar equations, we see that the equation in Ceva's theorem can be written as

$$\frac{\sin\alpha\sin\beta\sin\gamma}{\sin\alpha'\sin\beta'\sin\gamma'} = 1.$$

This is called the <u>*trigonometric form*</u> of the equation.)



Proof. Let O be the concurrent point. Through A, draw a line parallel to BE and let it intersect line BC at E'. Similarly, through A, draw a line parallel to CF and let it intersect line BC at F'. Then $AF/FB = F'C/CB$, $CE/EA = CB/BE'$ and $CD/CF' = OD/OA = BD/BE'$. So $BD/DC = BE'/F'C$. Therefore,

$$\frac{AF}{FB}\times\frac{BD}{DC}\times\frac{CE}{EA} = \frac{F'C}{CB}\times\frac{BE'}{F'C}\times\frac{CB}{BE'} = 1.$$

Converse of Ceva's Theorem. If $\frac{AF}{FB}\times\frac{BD}{DC}\times\frac{CE}{EA} = 1$, then lines AD, BE and CF are concurrent.

Proof. We will prove the converse by the <u>*method of false position*</u>. We are given $\frac{AF}{FB}\times\frac{BD}{DC}\times\frac{CE}{EA} = 1$. Let lines AD and BE intersect at X and lines CX and AB intersect at F'. By Ceva's theorem, $\frac{AF'}{F'B}\times\frac{BD}{DC}\times\frac{CE}{EA} = 1$. So $\frac{AF}{FB} = \frac{AF'}{F'B}$. Then $\frac{AB}{FB} = \frac{AF}{FB} + 1 = \frac{AF'}{F'B} + 1 = \frac{AB}{F'B}$, which implies $F = F'$.

Facts. (a) If *AD*, *BE* and *CF* are the median of ΔABC, then they concur at a point called the *centroid* of the triangle, which is commonly denoted by *G*. In this case, the Ceva equation is just $\frac{c/2}{c/2}\times\frac{a/2}{a/2}\times\frac{b/2}{b/2}=1$.

(b) If *AD*, *BE* and *CF* are the altitudes of ΔABC, then they concur at a point called the *orthocenter* of the triangle, which commonly denoted by *H*. In this case, the Ceva equation is just $\frac{b\cos A}{a\cos B}\times\frac{c\cos B}{b\cos C}\times\frac{a\cos C}{c\cos A}=1$.

(c) If *AD*, *BE* and *CF* are the angle bisectors of ΔABC, then they concur at a point called the *incenter* of the triangle, which is commonly denoted by *I*. In this case, the Ceva equation is just $\frac{bc/(a+b)}{ac/(a+b)}\times\frac{ca/(b+c)}{ba/(b+c)}\times\frac{ab/(c+a)}{cb/(c+a)}=1$. (Since points on an angle bisector are equidistant from the sides of the angle, *I* is equidistant from *AB*, *BC*, *CA* and hence it is the center of the inscribed circle in ΔABC.)

(d) The perpendicular bisectors of sides *AB*, *BC*, *CA* concur at a point called the *circumcenter* of the triangle, which is commonly denoted by *O*. (Since points on a perpendicular bisector of a line segment are equidistant from the endpoints, *O* is equidistant from *A*, *B*, *C* and hence it is the center of the circumscribed circle.)

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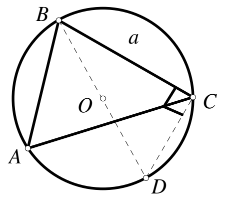
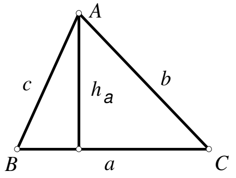
3. Formulas



Extended Sine Law. Let *R* be the radius of the circumcircle of ΔABC. Then

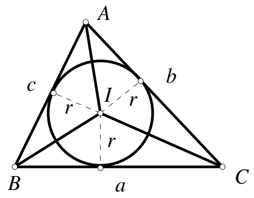
$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R.$$

Proof. Draw diameter *BD*. Then $\frac{a}{\sin A}=\frac{a}{\sin D}=BD=2R.$



Area of Triangle. Letting h_a denote the height from *A* to side *BC*, we have

$$[ABC]=\frac{ah_a}{2}=\frac{ab\sin C}{2}=\frac{abc}{4R}.$$



Let $s=(a+b+c)/2$ be the *semiperimeter* of ΔABC. Let *I* be the incenter of ΔABC and *r* be the radius of the incircle of ΔABC. Then

$$[ABC]=[AIB]+[BIC]+[CIA]=\frac{ar}{2}+\frac{br}{2}+\frac{cr}{2}=sr.$$

Heron's Formula. $[ABC]=\sqrt{s(s-a)(s-b)(s-c)}.$

Proof. By cosine law, $\cos C=(a^2+b^2-c^2)/(2ab)$. We have

$$\begin{aligned}[ABC]^2&=\frac{a^2b^2}{4}\sin^2 C=\frac{a^2b^2}{4}(1+\cos C)(1-\cos C)\\&=\frac{(2ab+a^2+b^2-c^2)(2ab-a^2-b^2+c^2)}{16}\\&=\frac{(a+b+c)}{2}\frac{(a+b-c)}{2}\frac{(c+a-b)}{2}\frac{(c-a+b)}{2}\\&=s(s-c)(s-b)(s-a).\end{aligned}$$

(*Remarks*. Combining with the formulas above, we get

$$R=\frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}\quad\text{and}\quad r=\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}\ .)$$

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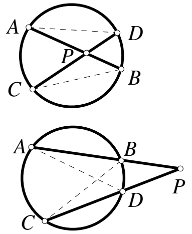
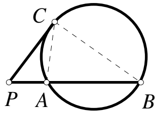
4. Theorems about Concyclic Points



Intersecting Chord Theorem.

(1) Let line segments *AB* and *CD* (or both extended to) intersect at point *P*. Then *A*,*B*,*C*,*D* are concyclic if and only if $PA\times PB=PC\times PD$.



(2) Let lines *PC* and *AB* intersect at *P*. Then *PC* is tangent to the circumcircle of ΔABC if and only if $PC^2 = PA\times PB$.

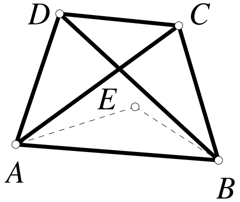
Proof. For (1), observe that $\angle APD = \angle CPB$. So *A*,*B*,*C*,*D* are concyclic $\Leftrightarrow$ $\angle APD = \angle CPB \Leftrightarrow \Delta APD \sim \Delta CPB \Leftrightarrow PA/PD = PC/PB \Leftrightarrow PA\times PB = PC\times PD$.

For (2), observe that $\angle APC = \angle CPB$. So *PC* is tangent to the circumcircle of $\Delta ABC \Leftrightarrow \angle ACP = \angle CBP \Leftrightarrow \Delta ACP \sim \Delta CBP \Leftrightarrow PC/PA = PB/PC \Leftrightarrow PC^2 = PA\times PB$.

Ptolemy's Theorem. *ABCD* is a cyclic quadrilateral if and only if

$$AB\times CD + AD\times BC = AC\times BD.$$

(For general quadrilateral, $AB\times CD + AD\times BC \geq AC\times BD.$)



Proof. On side *AB*, construct $\Delta BAE \sim \Delta CAD$. Then $AB/AC = AE/AD = BE/CD$ and $\angle BAE = \angle CAD$. So

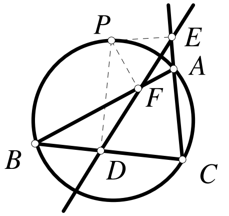
$$\angle BAC = \angle BAD - \angle CAD = \angle BAD - \angle BAE = \angle EAD.$$

Then $\Delta BAC \sim \Delta EAD$. So $BC/AC = ED/AD$. Then

$$AB\times CD + AD\times BC = AC\times BE + AC\times ED \geq AC\times BD.$$

Equality holds if and only if $BE+ED=BD$. This occurs if and only if $\angle DBA = \angle EBA = \angle DCA$, i.e. *ABCD* is cyclic.

Simson's Theorem. Let *P* be on the plane of ΔABC. Let *D*,*E*,*F* be the feet of the perpendiculars from *P* to lines *BC, CA, AB* respectively. If *P* is on the circumcircle of ΔABC, then *D*,*E*,*F* are collinear. The converse is also true.



Proof. Connect *PA* and *PB*. Since $\angle PFA = 90^\circ = \angle PEA$, so *P, F, A, E* are concyclic and hence $\angle PFE = \angle PAE$. Similarly, $\angle PDB = 90^\circ = \angle PFB$, so *P, B, D, F* are concyclic and hence

$$\angle PBD + \angle PFD = 180^\circ.$$

Now *P* is on the circumcircle $\Leftrightarrow \angle PAE = \angle PBC \Leftrightarrow \angle PFE = \angle PBD \Leftrightarrow \angle PFE + \angle PFD = \angle PBD + \angle PFD \Leftrightarrow \angle PFE + \angle PFD = 180^\circ \Leftrightarrow$ *D,E,F* are collinear.

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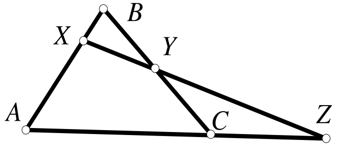
5. Menelaus' Theorem and Other Famous Theorems

Below we will write $P = WX \cap YZ$ to denote *P* is the point of intersection of lines *WX* and *YZ*. If points *A, B, C* are *collinear*, we will introduce the *sign convention* : $AB/BC = \overrightarrow{AB}/\overrightarrow{BC}$ (so if *B* is between *A* and *C*, then $AB/BC \geq 0$, *otherwise* $AB/BC \leq 0$).



Menelaus' Theorem Points *X, Y, Z* are taken from lines *AB, BC, CA* (which are the sides of $\triangle ABC$ extended) respectively. If there is a line passing through *X, Y, Z*, then

$$\frac{AX}{XB}\cdot\frac{BY}{YC}\cdot\frac{CZ}{ZA} = -1.$$

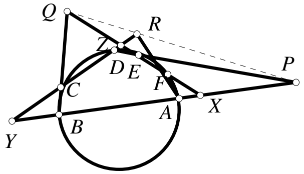
Proof. Let *L* be a line perpendicular to the line through *X, Y, Z* and intersect it at *O*. Let *A', B', C'* be the feet of the perpendiculars from *A, B, C* to *L* respectively. Then

$$\frac{AX}{XB} = \frac{A'O}{OB'}, \frac{BY}{YC} = \frac{B'O}{OC'}, \frac{CZ}{ZA} = \frac{C'O}{OA'}.$$

Multiplying these equations together, we get the result.

Converse of Menelaus' Theorem. If $\frac{AX}{XB}\cdot\frac{BY}{YC}\cdot\frac{CZ}{ZA} = -1$, then there is a line passing through *X, Y, Z* .

Proof (*by the method of false position*). To see this, let $Z' = XY \cap CA$. Then applying Menelaus theorem to the line through *X, Y, Z'* and comparing with the equation above, we get $CZ/ZA = CZ'/Z'A$. Then $CA/ZA = 1 + (CZ/ZA) = 1 + (CZ'/Z'A) = CA/ZA$. It follows $Z=Z'$.



Pascal's Theorem Let *A, B, C, D, E, F* be points on a circle (which are not necessarily in cyclic order). Let

$$P = AB\cap DE,\ Q = BC\cap EF,\ R = CD\cap FA.$$

Then *P,Q,R* are collinear.

Proof. Let $X = EF \cap AB$, $Y = AB \cap CD$, $Z = CD \cap EF$. Applying Menelaus' theorem respectively to lines BC, DE, FA cutting $\triangle XYZ$ extended, we have

$$\frac{ZQ}{QX}\cdot\frac{XB}{BY}\cdot\frac{YC}{CZ}=-1,\quad \frac{XP}{PY}\cdot\frac{YD}{DZ}\cdot\frac{ZE}{EX}=-1,\quad \frac{YR}{RZ}\cdot\frac{ZF}{FX}\cdot\frac{XA}{AY}=-1.$$

Multiplying these three equations together, then using the intersecting chord theorem to get $XA\cdot XB = XE\cdot XF$, $YC\cdot YD = YA\cdot YB$, $ZE\cdot ZF = ZC\cdot ZD$, we arrive at the equation

$$\frac{ZQ}{QX}\cdot\frac{XP}{PY}\cdot\frac{YR}{RZ}=-1.$$

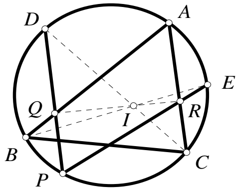
By the converse of Menelaus' theorem, this implies P, Q, R are collinear.

Remarks. There are limiting cases of Pascal's Theorem. For example, we may move A to approach B. In the limit, A and B will coincide and the line AB will become the tangent line at B.

Below we will give some examples of using Pascal's theorem in problems.

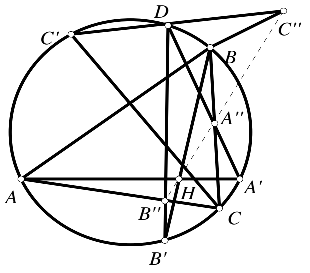
Example 1. (*2001 Macedonian Math Olympiad*) For the circumcircle of $\triangle ABC$, let D be the intersection of the tangent line at A with line BC, E be the intersection of the tangent line at B with line CA and F be the intersection of the tangent line at C with line AB. Prove that points D, E, F are collinear.

Solution Applying Pascal's theorem to A, A, B, B, C, C on the circumcircle, we easily get D, E, F are collinear.



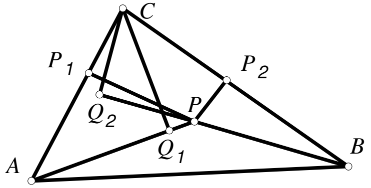
Example 2. Let D and E be the midpoints of the minor arcs AB and AC on the circumcircle of $\triangle ABC$, respectively. Let P be on the minor arc BC, $Q = DP \cap BA$ and $R = PE \cap AC$. Prove that line QR passes through the incenter I of $\triangle ABC$.

Solution Since D is the midpoint of arc AB, line CD bisects $\angle ACB$. Similarly, line EB bisects $\angle ABC$. So $I = CD \cap EB$. Applying Pascal's theorem to C, D, P, E, B, A, we get I, Q, R are collinear.



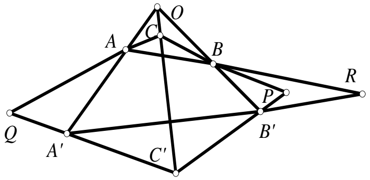
Example 3. (*2001 Australian Math Olympiad*) Let A,B,C,A',B',C' be points on a circle such that AA' is perpendicular to BC, BB' is perpendicular to CA, CC' is perpendicular to AB. Further, let D be a point on that circle and let DA' intersect BC in A'', DB' intersect CA in B'', and DC' intersect AB in C'', all segments being extended where required. Prove that A'', B'', C'' and the orthocenter of triangle ABC are collinear.

Solution Let H be the orthocenter of $\triangle ABC$. Applying Pascal's theorem to A, A', D, C', C, B, we see H, A'', C'' are collinear. Similarly, applying Pascal's theorem to B', D, C', C, A, B, we see B'', C'', H are collinear. So A'', B'', C'', H are collinear.



Example 4. (*1991 IMO unused problem*) Let ABC be any triangle and P any point in its interior. Let P_1, P_2 be the feet of the perpendiculars from P to the two sides AC and BC. Draw AP and BP and from C drop perpendiculars to AP and BP. Let Q_1 and Q_2 be the feet of these perpendiculars. If $Q_2 \neq P_1$ and $Q_1 \neq P_2$, then prove that the lines P_1Q_2, Q_1P_2 and AB are concurrent.

Solution Since $\angle CP_1P$, $\angle CP_2P$, $\angle CQ_2P$, $\angle CQ_1P$ are all right angles, we see that the points C, Q_1, P_1, P, P_2, Q_2 lie on a circle with CP as diameter. Note $A = CP_1 \cap PQ_1$ and $B = Q_2P \cap P_2C$. Applying Pascal's theorem to C, P_1, Q_2, P, Q_1, P_2, we see $X = P_1Q_2 \cap Q_1P_2$ is on line AB.



Desargues' Theorem For triangles ABC and $A'B'C'$, if lines AA', BB', CC' concur at a point O, then points P, Q, R are collinear, where $P = BC \cap B'C'$, $Q = CA \cap C'A'$, $R = AB \cap A'B'$.

Proof. Applying Menelaus' theorem respectively to line $A'B'$ cutting $\triangle OAB$ extended, line $B'C'$ cutting $\triangle OBC$ extended and the line $C'A'$ cutting $\triangle OCA$ extended, we have

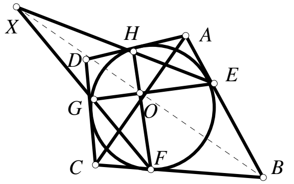
$$\frac{OA'}{A'A}\cdot\frac{AR}{RB}\cdot\frac{BB'}{B'O}=-1,\quad \frac{OB'}{B'B}\cdot\frac{BP}{PC}\cdot\frac{CC'}{C'O}=-1,\quad \frac{AA'}{A'O}\cdot\frac{OC'}{C'C}\cdot\frac{CQ}{QA}=-1.$$

Multiplying these three equations, $\frac{AR}{RB}\cdot\frac{BP}{PC}\cdot\frac{CQ}{QA}=-1.$

By the converse of Menelaus' theorem, this implies P,Q,R are collinear.

Converse of Desargues' Theorem. For triangles ABC and $A'B'C'$, if points P, Q, R are collinear, where $P = BC \cap B'C'$, $Q = CA \cap C'A'$, $R = AB \cap A'B'$, then lines AA', BB', CC' concur at a point O.

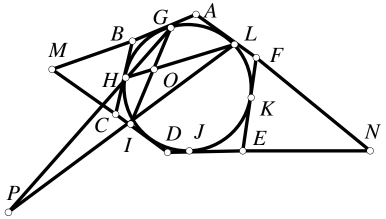
Proof. We can prove it as follow: let $O=BB'\cap CC'$. Consider $\triangle RBB'$ and $\triangle QCC'$. Since lines RQ, BC, $B'C'$ concur at P, and $A = RB \cap QC$, $O = BB' \cap CC'$, $A'=BR' \cap C'Q$, by Desargues' theorem, we have A,O,A' are collinear. Therefore, lines AA', BB', CC' concur at O.



Newton's Theorem A circle is inscribed in a quadrilateral $ABCD$ with sides AB, BC, CD, DA touch the circle at points E, F, G, H respectively. Then lines AC, EG, BD, FH are concurrent.

Proof. Let $O=EG\cap FH$ and $X=EH\cap FG$. Since D is the intersection of the tangent lines at G and at H to the circle, applying Pascal's theorem to E,G,G,F,H,H, we get O, D, X are collinear. Similarly, applying Pascal's theorem to E, E, H, F, F, G, we get B, X, O are collinear.

Then B,O,D are collinear and so lines EG, BD, FH are concurrent at O. Similarly, we can also obtain lines AC, EG, FH are concurrent at O. Then Newton's theorem follows.



Brianchon's Theorem Lines AB, BC, CD, DE, EF, FA are tangent to a circle at points G, H, I, J, K, L (not necessarily in cyclic order). Then lines AD, BE, CF are concurrent.

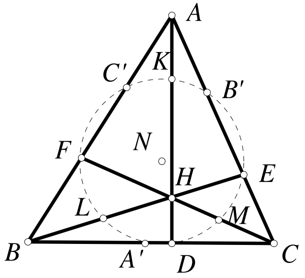
Proof. Let $M = AB \cap CD$, $N = DE \cap FA$. Applying Newton's theorem to quadrilateral $AMDN$, we see lines AD, IL, GJ concur at a point A'. Similarly, lines BE, HK, GJ concur at a point B' and lines CF, HK, IL concur at a point C'. Then A',B',G,J are collinear; B',C',H,K are collinear; C',A',I,L are collinear.

Next we apply Pascal's theorem to G, G, I, L, L, H and get points A, O, P are collinear, where $O = GI \cap LH$ and $P = IL \cap HG$. Applying Pascal's theorem again to H, H, L, I, I, G, we get C, O, P are collinear. Hence A, C, P are collinear.

Now $AB \cap A'B' = AB \cap GJ = G$, $BC \cap B'C' = BC \cap HK = H$ and $CA \cap C'A' = CA \cap IL = P$ are collinear. Applying the converse of Desargues' theorem to $\triangle ABC$ and $\triangle A'B'C'$, we get lines $AA'=AD$, $BB'=BE$, $CC'=CF$ are concurrent.

6. Nine Point Circle and Euler Line

The following theorem is a very interesting theorem. It is a high point in the history of geometry.

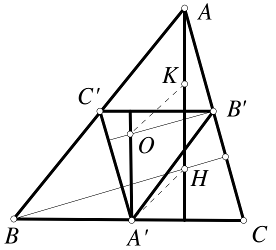


Nine Point Circle Theorem. For ΔABC, let A', B', C' be the midpoints of sides BC,CA,AB respectively. Let D,E,F be the feet of the altitudes to sides BC, CA, AB respectively. Let K,L,M be the midpoints of the line segments joining the orthocenter H to vertices A, B, C respectively.

Then A',B',C',D,E,F,K,L,M lie on a circle (called the *nine point circle* of ΔABC). The center N of this circle is the midpoint of OH and the radius of this circle is half the circumradius of ΔABC.

Proof. By the midpoint theorem, $\overrightarrow{C'B'}=\frac{1}{2}\overrightarrow{BC}=\overrightarrow{LM}$ and $\overrightarrow{C'L}=\frac{1}{2}\overrightarrow{AH}=\overrightarrow{B'M}$.

Since $AH\perp BC$, $B'C'LM$ is a rectangle. The circumcircle of $B'C'LM$ contains E, F since $B'L$, $C'M$ are diameters and $\angle B'EL = 90^\circ = \angle C'FM$. So B',C',E,F,L,M are concyclic. Similarly, A',B',D,E,K,L are concyclic. So, the nine points lie on a common circle. Note $B'L$, $C'M$, $A'K$ are diameters of the nine point circle.

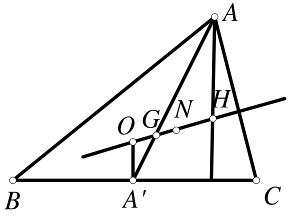


Next note O is the orthocenter of $\Delta A'B'C'$. Since $\Delta ABC \sim \Delta A'B'C'$, so $\Delta ABH \sim \Delta A'B'O$ and

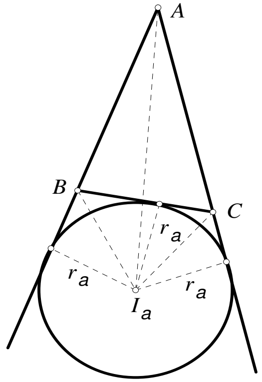
$$A'O/AH = A'B'/AB = 1/2 \Rightarrow A'O = \tfrac{1}{2}AH = KH.$$

Since $A'O \perp BC$ and $KH \perp BC$, so $A'O \parallel KH$. Then $A'OKH$ is a parallelogram. Since $A'K$ is a diameter of the nine point circle, its center is the midpoint of OH.

Finally, N,K are midpoints of OH, AH respectively. The radius NK of the nine point circle is half the circumradius OA by the midpoint theorem.



Euler Line of ΔABC. Since $A'O = \tfrac{1}{2}AH$ and $A'O \parallel AH$, so OH and AA' intersect at a point G such that $OG = \tfrac{1}{2}HG = \tfrac{1}{3}OH$. Similarly, this point G is on BB' and CC'. Hence G is the centroid. (Note $A'G = \tfrac{1}{3}A'A$.) The points O,G,N,H lie on a common line (called the <u>Euler line</u> of ΔABC). We have $OH = 2ON = 3OG$.

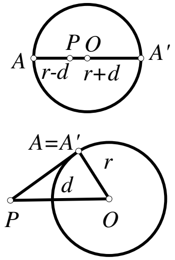


For ΔABC, there is an inscribed circle with center I and radius r. If we extend the sides of $\angle BAC$ beyond B and C, then the bisectors of $\angle BAC$ and the external angle bisectors of $\angle B$ and $\angle C$ are concurrent at a point I_a that is equidistant from side BC and the sides of $\angle BAC$. Hence, I_a is the center of a circle tangent to side BC and the sides of $\angle BAC$. This is called the <u>escribed circle</u> or <u>excircle</u> of ΔABC opposite A, I_a is called the <u>excenter</u> and its radius r_a is called the <u>exradius</u>. Similarly, there are excircles opposite to B and C.

Feuerbach's Theorem. For ΔABC, the inscribed circle is internally tangent to the nine point circle and the three escribed circles are externally tangent to the nine point circle.

7. Power of Points Respect to Circles

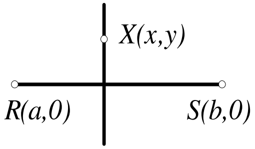
The <u>power</u> of a point P with respect to a circle is the number $d^2 - r^2$, where d is the distance from P to the center of the circle and r is the radius of the circle. (If P is outside the circle, the power is positive. If P is inside, the power is negative. If P is on the circle, the power is 0.)



In the intersecting chord theorem, if P is inside a circle and AA' is a chord through P, then the product $PA \times PA'$ is constant and can be determined by taking the case the chord AA' passes through P and the center O. This gives $PA \times PA' = r^2 - d^2$, where r is the radius of the circle and $d = OP$. In the case P is outside the circle, the product $PA \times PA'$ can be determined by taking the limiting case PA is tangent to the circle. Then $PA \times PA' = d^2 - r^2$. Thus, $PA \times PA'$ is the absolute value of the power of P with respect to the circle. This is known as the <u>power-of-a-point theorem</u>.

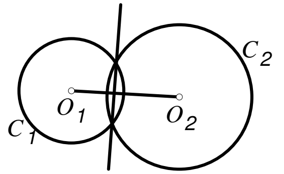
Next we will look at points having equal power with respect to two circles.

Theorem. On a plane, for distinct points R,S and a number m, the locus of all points X such that $RX^2 - SX^2 = m$ is the line perpendicular to line RS. Also, for distinct points R,S,X,Y, we have $RX^2 - SX^2 = RY^2 - SY^2$ if and only if $XY \perp RS$.

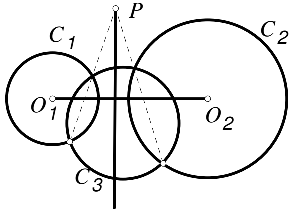


Proof. Let R,S have coordinate $(a,0),(b,0)$. Point X with coordinate (x,y) is on the locus if and only if $((a-x)^2+y^2) - ((b-x)^2+y^2) = m$, which is equivalent to $x = (a^2-b^2-m)/2(a-b)$, a line perpendicular to RS. The second statement follows by using Pythagoras' theorem for the if-part and letting $m = RY^2 - SY^2$ so that X,Y are both on the locus for the only-if-part.

Let circles C_1 and C_2 have distinct centers O_1 and O_2. By the theorem, the points X whose powers with respect to C_1 and C_2 are equal (i.e. $O_1X^2 - r_1^{\,2} = O_2X^2 - r_2^{\,2}$) form a line perpendicular to line O_1O_2. This line is called the <u>radical axis</u> of the two circles. In the case of three circles C_1, C_2, C_3 with noncollinear centers O_1, O_2, O_3, the three radical axes of the three pairs of circles intersect at a point called the <u>radical center</u> of the three circles. (This is because the intersection point of any two of these radical axes has equal power with respect to all three circles, hence it is on the third radical axis too.)



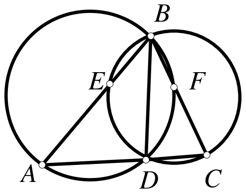
If two circles C_1 and C_2 intersect, their radical axis is the line through the intersection point(s) perpendicular to the line of the centers. (This is because the intersection point(s) have 0 power with respect to both circles, hence they are on the radical axis.)



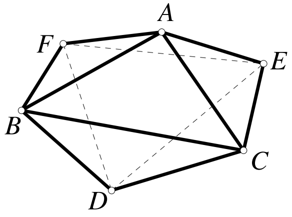
If the two circles do not intersect, their radical axis can be found by taking a third circle C_3 intersecting both C_1 and C_2. Let the radical axis of C_1 and C_3 intersect the radical axis of C_2 and C_3 at P. Then the radical axis of C_1 and C_2 is the line through P perpendicular to the line of centers of C_1 and C_2.

We will illustrate the usefulness of the intersecting chord theorem, the concepts of power of a point, radical axis and radical center in the following examples.

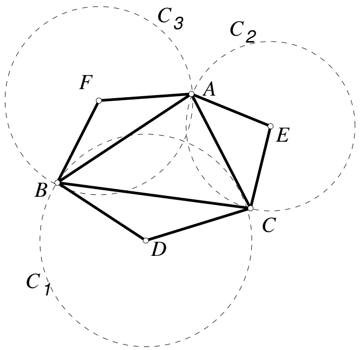
Example 1. (*1996 St. Petersburg City Math Olympiad*) Let BD be the angle bisector of angle B in triangle ABC with D on side AC. The circumcircle of triangle BDC meets AB at E, while the circumcircle of triangle ABD meets BC at F. Prove that $AE = CF$.



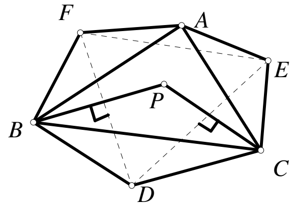
Solution. By the intersecting chord theorem, $AE \times AB = AD \times AC$ and $CF \times CB = CD \times CA$, so $AE/CF = (AD/CD)(BC/AB)$. However, $AB/CB = AD/CD$ by the angle bisector theorem. So $AE = CF$.



Example 2. (*1997 USA Math Olympiad*) Let ABC be a triangle, and draw isosceles triangles BCD, CAE, ABF externally to ABC, with BC, CA, AB as their respective bases. Prove the lines through A, B, C, perpendicular to the lines EF, FD, DE, respectively, are concurrent.



Solution 1. Let C_1 be the circle with center D and radius BD, C_2 be the circle with center E and radius CE, and C_3 be the circle with center F and radius AF. The line through A perpendicular to EF is the radical axis of C_2 and C_3, the line through B perpendicular to FD is the radical axis of C_3 and C_1 and the line through C perpendicular to DE is the radical axis of C_1 and C_2. These three lines concur at the radical center of the three circles.

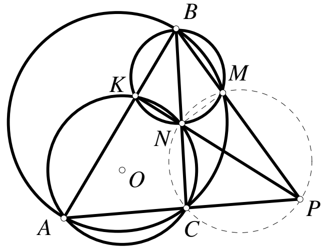


Solution 2. Let P be the intersection of the perpendicular line from B to FD with the perpendicular line from C to DE. Then $PB \perp FD$ and $PC \perp DE$. By the theorem above, we have

$$PF^2-PD^2 = BF^2-BD^2 \text{ and } PD^2-PE^2 = CD^2-CE^2.$$

Adding these and using $AF=BF$, $BD=CD$ and $CE=AE$, we get $PF^2-PE^2 = AF^2-AE^2$. So $PA \perp EF$ and P is the desired concurrent point.

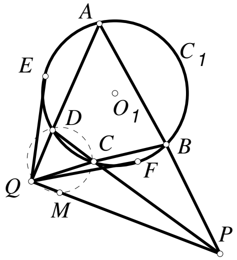
Example 3. (*1985 IMO*) A circle with center O passes through vertices A and C of triangle ABC and intersects side AB at K and side BC at N. Let the circumcircles of triangles ABC and KBN intersect at B and M. Prove that OM is perpendicular to BM.



Solution. For the three circles mentioned, the radical axes of the three pairs are lines AC, KN and BM. (The centers are noncollinear because two of them are on the perpendicular bisector of AC, but not the third.) So the axes will concur at the radical center P. Since $\angle PMN = \angle BKN = \angle NCA$, it follows that P, M, N, C are concyclic.

By power of a point, $BM \times BP = BN \times BC = BO^2 - r^2$ and $PM \times PB = PN \times PK = PO^2 - r^2$, where r is the radius of the circle through A, C, N, K. Then $PO^2 - BO^2 = BP(PM - BM) = PM^2 - BM^2$. By the theorem above, this implies $OM \perp PB$, which is the same as $OM \perp BM$.

Example 4. (*1997 Chinese Math Olympiad*) Let quadrilateral $ABCD$ be inscribed in a circle. Suppose lines AB and DC intersect at P and lines AD and BC intersect at Q. From Q, construct the tangents QE and QF to the circle, where E and F are the points of tangency. Prove that P, E, F are collinear.



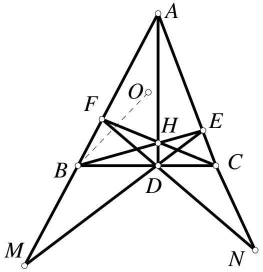
Solution. Let M be a point on PQ such that $\angle CMQ = \angle ADC$. Then D, C, M, Q are concyclic and also, B, C, M, P are concyclic. Let r_1 be the radius of the circumcircle C_1 of $ABCD$ and O_1be the center of C_1. By power of a point,

$$PO_1^2 - r_1^2 = PC \times PD = PM \times PQ$$

and

$$QO_1^2 - r_1^2 = QC \times QB = QM \times PQ.$$

Then $PO_1^2 - QO_1^2 = (PM - QM)\,PQ = PM^2 - QM^2$, which implies $O_1M \perp PQ$. The circle C_2 with QO_1 as diameter passes through M, E, F and intersects C_1 at E, F. If r_2 is the radius of C_2 and O_2 is the center of C_2, then $PO_1^2 - r_1^2 = PM \times PQ = PO_2^2 - r_2^2$. So P lies on the radical axis of C_1, C_2, which is the line EF.



Example 5. (*2001 Chinese National Senior High Math Competition*) As in the figure, in $\triangle ABC$, O is the circumcenter. The three altitudes AD, BE and CF intersect at H. Lines ED and AB intersect at M. Lines FD and AC intersect at N. Prove that (1) $OB \perp DF$, $OC \perp DE$; (2) $OH \perp MN$.

Solution. (1) Since $\angle AFC = 90^\circ = \angle ADC$, so A,C,D,F are concyclic. Then $\angle BDF = \angle BAC$. Also, $\angle OBC = \frac{1}{2}(180^\circ - \angle BOC) = 90^\circ - \angle BAC = 90^\circ - \angle BDF$ implies $OB \perp DF$. Similarly, $OC \perp DE$.

(2) We have

$$\begin{aligned}
CH \perp MA &\Leftrightarrow MC^2 - MH^2 = AC^2 - AH^2 && \text{(a)}\\
BH \perp NA &\Leftrightarrow NB^2 - NH^2 = AB^2 - AH^2 && \text{(b)}\\
DA \perp BC &\Leftrightarrow DB^2 - DC^2 = AB^2 - AC^2 && \text{(c)}\\
OB \perp DF = DN &\Leftrightarrow BN^2 - BD^2 = ON^2 - OD^2 && \text{(d)}\\
OC \perp DE = DM &\Leftrightarrow CM^2 - CD^2 = OM^2 - OD^2. && \text{(e)}
\end{aligned}$$

Doing (a)−(b)+(c)+(d)−(e), we get $NH^2 - MH^2 = ON^2 - OM^2$. So $OH \perp MN$.

Alternative Solution. (1) Let J be on line DM such that $JB \parallel DF$. Since $\angle AFC = 90^\circ = \angle ADC$, so A,C,D,F are concyclic. Then $\angle JBD = \angle BDF = \angle BAC$. So line BJ is tangent to the circumcircle of $\triangle ABC$. Then $OB \perp JB$. So $OB \perp DF$. Similarly, $OC \perp DE$.

(2) We note line OH contains the center of the circumcircle and the center of the nine point circle. We will show N, M are on the radical axis of these circles and hence $OH \perp MN$.

From (1), we know A,C,D,F are concyclic. By the intersecting chord theorem, $NA \times NC = NF \times ND$. Since AC is a chord of the circumcircle and FD is a chord on the nine-point circle, so N is on the radical axis of the circles and similarly for M.

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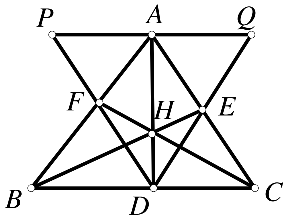
8. Miscellaneous Examples

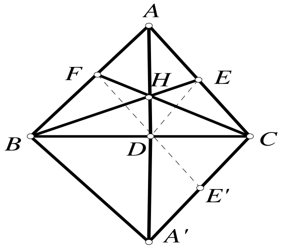
Example 1. (*1994 Canadian Math Olympiad*) Let ABC be an acute triangle. Let D be on side BC such that $AD \perp BC$. Let H be a point on the segment AD different from A and D. Let line BH intersect side AC at E and line CH intersect side AB at F. Prove that $\angle EDA = \angle FDA$.



Solution 1. Draw a line through A parallel to BC. Let the line intersect line DE at Q and line DF at P. Note $\Delta APF \sim \Delta BFD$. So $AP/BD = AF/BF$. Similarly, $AQ/CD = AE/CE$. Also, by Ceva's theorem,

$$\frac{AF}{FB}\frac{BD}{DC}\frac{CE}{EA} = 1 \Leftrightarrow \frac{AF}{FB}\cdot BD = \frac{AE}{CE}\cdot CD \Leftrightarrow AP = AQ.$$

This with $DA = DA$ and $\angle DAP = 90^\circ = \angle DAQ$ yield $\Delta DAP \sim \Delta DAQ$. Therefore, $\angle EDA = \angle FDA$.

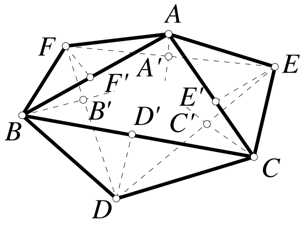


Solution 2. Let A', E' be the mirror image of A,E with respect to line BC. Since AF, CD, EH concur at B, by Ceva's theorem,

$$1 = \frac{AD}{DH}\frac{HF}{FC}\frac{CE}{EA} = \frac{-A'D}{DH}\frac{HF}{FC}\frac{CE'}{E'A}.$$

By the converse of Menelaus' theorem, D, F, E' are collinear. Hence, $\angle EDA = \angle E'DA' = \angle FDA$.

Example 2. (*1997 USA Math Olympiad*) Let ABC be a triangle, and draw isosceles triangles BCD, CAE, ABF externally to ABC, with BC, CA, AB as their respective bases. Prove the lines through A, B, C, perpendicular to the lines EF, FD, DE, respectively, are concurrent.



Solution. Let A',B',C' be points on FE,DF,ED respectively such that $AA'\perp FE$, $BB'\perp DF$ and $CC'\perp ED$. Let D',E',F' be points on CB,AC,BA respectively such that $DD'\perp CB$, $EE'\perp AC$ and $FF'\perp BA$. Now DD', EE', FF' are perpendicular bisectors of the sides of $\triangle ABC$. So they concur. By the trigonometric form of Ceva's theorem, we have

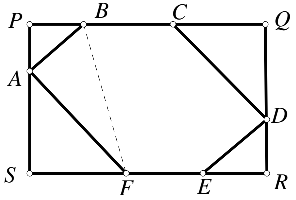
$$\frac{\sin\angle E'EF}{\sin\angle DEE'}\frac{\sin\angle F'FD}{\sin\angle EFF'}\frac{\sin\angle D'DE}{\sin\angle FDD'}=1.$$

Since $E'E\perp CA$ and $EF\perp AA'$, so $\angle E'EF\cong\angle CAA'$. Similarly, $\angle DEE'\cong\angle A'AB$, $\angle F'FD\cong\angle ABB'$, $\angle EFF'\cong\angle B'BC$, $\angle D'DE\cong\angle BCC'$ and $\angle FDD'\cong\angle B'BC$. So

$$\frac{\sin\angle CAA'}{\sin\angle A'AB}\frac{\sin\angle ABB'}{\sin\angle B'BC}\frac{\sin\angle BCC'}{\sin\angle B'BC}=1.$$

By the converse of Ceva's theorem, we get AA', BB', CC' are concurrent, which is the required conclusion.

Example 3. (*1996 IMO*) Let $ABCDEF$ be a convex hexagon such that AB is parallel to DE, BC is parallel to EF and CD is parallel to FA. Let R_A, R_C, R_E denote the circumradii of triangles FAB, BCD, DEF, respectively and let P denote the perimeter of the hexagon. Prove that $R_A+R_C+R_E\geq P/2$.



Solution. Due to the parallel opposite sides, we have $\angle A=\angle D$, $\angle B=\angle E$, $\angle C=\angle F$. Let $PQRS$ be the smallest rectangle containing the hexagon with side BC on PQ as shown. We have

$$\begin{aligned}2BF&\geq PS+QR=AP+AS+DQ+DR\\&=(AB\sin B+FA\sin C)+(CD\sin C+DE\sin B).\end{aligned}$$

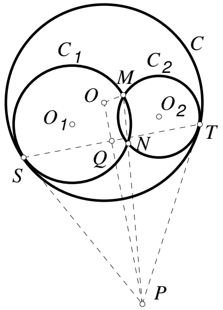
Similarly, $2DB\geq(CD\sin A+BC\sin B)+(EF\sin B+FA\sin A)$
and $2FD\geq(EF\sin C+DE\sin A)+(AB\sin A+BC\sin C)$.

By the extended sine law, $R_A=\dfrac{BF}{2\sin A},R_C=\dfrac{DB}{2\sin C},R_E=\dfrac{FD}{2\sin B}$. Dividing the first inequality by sin A, second inequality by sin C, third inequality by sin E and adding them, we get by the AM-GM inequality that

$$\begin{aligned}4(R_A+R_C+R_E)&\geq AB\left(\frac{\sin B}{\sin A}+\frac{\sin A}{\sin B}\right)+BC\left(\frac{\sin C}{\sin B}+\frac{\sin B}{\sin C}\right)+\cdots\\&\geq 2(AB+BC+CD+DE+EF+FA)=2P.\end{aligned}$$

The result follows.

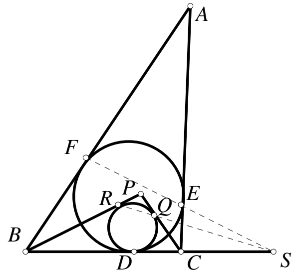
Example 4. (*1997 Chinese National Senior High Math Competition*) Circles C_1, C_2 with centers O_1, O_2 and distinct radii intersect at M, N. C_1, C_2 are internally tangent to a circle C with center O at S and T respectively. Prove that $OM\perp MN$ if and only if S,T,N are collinear.



Solution. Let the tangents at S and at T to circle C intersect at P. Let $Q=OP\cap ST$. Now $OP\perp ST$ and so $PQ\times PO=PS^2$. Also, $PS^2=PN\times PM$ by the intersecting chord theorem. Then $PQ\times PO=PN\times PM$, which implies O,Q,N,M concyclic. Hence,

$$\begin{aligned}\angle OMN=90^\circ&\Leftrightarrow\ \angle OQN=90^\circ\\&\Leftrightarrow\ S,T,N \text{ are collinear since } OQ\perp ST.\end{aligned}$$

Example 5. The inscribed circle of $\triangle ABC$ touches sides BC, CA, AB at D, E, F respectively. P is a point inside $\triangle ABC$. The inscribed circle of $\triangle PBC$ touches sides BC,CP,PB at D,Q,R respectively. Prove that E,F,R,Q are concyclic.

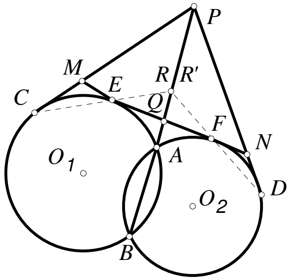


Solution. If $EF\parallel BC$, then $\angle AFE=\angle AEF$ implies $\angle ABC=\angle ACB$. So $\triangle ABC$ is isosceles. Then D is the midpoint of BC and P is on AD. Then $EFRQ$ is an isosceles trapezoid and so E,F,R,Q are concyclic.

If $EF\cap BC=S$, then by Menelaus theorem, $\dfrac{CS}{SB}\dfrac{BF}{FA}\dfrac{AE}{EC}=-1$. Since $AF=AE$,

BR=*BD*=*BF*, *CQ*=*CD*=*CE*, *PQ*=*PR*, we get $\frac{CS}{SB}\frac{BR}{RP}\frac{PQ}{QC}=-1$. So by the converse of Menelaus' theorem, we have *R, Q, S* collinear. By the intersecting chord theorem, $SE\times SF = SD^2 = SR\times SQ$. Then *E*,*F*,*R*,*Q* are concyclic.

Example 6. As in the figure, circles C_1, C_2 with centers O_1, O_2 respectively intersect at *A*, *B*. *P* is a point on line *AB*. From *P*, draw tangents to circle C_1 at *C* and circle C_2 at *D*. Let *EF* be a common tangent to both circles with *E* on C_1 and *F* on C_2. Prove that *AB, CE, DF* are concurrent.



Solution. Let $Q = AB\cap EF$. Let lines *CE, DF* intersect line *AB* at *R, R'* respectively. We have to show *R*=*R'*.

Let line *EF* intersect lines *PC, PD* at *M*, *N* respectively. Applying Menelaus' theorem to line *CER* through $\triangle QMP$, we get $\frac{MC}{CP}\frac{PR}{RQ}\frac{QE}{EM}=-1$. Since *MC*=*ME*, we get $\frac{PR}{RQ}=\frac{PC}{QE}$. Similarly, $\frac{PR'}{R'Q}=\frac{PD}{QF}$.

By the intersecting chord theorem, $PC^2=PA\times PB=PD^2$ and $QE^2=QA\times QB=QF^2$. So $\frac{PR}{RQ}=\frac{PR'}{R'Q}$. Therefore, *R*=*R'*.

Exercises

1. Let *a, b, c* denote the lengths of the sides *BC, CA, AB* respectively. Let h_a, h_b, h_c be the heights from *A, B, C* to the opposite sides respectively. Let *R* be the circumradius, *r* be the inradius and *s* be the semiperimeter of $\triangle ABC$.

   (a) For $\triangle ABC$, show that $\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r}$.

   (b) Show that $r=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$. (Hint: Show $\tan\frac{A}{2}=\frac{r}{s-a}$.)

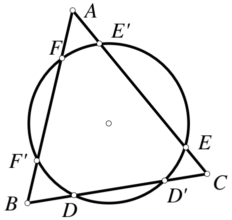
2. Show that among all triangles with the same perimeter, the equilateral triangle has the largest area.

3. (*1996 Iranian Math Olympiad*) Let *ABC* be a scalene triangle (i.e. no two sides equal). The medians from *A, B, C* meet the circumcircle again at *L, M, N* respectively. If *LM*=*LN*, prove that $2BC^2=AB^2+AC^2$. (*Hint*: Show $\frac{LN}{AC}=\frac{LG}{CG}$ first.)

4. (*1996 St. Petersburg Math Olympiad*) Let *BD* be the bisector of angle *B* in $\triangle ABC$. The circumcircle of $\triangle BDC$ meets *AB* at *E*, while the circumcircle of $\triangle ABD$ meet *BC* at *F*. Prove that *AE*=*CF*.

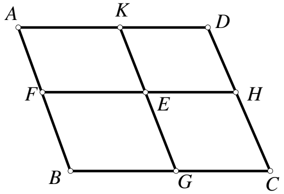
5. (*1995 IMO*) Let *ABCDEF* be a convex hexagon with *AB* = *BC* = *CD*, *DE* = *EF* = *FA* and $\angle BCD = \angle EFA = 60^\circ$. Prove that $AG+GB+GH+DH+HE \geq CF$.

6\.



In the figure, if lines *AD, BE, CF* are concurrent, show that lines *AD', BE', CF'* are concurrent.

7\.



In the figure, *ABCD, AFHD, KGCD* are parallelograms and *KG, FH* intersect at point *E*. Show that lines *FK, BD, GH* are concurrent. (*Hint*: There are more than one ways of solving this. One way is to let *FK, BD* intersect at *X*, then show *G,H,X* are collinear.)

8. (a) Let the angle bisector of $\angle BAC$ intersect the circumcircle of $\triangle ABC$ at *D*. Show that if point *I* on the line segment *AD* is the incenter of $\triangle ABC$, then *BD* = *ID* = *CD*. (Remark: The converse is also true.)

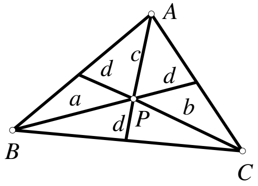
(b) Show that $OI^2 = R^2 - 2Rr$, where O is the circumcenter, R is the circumradius and r is the inradius of $\triangle ABC$.

9. Let a,b,c and a', b', c' be the lengths of two triangles. Let $K(x,y,z)$ be the area of a triangle with side lengths x,y,z. Show that

$$\sqrt{K(a+a',b+b',c+c')} \geq \sqrt{K(a,b,c)} + \sqrt{K(a',b'c')}.$$

When does equality hold?

10.



(*Hong Kong IMO Prelim Contest 90-91*) Let P be an interior point of $\triangle ABC$ and extend lines from the vertices through P to the opposite sides. Let a,b,c,d denote the lengths of the line segments indicated in the figure. Find abc if $a+b+c = 43$ and $d = 3$.