Inequalities (Part III)

§1. Tangent Line Method

Below we will give some examples, where finding the equation of a tangent line is the critical step to solving the problems.

Example 1. Let $a,b,c,d>0$ and $a+b+c+d=1$. Prove that

$$6(a^3+b^3+c^3+d^3) \ge (a^2+b^2+c^2+d^2) + 1/8.$$

Solution. We have $0 < a,b,c,d < 1$. Let $f(x) = 6x^3-x^2$. Then we have to prove $f(a)+f(b)+f(c)+f(d) \ge 1/8$ (*Thoughts*: Since there is equality when $a=b=c=d=1/4$, we consider the graph of $f(x)$ and its tangent line at $x = 1/4$. By a simple sketch, it seems the tangent line is below the graph of $f(x)$ on the interval (0,1). By calculus, the equation of the tangent line at $x = 1/4$ is $y = (5x-1)/8$.)

We claim that for $0 < x < 1$, $f(x) = 6x^3-x^2 \ge (5x-1)/8$. This is equivalent to $48x^3-8x^2-5x+1 \ge 0$. (*Thought*: Since the graphs intersect at $x = 1/4$, we expect $4x-1$ is a factor.) Indeed, $48x^3-8x^2-5x+1 = (4x-1)^2(3x+1) \ge 0$ for $0 < x < 1$. So the claim is true.

Therefore,

$$f(a)+f(b)+f(c)+f(d) \ge 5(a+b+c+d)/8-4/8 = 1/8.$$

Example 2. (*2003 USA Math Olympiad*) Let $a,b,c > 0$. Prove that

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2}+\frac{(2b+c+a)^2}{2b^2+(c+a)^2}+\frac{(2c+a+b)^2}{2c^2+(a+b)^2} \le 8.$$

Solution. Setting $a' = a/(a+b+c)$, $b' = b/(a+b+c)$, $c' = c/(a+b+c)$ if necessary, we may assume $0 < a,b,c < 1$ and $a+b+c = 1$. Then the first term on the left side of the inequality is equal to

$$f(a) = \frac{(a+1)^2}{2a^2+(1-a)^2} = \frac{a^2+2a+1}{3a^2-2a+1}.$$

(*Note*: When $a = b = c = 1/3$, there is equality. A simple sketch of $f(x)$ on [0,1] shows the curve is below the tangent line at $x = 1/3$, which has the equation $y = \frac{12x+4}{3}$.) So we claim that $\frac{a^2+2a+1}{3a^2-2a+1} \le \frac{12a+4}{3}$ for $0 < a < 1$.

Multiplying out, we see this is equivalent to $36a^3-15a^2-2a+1 \ge 0$ for $0 < a < 1$. (*Note*: Since the curve and the line intersect at $a = 1/3$, we expect $3a-1$ is a factor.) Indeed, $36a^3-15a^2-2a+1 = (3a-1)^2(4a+1) \ge 0$ for $0 < a < 1$. Finally adding the similar inequalities for b and c, we get the desired inequality.

Example 3. (*1997 Japanese Math Olympiad*) Let $a,b,c > 0$. Prove that

$$\frac{(b+c-a)^2}{(b+c)^2+a^2}+\frac{(c+a-b)^2}{(c+a)^2+b^2}+\frac{(a+b-c)^2}{(a+b)^2+c^2} \ge \frac{3}{5}.$$

Solution. (Due to Titu Andreescu and Gabriel Dospinescu) As in the last example, we may assume $0 < a,b,c < 1$ and $a+b+c = 1$. Then the first term on the left become $\frac{(1-2a)^2}{(1-a)^2+a^2} = 2-\frac{2}{1+(1-2a)^2}$.

Let $x_1 = 1-2a$, $x_2 = 1-2b$, $x_3 = 1-2c$, then $x_1+x_2+x_3 = 1$, but $-1 < x_1, x_2, x_3 < 1$. In terms of x_1, x_2, x_3, the desired inequality is

$$\frac{1}{1+x_1^2}+\frac{1}{1+x_2^2}+\frac{1}{1+x_3^2} \le \frac{27}{10}.$$

(*Note*: As in the last example, we consider the equation of the tangent line to $f(x) = 1/(1+x^2)$ at $x = 1/3$, which is $y = 27(-x+2)/50$.)

So we claim that $f(x) \le 27(-x+2)/50$ for $-1 < x < 1$. This is equivalent to $(3x-1)^2(4-3x) \ge 0$. Hence the claim is true for $-1 < x < 1$. Then $f(x_1)+f(x_2)+f(x_3) \le 27/10$ and the desired inequality follows.

Exercise. (2008 *CWMO*) Let a,b,c be real numbers with $a+b+c = 3$. Prove that

$$\frac{1}{5a^2-4a+11}+\frac{1}{5b^2-4b+11}+\frac{1}{5c^2-4c+11} \le \frac{1}{4}.$$

§2. Schur's Inequality

Sometimes in proving an inequality, we do not see any easy way. It will be good to know some brute force methods in such situation. Below we will introduce a simple inequality that turns out to be useful in proving inequalities by brute force.

Schur's Inequality. For any $x, y, z \ge 0$ and real r,

$$x^r(x-y)(x-z) + y^r(y-x)(y-z) + z^r(z-x)(z-y) \ge 0.$$

Equality holds if and only if $x = y = z$ or two of x, y, z are equal and the third is 0.

Proof. Observe that the inequality is symmetric in x, y, z. So without loss of generality, we may assume $x \ge y \ge z$. Then $a = x-y \ge 0$ and $b = y-z \ge 0$. Now the left hand side of the inequality can be written as

$$x^r a(a+b) - y^r ab + z^r b(a+b) = x^r a^2 + (x^r - y^r + z^r)ab + z^r b^2.$$

If $r \ge 0$, then $x^r \ge y^r$. If $r < 0$, then $z^r \ge y^r$. So $x^r - y^r + z^r \ge 0$ and Schur's inequality follows. Now $a > 0$ implies $x > y \ge 0$ and $x^r a^2 > 0$. So, equality holds in Schur's inequality if and only if $a = 0$ and $z^r b^2 = 0$, that is $x = y$ and either $y = z$ or $z = 0$.

In using the Schur's inequality, we often expand out expressions. So to simplify writing, we introduce the *symmetric sum* notation $\sum_{sym} f(x,y,z)$ to denote

$$f(x,y,z)+f(x,z,y)+f(y,z,x)+f(y,x,z)+f(z,x,y)+f(z,y,x).$$

For example, $\sum_{sym} x^3 = 2x^3 + 2y^3 + 2z^3$, $\sum_{sym} x^2y = x^2y + x^2z + y^2z + y^2x + z^2x + z^2y$ and $\sum_{sym} xyz = 6xyz$. Similarly, for a function of n variables, the symmetric sum is the sum of all $n!$ terms, where we take all possible permutations of the n variables.

After expanding the $r = 1$ case of Schur's inequality, we get

$$x^3+y^3+z^3-(x^2y+x^2z+y^2x+y^2z+z^2x+z^2y)+3xyz \ge 0.$$

In symmetric sum notation, it is $\sum_{sym}(x^3 - 2x^2y + xyz) \ge 0$.

By expanding both sides and rearranging terms, each of the following inequalities is equivalent to the $r = 1$ case of Schur's inequality. These are common disguises.

a) $x^3+y^3+z^3+3xyz \ge xy(x+y)+yz(y+z)+zx(z+x)$,

b) $xyz \ge (x+y-z)(y+z-x)(z+x-y)$,

c) $4(x+y+z)(xy+yz+zx) \le (x+y+z)^3+9xyz$.

Example 1. (*2000 IMO*) Let $a, b, c > 0$ and $abc = 1$. Prove that

$$(a-1+\frac{1}{b})(b-1+\frac{1}{c})(c-1+\frac{1}{a}) \le 1.$$

Solution. Let $x = a$, $y = 1$, $z = 1/b = ac$. Then $a = x/y$, $b = y/z$ and $c = z/x$. Substituting these into the desired inequality, we get

$$\frac{(x-y+z)}{y}\frac{(y-z+x)}{z}\frac{(z-x+y)}{x} \le 1,$$

which is disguise *b*) of the $r = 1$ case of Schur's inequality.

Example 2. (*1984 IMO*) Prove that

$$0 \le yz+zx+xy-2xyz \le 7/27,$$

where x, y, z are nonnegative real numbers such that $x+y+z = 1$.

Solution. In Schur's inequality, all terms are of the same degree. So we first change the desired inequality to one where all terms are of the same degree. Since $x+y+z = 1$, the desired inequality is the same as

$$0 \le (x+y+z)(yz+zx+xy) - 2xyz \le \frac{7(x+y+z)^3}{27}.$$

Expanding the middle expression, we get $xyz + \sum_{sym} x^2y$, which is nonnegative and the left inequality is proved. Expanding the rightmost expression and subtracting the middle expression, we get

$$\frac{7}{54}\sum_{sym}(x^3-\frac{12}{7}x^2y+\frac{5}{7}xyz). \qquad (1)$$

By Schur's inequality, we have

$$\sum_{sym}(x^3-2x^2y+xyz)\ge 0. \qquad (2)$$

Note $(x^3-\frac{12}{7}x^2y+\frac{5}{7}xyz)-(x^3-2x^2y+xyz)=\frac{2}{7}(x^2y-xyz).$ By the AM-GM inequality, we have $\sum_{sym}x^2y\ge 6(x^6y^6z^6)^{1/6}=\sum_{sym}xyz,$ which is the same as

$$\sum_{sym}(x^2y-xyz)\ge 0. \qquad (3)$$

Multiplying (3) by 2/7 and adding it to (2), we see the symmetric sum in (1) is nonnegative. This yields the inequality on the right.

Example 3. (*2004 APMO*) Prove that

$$(a^2+2)(b^2+2)(c^2+2)\ge 9(ab+bc+ca)$$

for any positive real numbers *a,b,c*.

Solution. Expanding and expressing in symmetric sum notation, the desired inequality is $(abc)^2+\sum_{sym}(a^2b^2+2a^2)+8\ge\frac{9}{2}\sum_{sym}ab.$

Since $a^2+b^2\ge 2ab$, we get $\sum_{sym}a^2\ge\sum_{sym}ab$. Also, from $a^2b^2+1\ge 2ab$, we get $\sum_{sym}a^2b^2+6\ge 2\sum_{sym}ab$. Using these, the problem is reduced to showing

$$(abc)^2+2\ge\sum_{sym}(ab-\tfrac{1}{2}a^2).$$

To prove this, we apply the AM-GM inequality twice and disguise c) of the $r=1$ case of Schur's inequality as follow:

$$\begin{aligned}(abc)^2+2&\ge 3(abc)^{2/3}\ge 9abc/(a+b+c)\\&\ge 4(ab+bc+ca)-(a+b+c)^2\\&=2(ab+bc+ca)-(a^2+b^2+c^2)=\sum_{sym}(ab-\tfrac{1}{2}a^2).\end{aligned}$$

Example 4. (*2000 USA Team Selection Test*) Prove that for any positive real numbers *a, b, c*, the following inequality holds

$$\frac{a+b+c}{3}-\sqrt[3]{abc}\le\max\{(\sqrt{a}-\sqrt{b})^2,(\sqrt{b}-\sqrt{c})^2,(\sqrt{c}-\sqrt{a})^2\}.$$

Solution. From the last part of the solution of example 3, we get

$$3(xyz)^{2/3}\ge 2(xy+yz+zx)-(x^2+y^2+z^2)$$

for any $x,y,z>0$. (*Note*: this used Schur's inequality.) Let $x=\sqrt{a}$, $y=\sqrt{b}$ and $z=\sqrt{c}$. Arranging terms, we get

$$\begin{aligned}a+b+c-3\sqrt[3]{abc}&\le 2(a+b+c-\sqrt{ab}-\sqrt{bc}-\sqrt{ca})\\&=(\sqrt{a}-\sqrt{b})^2+(\sqrt{b}-\sqrt{c})^2+(\sqrt{c}-\sqrt{a})^2\\&\le 3\max\{(\sqrt{a}-\sqrt{b})^2,(\sqrt{b}-\sqrt{c})^2,(\sqrt{c}-\sqrt{a})^2\}.\end{aligned}$$

Dividing by 3, we get the desired inequality.

Example 5. (*2003 USA Team Selection Test*) Let *a,b,c* be real numbers in the interval $(0, \pi/2)$. Prove that

$$\frac{\sin a\sin(a-b)\sin(a-c)}{\sin(b+c)}+\frac{\sin b\sin(b-c)\sin(b-a)}{\sin(c+a)}+\frac{\sin c\sin(c-a)\sin(c-b)}{\sin(a+b)}\ge 0.$$

Solution. Observe that $\sin(u-v)\sin(u+v)=(\cos 2v-\cos 2u)/2=\sin^2 u-\sin^2 v$. Let $x=\sin^2 a$, $y=\sin^2 b$, $z=\sin^2 c$. In adding up the terms, the left side of the inequality becomes

$$\frac{\sqrt{x}(x-y)(x-z)+\sqrt{y}(y-z)(y-x)+\sqrt{z}(z-x)(z-y)}{\sin(b+c)\sin(c+a)\sin(a+b)}.$$

This is nonnegative by the $r=1/2$ case of Schur's inequality.

§3. Muirhead's Inequality

Muirhead's inequality is an important generalization of the AM-GM inequality. It is a powerful tool for solving inequality problem. First we give a definition which is a generalization of arithmetic and geometric means.

Definition. Let $x_1, x_2, \ldots, x_n$ be positive real numbers and $p = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n$. The ***p-mean*** of $x_1, x_2, \ldots, x_n$ is defined by

$$[p] = \frac{1}{n!}\sum_{\sigma\in S_n} x_{\sigma(1)}^{p_1} x_{\sigma(2)}^{p_2}\cdots x_{\sigma(n)}^{p_n},$$

where S_n is the set of all permutations of $\{1,2,\ldots, n\}$. (The summation sign means to sum $n!$ terms, one term for each permutation σ in S_n.)

For example, $[(1,0,\ldots,0)] = \frac{1}{n!}\sum_{\sigma\in S_n} x_{\sigma(1)} = \frac{1}{n}\sum_{i=1}^{n} x_i$ is the arithmetic mean of $x_1, x_2, \ldots, x_n$ and $\left[\left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right)\right] = x_1^{1/n} x_2^{1/n}\cdots x_n^{1/n}$ is their geometric mean.

Theorem (Muirhead's Inequality). Let $x_1, x_2, \ldots, x_n$ be positive real numbers and $p, q \in \mathbb{R}^n$. If $p \succ q$, then $[p] \geq [q]$. Furthermore, for $p \neq q$, equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

Since $(1,0,\ldots,0) \succ (1/n, 1/n, \ldots, 1/n)$, the AM-GM inequality is a corollary of Muirhead's inequality.

Example 1. For any $a, b, c > 0$, prove that
$$(a+b)(b+c)(c+a) \geq 8abc.$$

Solution. Expanding both sides, the desired inequality is
$$a^2b+a^2c+b^2c+b^2a+c^2a+c^2b \geq 6abc$$
or $[(2,1,0)] \geq [(1,1,1)]$. This is Muirhead's inequality for $(2,1,0) \succ (1,1,1)$.

For the next example, we would like to point out a useful trick. When the product of $x_1, x_2, \ldots, x_n$ is 1, we have
$$[(p_1, p_2, \ldots, p_n)] = [(p_1-r, p_2-r, \ldots, p_n-r)]$$
for any real number r.

Example 2. (*1995 IMO*) For any $a, b, c > 0$ with $abc = 1$, prove that
$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

Solution. Multiplying by the common denominator and expanding both sides, the desired inequality is
$$\begin{aligned} &2(a^4b^4 + b^4c^4 + c^4a^4) + 2(a^4b^3c + a^4c^3b + b^4c^3a + b^4a^3c + c^4a^3b + c^4b^3a) \\ &\quad + 2(a^3b^3c^2 + b^3c^3a^2 + c^3a^3b^2) \\ &\geq 3(a^5b^4c^3 + a^5c^4b^3 + b^5c^4a^3 + b^5a^4c^3 + c^5a^4b^3 + c^5b^4a^3) + 6a^4b^4c^4. \end{aligned}$$
This is equivalent to $[(4,4,0)]+2[(4,3,1)] + [(3,3,2)] \geq 3[(5,4,3)] + [(4,4,4)]$. Note $4+4+0 = 4+3+1 = 3+3+2 = 8$, but $5+4+3 = 4+4+4 = 12$. So we can set $r = 4/3$ and use the trick above to get $[(5,4,3)] = [(11/3,8/3,5/3)]$ and also $[(4,4,4)] = [(8/3,8/3,8/3)]$.

Observe that $(4,4,0) \succ (11/3,8/3,5/3)$, $(4,3,1) \succ (11/3,8/3,5/3)$ and $(3,3,2) \succ (8/3,8/3,8/3)$. So applying Muirhead's inequality to these three majorizations and adding the inequalities, we get the desired inequality.

Example 3. (*1990 IMO Shortlisted Problem*) For any $x, y, z > 0$ with $xyz = 1$, prove that
$$\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+z)(1+x)} + \frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}.$$

Solution. Multiplying by the common denominator and expanding both sides, the desired inequality is
$$4(x^4+y^4+z^4+x^3+y^3+z^3) \geq 3(1+x+y+z+xy+yz+zx+xyz)$$
or $4[(4,0,0)] + 4[(3,0,0)] \geq [(0,0,0)] + 3[(1,0,0)] + 3[(1,1,0)] + [(1,1,1)]$.

For this, we apply Muirhead's inequality and the trick as follow:

$$[(4,0,0)] \geq [(4/3,4/3,4/3)] = [(0,0,0)],$$
$$3[(4,0,0)] \geq 3[(2,1,1)] = 3[(1,0,0)],$$
$$3[(3,0,0)] \geq 3[(4/3,4/3,1/3)] = 3[(1,1,0)]$$
and $[(3,0,0)] \geq [(1,1,1)]$.

Adding these, we get the desired inequality.

Remark. For the following example, we will modify the trick above. In case $xyz \geq 1$, we have $[(p_1, p_2, p_3)] \geq [(p_1-r, p_2-r, p_3-r)]$ for every $r \geq 0$. Also, we will use the following

Fact. For $p, q \in \mathbb{R}^n$, we have $\frac{[p]+[q]}{2} \geq \left[\frac{p+q}{2}\right]$. (This is due to the AM-GM inequality $\frac{x_{\sigma(1)}^{p_1} \cdots x_{\sigma(n)}^{p_n} + x_{\sigma(1)}^{q_1} \cdots x_{\sigma(n)}^{q_n}}{2} \geq x_{\sigma(1)}^{(p_1+q_1)/2} \cdots x_{\sigma(n)}^{(p_n+q_n)/2}$. Summing over $\sigma \in S_n$ and dividing by $n!$, we get the inequality.)

Example 4. (*2005 IMO*) For any $x, y, z > 0$ with $xyz \geq 1$, prove that

$$\frac{x^5-x^2}{x^5+y^2+z^2} + \frac{y^5-y^2}{y^5+z^2+x^2} + \frac{z^5-z^2}{z^5+x^2+y^2} \geq 0.$$

Solution. Multiplying by the common denominator and expanding both sides, the desired inequality is equivalent to $[(9,0,0)]+4[(7,5,0)]+[(5,2,2)]+[(5,5,5)] \geq [(6,0,0)] + [(5,5,2)] + 2[(5,4,0)] +2[(4,2,0)] + [(2,2,2)]$.

To prove this, we note that

(1) $[(9,0,0)] \geq [(7,1,1)] \geq [(6,0,0)]$

(2) $[(7,5,0)] \geq [(5,5,2)]$

(3) $2[(7,5,0)] \geq 2[(6,5,1)] \geq 2[(5,4,0)]$

(4) $[(7,5,0)] + [(5,2,2)] \geq 2[(6,7/2,1)] \geq 2[(9/2,2,-1/2)] \geq 2[(4,2,0)]$

(5) $[(5,5,5)] \geq [(2,2,2)]$,

where (1) and (3) are by Muirhead's inequality and the remark, (2) is by Muirhead's inequality, (4) is by the fact, Muirhead's inequality and the remark and (5) is by the remark.

Considering the sum of the leftmost parts of these inequalities is greater than or equal to the sum of the rightmost parts of these inequalities, we get the desired inequalities.

Alternate Solution. Since

$$\frac{x^5-x^2}{x^5+y^2+z^2} - \frac{x^5-x^2}{x^3(x^2+y^2+z^2)} = \frac{(x^3-1)^2(y^2+z^2)}{x(x^2+y^2+z^2)(x^5+y^2+z^2)} \geq 0,$$

we have

$$\begin{aligned}\frac{x^5-x^2}{x^5+y^2+z^2} + \frac{y^5-y^2}{y^5+z^2+x^2} + \frac{z^5-z^2}{z^5+x^2+y^2} &\geq \frac{x^5-x^2}{x^3(x^2+y^2+z^2)} + \frac{y^5-y^2}{y^3(y^2+z^2+x^2)} + \frac{z^5-z^2}{z^3(z^2+x^2+y^2)} \\ &\geq \frac{1}{x^2+y^2+z^2}\left(x^2-\frac{1}{x}+y^2-\frac{1}{y}+z^2-\frac{1}{z}\right) \\ &\geq \frac{1}{x^2+y^2+z^2}(x^2+y^2+z^2-yz-zx-xy) \\ &= \frac{(x-y)^2+(y-z)^2+(z-x)^2}{2(x^2+y^2+z^2)} \geq 0.\end{aligned}$$

Proofs of Muirhead's Inequality

Let $p \succ q$ and $p \neq q$. From $i = 1$ to n, the first nonzero $p_i - q_i$ is positive. So there will be a negative $p_i - q_i$ later. It follows that there are $j < k$ such that $p_j > q_j$, $p_k < q_k$ and $p_i = q_i$ for any possible i between j, k.

Let $b = (p_j+p_k)/2$, $d = (p_j-p_k)/2$ so that $[b-d,b+d] = [p_k, p_j] \supset [q_k, q_j]$. Let c be the maximum of $|q_j-b|$ and $|q_k-b|$, then $0 \leq d < c$. Let $r = (r_1,\ldots,r_n)$ be defined by $r_i = p_i$ except $r_j = b + c$ and $r_k = b - c$. By the definition of c, either $r_j = q_j$ or $r_k = q_k$. Also, by the definitions of b, c, d, we get $p \succ r$, $p \neq r$ and $r \succ q$. Now

$$n!([p]-[r]) = \sum_{\sigma\in S_n} x_\sigma (x_{\sigma(j)}^{p_j} x_{\sigma(k)}^{p_k} - x_{\sigma(j)}^{r_j} x_{\sigma(k)}^{r_k}) = \sum_{\sigma\in S_n} x_\sigma (u^{b+d} v^{b-d} - u^{b+c} v^{b-c}),$$

where x_σ is the product of $x_{\sigma(i)}^{p_i}$ for $i \neq j$, k and $u = x_{\sigma(j)}$, $v = x_{\sigma(k)}$. For each permutation σ, there is a permutation ρ such that $\sigma(i) = \rho(i)$ for $i \neq j$, k and $\sigma(j) = \rho(k)$, $\sigma(k) = \rho(j)$. In the above sum, if we pair the terms for σ and ρ, then $x_\sigma = x_\rho$ and combining the parenthetical factors for the σ and ρ terms, we have

$$(u^{b+d} v^{b-d} - u^{b+c} v^{b-c}) + (v^{b+d} u^{b-d} - v^{b+c} u^{b-c}) = u^{b-d} v^{b-d} (u^{d+c} - v^{d+c}) (u^{d-c} - v^{d-c}) \geq 0.$$

So the above sum is nonnegative. Then $[p] \geq [r]$.

Equality holds if and only if $u = v$ for all pairs of σ and ρ, which means $x_1 = x_2 = \cdots = x_n$. Finally we recall r has at least one more coordinate in agreement with q than p. So repeating this process finitely many times, we will eventually get the case $r = q$. Then we are done.