Math Problem Book I

compiled by

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Preface

There are over fifty countries in the world nowadays that hold mathematical olympiads at the secondary school level annually. In Hungary, Russia and Romania, mathematical competitions have a long history, dating back to the late 1800's in Hungary's case. Many professional or amateur mathematicians developed their interest in math by working on these olympiad problems in their youths and some in their adulthoods as well.

The problems in this book came from many sources. For those involved in international math competitions, they no doubt will recognize many of these problems. We tried to identify the sources whenever possible, but there are still some that escape us at the moment. Hopefully, in future editions of the book we can fill in these missing sources with the help of the knowledgeable readers.

This book is for students who have creative minds and are interested in mathematics. Through problem solving, they will learn a great deal more than school curricula can offer and will sharpen their analytical skills. We hope the problems collected in this book will stimulate them and seduce them to deeper understanding of what mathematics is all about. We hope the international math communities support our efforts for using these brilliant problems and solutions to attract our young students to mathematics.

Most of the problems have been used in practice sessions for students participated in the Hong Kong IMO training program. We are especially pleased with the efforts of these students. In fact, the original motivation for writing the book was to reward them in some ways, especially those who worked so hard to become reserve or team members. It is only fitting to list their names along with their solutions. Again there are unsung heros

who contributed solutions, but whose names we can only hope to identify in future editions.

As the title of the book suggest, this is a problem book. So very little introduction materials can be found. We do promise to write another book presenting the materials covered in the Hong Kong IMO training program. This, for certain, will involve the dedication of more than one person. Also, this is the first of a series of problem books we hope. From the results of the Hong Kong IMO preliminary contests, we can see waves of new creative minds appear in the training program continuously and they are younger and younger. Maybe the next problem book in the series will be written by our students.

Finally, we would like to express deep gratitude to the Hong Kong Quality Education Fund, which provided the support that made this book possible.

Kin Y. Li
Hong Kong
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Advices to the Readers

The only way to learn mathematics is to do mathematics. In this book, you will find many math problems, ranging from simple to challenging problems. You may not succeed in solving all the problems. Very few people can solve them all. The purposes of the book are to expose you to many interesting and useful mathematical ideas, to develop your skills in analyzing problems and most important of all, to unleash your potential of creativity. While thinking about the problems, you may discover things you never know before and putting in your ideas, you can create something you can be proud of.

To start thinking about a problem, very often it is helpful to look at the initial cases, such as when $n = 2, 3, 4, 5$. These cases are simple enough to let you get a feeling of the situations. Sometimes, the ideas in these cases allow you to see a pattern, which can solve the whole problem. For geometry problems, always draw a picture as accurate as possible first. Have protractor, ruler and compass ready to measure angles and lengths.

Other things you can try in tackling a problem include changing the given conditions a little or experimenting with some special cases first. Sometimes may be you can even guess the answers from some cases, then you can study the form of the answers and trace backward.

Finally, when you figure out the solutions, don't just stop there. You should try to generalize the problem, see how the given facts are necessary for solving the problem. This may help you to solve related problems later on. Always try to write out your solution in a clear and concise manner. Along the way, you will polish the argument and see the steps of the solutions more clearly. This helps you to develop strategies for dealing with other problems.

The solutions presented in the book are by no means the only ways to do the problems. If you have a nice elegant solution to a problem and would like to share with others (in future editions of this book), please send it to us by email at *makyli@ust.hk* . Also if you have something you cannot understand, please feel free to contact us by email. We hope this book will increase your interest in math.

Finally, we will offer one last advice. <u>Don't start with problem 1.</u> Read the statements of the problems and start with the ones that interest you the most. We recommend inspecting the list of miscellaneous problems first.

Have a fun time.

Table of Contents

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Problems

Algebra Problems

Polynomials

1. (Crux Mathematicorum, Problem 7) Find (without calculus) a fifth degree polynomial $p(x)$ such that $p(x)+1$ is divisible by $(x-1)^3$ and $p(x)-1$ is divisible by $(x+1)^3$.

2. A polynomial $P(x)$ of the n-th degree satisfies $P(k)=2^k$ for $k=0,1,2,\ldots,n$. Find the value of $P(n+1)$.

3. (1999 Putnam Exam) Let $P(x)$ be a polynomial with real coefficients such that $P(x)\geq 0$ for every real x. Prove that

$$P(x)=f_1(x)^2+f_2(x)^2+\cdots+f_n(x)^2$$

for some polynomials $f_1(x),f_2(x),\ldots,f_n(x)$ with real coefficients.

4. (1995 Russian Math Olympiad) Is it possible to find three quadratic polynomials $f(x),g(x),h(x)$ such that the equation $f(g(h(x)))=0$ has the eight roots $1,2,3,4,5,6,7,8$?

5. (1968 Putnam Exam) Determine all polynomials whose coefficients are all ± 1 that have only real roots.

6. (1990 Putnam Exam) Is there an infinite sequence $a_0,a_1,a_2,\ldots$ of nonzero real numbers such that for $n=1,2,3,\ldots,$ the polynomial $P_n(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n$ has exactly n distinct real roots?

7. (1991 Austrian-Polish Math Competition) Let $P(x)$ be a polynomial with real coefficients such that $P(x)\geq 0$ for $0\leq x\leq 1$. Show that there are polynomials $A(x),B(x),C(x)$ with real coefficients such that

   (a) $A(x)\geq 0, B(x)\geq 0, C(x)\geq 0$ for all real x and
   (b) $P(x)=A(x)+xB(x)+(1-x)C(x)$ for all real x.

   (For example, if $P(x)=x(1-x)$, then $P(x)=0+x(1-x)^2+(1-x)x^2$.)

8. (1993 IMO) Let $f(x)=x^n+5x^{n-1}+3$, where $n>1$ is an integer. Prove that $f(x)$ cannot be expressed as a product of two polynomials, each has integer coefficients and degree at least 1.

9. Prove that if the integer a is not divisible by 5, then $f(x)=x^5-x+a$ cannot be factored as the product of two nonconstant polynomials with integer coefficients.

10. (1991 Soviet Math Olympiad) Given $2n$ distinct numbers $a_1,a_2,\ldots,a_n$, $b_1,b_2,\ldots,b_n$, an $n\times n$ table is filled as follows: into the cell in the i-th row and j-th column is written the number a_i+b_j. Prove that if the product of each column is the same, then also the product of each row is the same.

11. Let $a_1,a_2,\ldots,a_n$ and $b_1,b_2,\ldots,b_n$ be two distinct collections of n positive integers, where each collection may contain repetitions. If the two collections of integers $a_i+a_j (1\leq i<j\leq n)$ and b_i+b_j $(1\leq i<j\leq n)$ are the same, then show that n is a power of 2.

Recurrence Relations

12. The sequence x_n is defined by

$$x_1=2,\quad x_{n+1}=\frac{2+x_n}{1-2x_n},\quad n=1,2,3,\ldots.$$

Prove that $x_n\neq\frac{1}{2}$ or 0 for all n and the terms of the sequence are all distinct.

13. (1988 Nanchang City Math Competition) Define $a_1=1,\ a_2=7$ and $a_{n+2}=\dfrac{a_{n+1}^2-1}{a_n}$ for positive integer n. Prove that $9a_na_{n+1}+1$ is a perfect square for every positive integer n.

14. (Proposed by Bulgaria for 1988 IMO) Define $a_0=0,a_1=1$ and $a_n=2a_{n-1}+a_{n-2}$ for $n>1$. Show that for positive integer k, a_n is divisible by 2^k if and only if n is divisible by 2^k.

15. (American Mathematical Monthly, Problem E2998) Let x and y be distinct complex numbers such that $\dfrac{x^n - y^n}{x - y}$ is an integer for some four consecutive positive integers n. Show that $\dfrac{x^n - y^n}{x - y}$ is an integer for all positive integers n.

Inequalities

16. For real numbers $a_1, a_2, a_3, \ldots$, if $a_{n-1} + a_{n+1} \geq 2a_n$ for $n = 2, 3, \ldots$, then prove that

$$A_{n-1} + A_{n+1} \geq 2A_n \quad \text{for} \quad n = 2, 3, \ldots,$$

where A_n is the average of $a_1, a_2, \ldots, a_n$.

17. Let $a, b, c > 0$ and $abc \leq 1$. Prove that

$$\frac{a}{c} + \frac{b}{a} + \frac{c}{b} \geq a + b + c.$$

18. (1982 Moscow Math Olympiad) Use the identity $1^3 + 2^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$ to prove that for distinct positive integers $a_1, a_2, \ldots, a_n$,

$$(a_1^7 + a_2^7 + \cdots + a_n^7) + (a_1^5 + a_2^5 + \cdots + a_n^5) \geq 2(a_1^3 + a_2^3 + \cdots + a_n^3)^2.$$

Can equality occur?

19. (1997 IMO shortlisted problem) Let $a_1 \geq \cdots \geq a_n \geq a_{n+1} = 0$ be a sequence of real numbers. Prove that

$$\sqrt{\sum_{k=1}^{n} a_k} \leq \sum_{k=1}^{n} \sqrt{k}(\sqrt{a_k} - \sqrt{a_{k+1}}).$$

20. (1994 Chinese Team Selection Test) For $0 \leq a \leq b \leq c \leq d \leq e$ and $a + b + c + d + e = 1$, show that

$$ad + dc + cb + be + ea \leq \frac{1}{5}.$$

21. (1985 Wuhu City Math Competition) Let x, y, z be real numbers such that $x + y + z = 0$. Show that

$$6(x^3 + y^3 + z^3)^2 \leq (x^2 + y^2 + z^2)^3.$$

22. (1999 IMO) Let n be a fixed integer, with $n \geq 2$.

    (a) Determine the least constant C such that the inequality

$$\sum_{1 \leq i < j \leq n} x_i x_j (x_i^2 + x_j^2) \leq C \Big(\sum_{1 \leq i \leq n} x_i \Big)^4$$

    holds for all nonnegative real numbers $x_1, x_2, \ldots, x_n$.

    (b) For this constant C, determine when equality holds.

23. (1995 Bulgarian Math Competition) Let $n \geq 2$ and $0 \leq x_i \leq 1$ for $i = 1, 2, \ldots, n$. Prove that

$$(x_1 + x_2 + \cdots + x_n) - (x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1) \leq \left[\frac{n}{2}\right],$$

where $[x]$ is the greatest integer less than or equal to x.

24. For every triplet of functions $f, g, h : [0, 1] \to R$, prove that there are numbers x, y, z in $[0, 1]$ such that

$$|f(x) + g(y) + h(z) - xyz| \geq \frac{1}{3}.$$

25. (Proposed by Great Britain for 1987 IMO) If x, y, z are real numbers such that $x^2 + y^2 + z^2 = 2$, then show that $x + y + z \leq xyz + 2$.

26. (Proposed by USA for 1993 IMO) Prove that for positive real numbers a, b, c, d,

$$\frac{a}{b+2c+3d}+\frac{b}{c+2d+3a}+\frac{c}{d+2a+3b}+\frac{d}{a+2b+3c}\geq\frac{2}{3}.$$

27. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be $2n$ positive real numbers such that

 (a) $a_1 \geq a_2 \geq \cdots \geq a_n$ and
 (b) $b_1 b_2 \cdots b_k \geq a_1 a_2 \cdots a_k$ for all $k, 1 \leq k \leq n$.

 Show that $b_1 + b_2 + \cdots + b_n \geq a_1 + a_2 + \cdots + a_n$.

28. (Proposed by Greece for 1987 IMO) Let $a, b, c > 0$ and m be a positive integer, prove that

$$\frac{a^m}{b+c}+\frac{b^m}{c+a}+\frac{c^m}{a+b}\geq\frac{3}{2}\left(\frac{a+b+c}{3}\right)^{m-1}.$$

29. Let $a_1, a_2, \ldots, a_n$ be distinct positive integers, show that

$$\frac{a_1}{2}+\frac{a_2}{8}+\cdots+\frac{a_n}{n2^n}\geq 1-\frac{1}{2^n}.$$

30. (1982 West German Math Olympiad) If $a_1, a_2, \ldots, a_n > 0$ and $a = a_1 + a_2 + \cdots + a_n$, then show that

$$\sum_{i=1}^{n}\frac{a_i}{2a-a_i}\geq\frac{n}{2n-1}.$$

31. Prove that if $a, b, c > 0$, then $\dfrac{a^3}{b+c}+\dfrac{b^3}{c+a}+\dfrac{c^3}{a+b}\geq\dfrac{a^2+b^2+c^2}{2}$.

32. Let $a, b, c, d > 0$ and

$$\frac{1}{1+a^4}+\frac{1}{1+b^4}+\frac{1}{1+c^4}+\frac{1}{1+d^4}=1.$$

 Prove that $abcd \geq 3$.

33. (Due to Paul Erdös) Each of the positive integers $a_1, \ldots, a_n$ is less than 1951. The least common multiple of any two of these is greater than 1951. Show that

$$\frac{1}{a_1}+\cdots+\frac{1}{a_n}<1+\frac{n}{1951}.$$

34. A sequence (P_n) of polynomials is defined recursively as follows:

$$P_0(x)=0 \quad \text{and for } n\geq 0, \quad P_{n+1}(x)=P_n(x)+\frac{x-P_n(x)^2}{2}.$$

 Prove that

$$0\leq\sqrt{x}-P_n(x)\leq\frac{2}{n+1}$$

 for every nonnegative integer n and all x in $[0, 1]$.

35. (1996 IMO shortlisted problem) Let $P(x)$ be the real polynomial function, $P(x) = ax^3 + bx^2 + cx + d$. Prove that if $|P(x)| \leq 1$ for all x such that $|x| \leq 1$, then

$$|a|+|b|+|c|+|d|\leq 7.$$

36. (American Mathematical Monthly, Problem 4426) Let $P(z) = az^3 + bz^2 + cz + d$, where a, b, c, d are complex numbers with $|a| = |b| = |c| = |d| = 1$. Show that $|P(z)| \geq \sqrt{6}$ for at least one complex number z satisfying $|z| = 1$.

37. (1997 Hungarian-Israeli Math Competition) Find all real numbers α with the following property: for any positive integer n, there exists an integer m such that $\left|\alpha-\dfrac{m}{n}\right|<\dfrac{1}{3n}$?

38. (1979 British Math Olympiad) If n is a positive integer, denote by $p(n)$ the number of ways of expressing n as the sum of one or more positive integers. Thus $p(4) = 5$, as there are five different ways of expressing 4 in terms of positive integers; namely

$$1+1+1+1,\;\; 1+1+2,\;\; 1+3,\;\; 2+2,\;\; \text{and}\;\; 4.$$

Prove that $p(n+1) - 2p(n) + p(n-1) \geq 0$ for each $n > 1$.

Functional Equations

39. Find all polynomials f satisfying $f(x^2) + f(x)f(x+1) = 0$.

40. (1997 Greek Math Olympiad) Let $f : (0, \infty) \to R$ be a function such that
 (a) f is strictly increasing,
 (b) $f(x) > -\frac{1}{x}$ for all $x > 0$ and
 (c) $f(x)f(f(x) + \frac{1}{x}) = 1$ for all $x > 0$.

 Find $f(1)$.

41. (1979 Eötvös-Kürschák Math Competition) The function f is defined for all real numbers and satisfies $f(x) \leq x$ and $f(x+y) \leq f(x) + f(y)$ for all real x, y. Prove that $f(x) = x$ for every real number x.

42. (Proposed by Ireland for 1989 IMO) Suppose $f : R \to R$ satisfies $f(1) = 1, f(a+b) = f(a) + f(b)$ for all $a, b \in R$ and $f(x)f(\frac{1}{x}) = 1$ for $x \neq 0$. Show that $f(x) = x$ for all x.

43. (1992 Polish Math Olympiad) Let Q^+ be the positive rational numbers. Determine all functions $f : Q^+ \to Q^+$ such that $f(x+1) = f(x) + 1$ and $f(x^3) = f(x)^3$ for every $x \in Q^+$.

44. (1996 IMO shortlisted problem) Let R denote the real numbers and $f : R \to [-1, 1]$ satisfy

$$f\left(x + \frac{13}{42}\right) + f(x) = f\left(x + \frac{1}{6}\right) + f\left(x + \frac{1}{7}\right)$$

 for every $x \in R$. Show that f is a periodic function, i.e. there is a nonzero real number T such that $f(x+T) = f(x)$ for every $x \in R$.

45. Let N denote the positive integers. Suppose $s : N \to N$ is an increasing function such that $s(s(n)) = 3n$ for all $n \in N$. Find all possible values of $s(1997)$.

46. Let N be the positive integers. Is there a function $f : N \to N$ such that $f^{(1996)}(n) = 2n$ for all $n \in N$, where $f^{(1)}(x) = f(x)$ and $f^{(k+1)}(x) = f(f^{(k)}(x))$?

47. (American Mathematical Monthly, Problem E984) Let R denote the real numbers. Find all functions $f : R \to R$ such that $f(f(x)) = x^2 - 2$ or show no such function can exist.

48. Let R be the real numbers. Find all functions $f : R \to R$ such that for all real numbers x and y,

$$f\big(xf(y) + x\big) = xy + f(x).$$

49. (1999 IMO) Determine all functions $f : R \to R$ such that

$$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$$

 for all x, y in R.

50. (1995 Byelorussian Math Olympiad) Let R be the real numbers. Find all functions $f : R \to R$ such that

$$f(f(x+y)) = f(x+y) + f(x)f(y) - xy$$

 for all $x, y \in R$.

51. (1993 Czechoslovak Math Olympiad) Let Z be the integers. Find all functions $f : Z \to Z$ such that

$$f(-1) = f(1) \text{ and } f(x) + f(y) = f(x + 2xy) + f(y - 2xy)$$

 for all integers x, y.

52. (1995 South Korean Math Olympiad) Let A be the set of non-negative integers. Find all functions $f : A \to A$ satisfying the following two conditions:

 (a) For any $m, n \in A$, $2f(m^2 + n^2) = (f(m))^2 + (f(n))^2$.

(b) For any $m, n \in A$ with $m \geq n$, $f(m^2) \geq f(n^2)$.

53. (American Mathematical Monthly, Problem E2176) Let Q denote the rational numbers. Find all functions $f : Q \to Q$ such that

$$f(2) = 2 \qquad \text{and} \qquad f\left(\frac{x+y}{x-y}\right) = \frac{f(x)+f(y)}{f(x)-f(y)} \quad \text{for } x \neq y.$$

54. (Mathematics Magazine, Problem 1552) Find all functions $f : R \to R$ such that

$$f(x + yf(x)) = f(x) + xf(y) \quad \text{for all } x, y \text{ in } R.$$

Maximum/Minimum

55. (1985 Austrian Math Olympiad) For positive integers n, define

$$f(n) = 1^n + 2^{n-1} + 3^{n-2} + \cdots + (n-2)^3 + (n-1)^2 + n.$$

What is the minimum of $f(n+1)/f(n)$?

56. (1996 Putnam Exam) Given that $\{x_1, x_2, \ldots, x_n\} = \{1, 2, \ldots, n\}$, find the largest possible value of $x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1$ in terms of n (with $n \geq 2$).

Geometry Problems

57. (1995 British Math Olympiad) Triangle ABC has a right angle at C. The internal bisectors of angles BAC and ABC meet BC and CA at P and Q respectively. The points M and N are the feet of the perpendiculars from P and Q to AB. Find angle MCN.

58. (1988 Leningrad Math Olympiad) Squares $ABDE$ and $BCFG$ are drawn outside of triangle ABC. Prove that triangle ABC is isosceles if DG is parallel to AC.

59. AB is a chord of a circle, which is not a diameter. Chords A_1B_1 and A_2B_2 intersect at the midpoint P of AB. Let the tangents to the circle at A_1 and B_1 intersect at C_1. Similarly, let the tangents to the circle at A_2 and B_2 intersect at C_2. Prove that C_1C_2 is parallel to AB.

60. (1991 Hunan Province Math Competition) Two circles with centers O_1 and O_2 intersect at points A and B. A line through A intersects the circles with centers O_1 and O_2 at points Y, Z, respectively. Let the tangents at Y and Z intersect at X and lines YO_1 and ZO_2 intersect at P. Let the circumcircle of $\triangle O_1O_2B$ have center at O and intersect line XB at B and Q. Prove that PQ is a diameter of the circumcircle of $\triangle O_1O_2B$.

61. (1981 Beijing City Math Competition) In a disk with center O, there are four points such that the distance between every pair of them is greater than the radius of the disk. Prove that there is a pair of perpendicular diameters such that exactly one of the four points lies inside each of the four quarter disks formed by the diameters.

62. The lengths of the sides of a quadrilateral are positive integers. The length of each side divides the sum of the other three lengths. Prove that two of the sides have the same length.

63. (1988 Sichuan Province Math Competition) Suppose the lengths of the three sides of $\triangle ABC$ are integers and the inradius of the triangle is 1. Prove that the triangle is a right triangle.

Geometric Equations

64. (1985 IMO) A circle has center on the side AB of the cyclic quadrilateral $ABCD$. The other three sides are tangent to the circle. Prove that $AD + BC = AB$.

65. (1995 Russian Math Olympiad) Circles S_1 and S_2 with centers O_1, O_2 respectively intersect each other at points A and B. Ray O_1B intersects S_2 at point F and ray O_2B intersects S_1 at point E. The line parallel to EF and passing through B intersects S_1 and S_2 at points M and N, respectively. Prove that (B is the incenter of $\triangle EAF$ and) $MN = AE + AF$.

66. Point C lies on the minor arc AB of the circle centered at O. Suppose the tangent line at C cuts the perpendiculars to chord AB through A at E and through B at F. Let D be the intersection of chord AB and radius OC. Prove that $CE \cdot CF = AD \cdot BD$ and $CD^2 = AE \cdot BF$.

67. Quadrilaterals $ABCP$ and $A'B'C'P'$ are inscribed in two concentric circles. If triangles ABC and $A'B'C'$ are equilateral, prove that

$$P'A^2 + P'B^2 + P'C^2 = PA'^2 + PB'^2 + PC'^2.$$

68. Let the inscribed circle of triangle ABC touchs side BC at D, side CA at E and side AB at F. Let G be the foot of perpendicular from D to EF. Show that $\dfrac{FG}{EG} = \dfrac{BF}{CE}$.

69. (1998 IMO shortlisted problem) Let $ABCDEF$ be a convex hexagon such that

$$\angle B + \angle D + \angle F = 360^\circ \quad \text{and} \quad \frac{AB}{BC} \cdot \frac{CD}{DE} \cdot \frac{EF}{FA} = 1.$$

Prove that

$$\frac{BC}{CA} \cdot \frac{AE}{EF} \cdot \frac{FD}{DB} = 1.$$

Similar Triangles

70. (1984 British Math Olympiad) P, Q, and R are arbitrary points on the sides BC, CA, and AB respectively of triangle ABC. Prove that the three circumcentres of triangles AQR, BRP, and CPQ form a triangle similar to triangle ABC.

71. Hexagon $ABCDEF$ is inscribed in a circle so that $AB = CD = EF$. Let P, Q, R be the points of intersection of AC and BD, CE and DF, EA and FB respectively. Prove that triangles PQR and BDF are similar.

72. (1998 IMO shortlisted problem) Let $ABCD$ be a cyclic quadrilateral. Let E and F be variable points on the sides AB and CD, respectively, such that $AE : EB = CF : FD$. Let P be the point on the segment EF such that $PE : PF = AB : CD$. Prove that the ratio between the areas of triangles APD and BPC does not depend on the choice of E and F.

Tangent Lines

73. Two circles intersect at points A and B. An arbitrary line through B intersects the first circle again at C and the second circle again at D. The tangents to the first circle at C and to the second circle at D intersect at M. The parallel to CM which passes through the point of intersection of AM and CD intersects AC at K. Prove that BK is tangent to the second circle.

74. (1999 IMO) Two circles Γ_1 and Γ_2 are contained inside the circle Γ, and are tangent to Γ at the distinct points M and N, respectively. Γ_1 passes through the center of Γ_2. The line passing through the two points of intersection of Γ_1 and Γ_2 meets Γ at A and B, respectively. The lines MA and MB meets Γ_1 at C and D, respectively. Prove that CD is tangent to Γ_2.

75. (Proposed by India for 1992 IMO) Circles G_1 and G_2 touch each other externally at a point W and are inscribed in a circle G. A, B, C are

points on G such that A, G_1 and G_2 are on the same side of chord BC, which is also tangent to G_1 and G_2. Suppose AW is also tangent to G_1 and G_2. Prove that W is the incenter of triangle ABC.

Locus

76. Perpendiculars from a point P on the circumcircle of $\triangle ABC$ are drawn to lines AB, BC with feet at D, E, respectively. Find the locus of the circumcenter of $\triangle PDE$ as P moves around the circle.

77. Suppose A is a point inside a given circle and is different from the center. Consider all chords (excluding the diameter) passing through A. What is the locus of the intersection of the tangent lines at the endpoints of these chords?

78. Given $\triangle ABC$. Let line EF bisects $\angle BAC$ and $AE \cdot AF = AB \cdot AC$. Find the locus of the intersection P of lines BE and CF.

79. (1996 Putnam Exam) Let C_1 and C_2 be circles whose centers are 10 units apart, and whose radii are 1 and 3. Find the locus of all points M for which there exists points X on C_1 and Y on C_2 such that M is the midpoint of the line segment XY.

Collinear or Concyclic Points

80. (1982 IMO) Diagonals AC and CE of the regular hexagon $ABCDEF$ are divided by the inner points M and N, respectively, so that

$$\frac{AM}{AC} = \frac{CN}{CE} = r.$$

Determine r if B, M and N are collinear.

81. (1965 Putnam Exam) If A, B, C, D are four distinct points such that every circle through A and B intersects or coincides with every circle through C and D, prove that the four points are either collinear or concyclic.

82. (1957 Putnam Exam) Given an infinite number of points in a plane, prove that if all the distances between every pair are integers, then the points are collinear.

83. (1995 IMO shortlisted problem) The incircle of triangle ABC touches BC, CA and AB at D, E and F respectively. X is a point inside triangle ABC such that the incircle of triangle XBC touches BC at D also, and touches CX and XB at Y and Z respectively. Prove that $EFZY$ is a cyclic quadrilateral.

84. (1998 IMO) In the convex quadrilateral $ABCD$, the diagonals AC and BD are perpendicular and the opposite sides AB and DC are not parallel. Suppose the point P, where the perpendicular bisectors of AB and DC meet, is inside $ABCD$. Prove that $ABCD$ is a cyclic quadrilateral if and only if the triangles ABP and CDP have equal areas.

85. (1970 Putnam Exam) Show that if a convex quadrilateral with side-lengths a, b, c, d and area $\sqrt{abcd}$ has an inscribed circle, then it is a cyclic quadrilateral.

Concurrent Lines

86. In $\triangle ABC$, suppose $AB > AC$. Let P and Q be the feet of the perpendiculars from B and C to the angle bisector of $\angle BAC$, respectively. Let D be on line BC such that $DA \perp AP$. Prove that lines BQ, PC and AD are concurrent.

87. (1990 Chinese National Math Competition) Diagonals AC and BD of a cyclic quadrilateral $ABCD$ meets at P. Let the circumcenters of $ABCD, ABP, BCP, CDP$ and DAP be O, O_1, O_2, O_3 and O_4, respectively. Prove that OP, O_1O_3, O_2O_4 are concurrent.

88. (1995 IMO) Let A, B, C and D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at the points X and Y. The line XY meets BC at the point Z. Let P be a point on the line XY different from Z. The line CP intersects the circle with

diameter AC at the points C and M, and the line BP intersects the circle with diameter BD at the points B and N. Prove that the lines AM, DN and XY are concurrent.

89. AD, BE, CF are the altitudes of $\triangle ABC$. If P, Q, R are the midpoints of DE, EF, FD, respectively, then show that the perpendicular from P, Q, R to AB, BC, CA, respectively, are concurrent.

90. (1988 Chinese Math Olympiad Training Test) $ABCDEF$ is a hexagon inscribed in a circle. Show that the diagonals AD, BE, CF are concurrent if and only if $AB \cdot CD \cdot EF = BC \cdot DE \cdot FA$.

91. A circle intersects a triangle ABC at six points $A_1, A_2, B_1, B_2, C_1, C_2$, where the order of appearance along the triangle is $A, C_1, C_2, B, A_1, A_2, C, B_1, B_2, A$. Suppose B_1C_1, B_2C_2 meets at X, C_1A_1, C_2A_2 meets at Y and A_1B_1, A_2B_2 meets at Z. Show that AX, BY, CZ are concurrent.

92. (1995 IMO shortlisted problem) A circle passing through vertices B and C of triangle ABC intersects sides AB and AC at C' and B', respectively. Prove that BB', CC' and HH' are concurrent, where H and H' are the orthocenters of triangles ABC and $AB'C'$, respectively.

<u>*Perpendicular Lines*</u>

93. (1998 APMO) Let ABC be a triangle and D the foot of the altitude from A. Let E and F be on a line passing through D such that AE is perpendicular to BE, AF is perpendicular to CF, and E and F are different from D. Let M and N be the midpoints of the line segments BC and EF, respectively. Prove that AN is perpendicular to NM.

94. (2000 APMO) Let ABC be a triangle. Let M and N be the points in which the median and the angle bisector, respectively, at A meet the side BC. Let Q and P be the points in which the perpendicular at N to NA meets MA and BA, respectively, and O the point in which the perpendicular at P to BA meets AN produced. Prove that QO is perpendicular to BC.

95. Let BB' and CC' be altitudes of triangle ABC. Assume that $AB \neq AC$. Let M be the midpoint of BC, H the orthocenter of ABC and D the intersection of $B'C'$ and BC. Prove that $DH \perp AM$.

96. (1996 Chinese Team Selection Test) The semicircle with side BC of $\triangle ABC$ as diameter intersects sides AB, AC at points D, E, respectively. Let F, G be the feet of the perpendiculars from D, E to side BC respectively. Let M be the intersection of DG and EF. Prove that $AM \perp BC$.

97. (1985 IMO) A circle with center O passes through the vertices A and C of triangle ABC and intersects the segments AB and AC again at distinct points K and N, respectively. The circumcircles of triangles ABC and KBN intersect at exactly two distinct points B and M. Prove that $OM \perp MB$.

98. (1997 Chinese Senoir High Math Competition) A circle with center O is internally tangent to two circles inside it at points S and T. Suppose the two circles inside intersect at M and N with N closer to ST. Show that $OM \perp MN$ if and only if S, N, T are collinear.

99. AD, BE, CF are the altitudes of $\triangle ABC$. Lines EF, FD, DE meet lines BC, CA, AB in points L, M, N, respectively. Show that L, M, N are collinear and the line through them is perpendicular to the line joining the orthocenter H and circumcenter O of $\triangle ABC$.

<u>*Geometric Inequalities, Maximum/Minimum*</u>

100. (1973 IMO) Let $P_1, P_2, \ldots, P_{2n+1}$ be distinct points on some half of the unit circle centered at the origin O. Show that

$$|\overrightarrow{OP_1} + \overrightarrow{OP_2} + \cdots + \overrightarrow{OP_{2n+1}}| \geq 1.$$

101. Let the angle bisectors of $\angle A, \angle B, \angle C$ of triangle ABC intersect its circumcircle at P, Q, R, respectively. Prove that

$$AP + BQ + CR > BC + CA + AB.$$

102. (1997 APMO) Let ABC be a triangle inscribed in a circle and let $l_a = m_a/M_a, l_b = m_b/M_b, l_c = m_c/M_c$, where m_a, m_b, m_c are the lengths of the angle bisectors (internal to the triangle) and M_a, M_b, M_c are the lengths of the angle bisectors extended until they meet the circle. Prove that

$$\frac{l_a}{\sin^2 A} + \frac{l_b}{\sin^2 B} + \frac{l_c}{\sin^2 C} \geq 3,$$

and that equality holds iff ABC is equilateral.

103. (Mathematics Magazine, Problem 1506) Let I and O be the incenter and circumcenter of $\triangle ABC$, respectively. Assume $\triangle ABC$ is not equilateral (so $I \neq O$). Prove that

$$\angle AIO \leq 90^\circ \quad \text{if and only if} \quad 2BC \leq AB + CA.$$

104. Squares $ABDE$ and $ACFG$ are drawn outside $\triangle ABC$. Let P, Q be points on EG such that BP and CQ are perpendicular to BC. Prove that $BP + CQ \geq BC + EG$. When does equality hold?

105. Point P is inside $\triangle ABC$. Determine points D on side AB and E on side AC such that $BD = CE$ and $PD + PE$ is minimum.

Solid or Space Geometry

106. (Proposed by Italy for 1967 IMO) Which regular polygons can be obtained (and how) by cutting a cube with a plane?

107. (1995 Israeli Math Olympiad) Four points are given in space, in general position (i.e., they are not coplanar and any three are not collinear). A plane π is called an *equalizing* plane if all four points have the same distance from π. Find the number of equalizing planes.

Number Theory Problems

Digits

108. (1956 Putnam Exam) Prove that every positive integer has a multiple whose decimal representation involves all ten digits.

109. Does there exist a positive integer a such that the sum of the digits (in base 10) of a is 1999 and the sum of the digits (in base 10) of a^2 is 1999^2?

110. (Proposed by USSR for 1991 IMO) Let a_n be the last nonzero digit in the decimal representation of the number $n!$. Does the sequence $a_1, a_2, \ldots, a_n, \ldots$ become periodic after a finite number of terms?

Modulo Arithmetic

111. (1956 Putnam Exam) Prove that the number of odd binomial coefficients in any row of the Pascal triangle is a power of 2.

112. Let $a_1, a_2, a_3, \ldots, a_{11}$ and $b_1, b_2, b_3, \ldots, b_{11}$ be two permutations of the natural numbers $1, 2, 3, \ldots, 11$. Show that if each of the numbers $a_1 b_1$, $a_2 b_2, a_3 b_3, \ldots, a_{11} b_{11}$ is divided by 11, then at least two of them will have the same remainder.

113. (1995 Czech-Slovak Match) Let $a_1, a_2, \ldots$ be a sequence satisfying $a_1 = 2, a_2 = 5$ and

$$a_{n+2} = (2 - n^2) a_{n+1} + (2 + n^2) a_n$$

for all $n \geq 1$. Do there exist indices p, q and r such that $a_p a_q = a_r$?

Prime Factorization

114. (American Mathematical Monthly, Problem E2684) Let A_n be the set of positive integers which are less than n and are relatively prime to n. For which $n > 1$, do the integers in A_n form an arithmetic progression?

115. (1971 IMO) Prove that the set of integers of the form $2^k - 3$ ($k = 2, 3, \dots$) contains an infinite subset in which every two members are relatively prime.

116. (1988 Chinese Math Olympiad Training Test) Determine the smallest value of the natural number $n > 3$ with the property that whenever the set $S_n = \{3, 4, \dots, n\}$ is partitioned into the union of two subsets, at least one of the subsets contains three numbers a, b and c (not necessarily distinct) such that $ab = c$.

Base n Representations

117. (1983 IMO) Can you choose 1983 pairwise distinct nonnegative integers less than 10^5 such that no three are in arithmetic progression?

118. (American Mathematical Monthly, Problem 2486) Let p be an odd prime number and r be a positive integer *not* divisible by p. For any positive integer k, show that there exists a positive integer m such that the rightmost k digits of m^r, when expressed in the base p, are all 1's.

119. (Proposed by Romania for 1985 IMO) Show that the sequence $\{a_n\}$ defined by $a_n = [n\sqrt{2}]$ for $n = 1, 2, 3, \dots$ (where the brackets denote the greatest integer function) contains an infinite number of integral powers of 2.

Representations

120. Find all (even) natural numbers n which can be written as a sum of two odd composite numbers.

121. Find all positive integers which cannot be written as the sum of two or more consecutive positive integers.

122. (Proposed by Australia for 1990 IMO) Observe that $9 = 4+5 = 2+3+4$. Is there an integer N which can be written as a sum of 1990 consecutive positive integers and which can be written as a sum of (more than one) consecutive integers in exactly 1990 ways?

123. Show that if $p > 3$ is prime, then p^n cannot be the sum of two positive cubes for any $n \geq 1$. What about $p = 2$ or 3?

124. (Due to Paul Erdös and M. Surányi) Prove that every integer k can be represented in infinitely many ways in the form $k = \pm 1^2 \pm 2^2 \pm \cdots \pm m^2$ for some positive integer m and some choice of signs + or −.

125. (1996 IMO shortlisted problem) A finite sequence of integers $a_0, a_1, \dots, a_n$ is called *quadratic* if for each $i \in \{1, 2, \dots, n\}$, $|a_i - a_{i-1}| = i^2$.

   (a) Prove that for any two integers b and c, there exists a natural number n and a quadratic sequence with $a_0 = b$ and $a_n = c$.

   (b) Find the least natural number n for which there exists a quadratic sequence with $a_0 = 0$ and $a_n = 1996$.

126. Prove that every integer greater than 17 can be represented as a sum of three integers > 1 which are pairwise relatively prime, and show that 17 does not have this property.

Chinese Remainder Theorem

127. (1988 Chinese Team Selection Test) Define $x_n = 3x_{n-1} + 2$ for all positive integers n. Prove that an integer value can be chosen for x_0 so that x_{100} is divisible by 1998.

128. (Proposed by North Korea for 1992 IMO) Does there exist a set M with the following properties:

   (a) The set M consists of 1992 natural numbers.

   (b) Every element in M and the sum of any number of elements in M have the form m^k, where m, k are positive integers and $k \geq 2$?

Divisibility

129. Find all positive integers a, b such that $b > 2$ and $2^a + 1$ is divisible by $2^b - 1$.

130. Show that there are infinitely many composite n such that $3^{n-1}-2^{n-1}$ is divisible by n.

131. Prove that there are infinitely many positive integers n such that 2^n+1 is divisible by n. Find all such n's that are prime numbers.

132. (1998 Romanian Math Olympiad) Find all positive integers (x, n) such that $x^n + 2^n + 1$ is a divisor of $x^{n+1} + 2^{n+1} + 1$.

133. (1995 Bulgarian Math Competition) Find all pairs of positive integers (x, y) for which $\dfrac{x^2+y^2}{x-y}$ is an integer and divides 1995.

134. (1995 Russian Math Olympiad) Is there a sequence of natural numbers in which every natural number occurs just once and moreover, for any $k = 1, 2, 3, \ldots$ the sum of the first k terms is divisible by k?

135. (1998 Putnam Exam) Let $A_1 = 0$ and $A_2 = 1$. For $n > 2$, the number A_n is defined by concatenating the decimal expansions of A_{n-1} and A_{n-2} from left to right. For example, $A_3 = A_2A_1 = 10$, $A_4 = A_3A_2 = 101$, $A_5 = A_4A_3 = 10110$, and so forth. Determine all n such that A_n is divisible by 11.

136. (1995 Bulgarian Math Competition) If $k > 1$, show that k does not divide $2^{k-1} + 1$. Use this to find all prime numbers p and q such that $2^p + 2^q$ is divisible by pq.

137. Show that for any positive integer n, there is a number whose decimal representation contains n digits, each of which is 1 or 2, and which is divisible by 2^n.

138. For a positive integer n, let $f(n)$ be the largest integer k such that 2^k divides n and $g(n)$ be the sum of the digits in the binary representation of n. Prove that for any positive integer n,

    (a) $f(n!) = n - g(n)$;
    (b) 4 divides $\dbinom{2n}{n} = \dfrac{(2n)!}{n!n!}$ if and only if n is not a power of 2.

139. (Proposed by Australia for 1992 IMO) Prove that for any positive integer m, there exist an infinite number of pairs of integers (x, y) such that

    (a) x and y are relatively prime;
    (b) y divides $x^2 + m$;
    (c) x divides $y^2 + m$.

140. Find all integers $n > 1$ such that $1^n + 2^n + \cdots + (n-1)^n$ is divisible by n.

141. (1972 Putnam Exam) Show that if n is an integer greater than 1, then n does not divide $2^n - 1$.

142. (Proposed by Romania for 1985 IMO) For $k \geq 2$, let $n_1, n_2, \ldots, n_k$ be positive integers such that

$$n_2 \mid (2^{n_1}-1),\ n_3 \mid (2^{n_2}-1),\ \ldots,\ n_k \mid (2^{n_{k-1}}-1),\ n_1 \mid (2^{n_k}-1).$$

Prove that $n_1 = n_2 = \cdots = n_k = 1$.

143. (1998 APMO) Determine the largest of all integer n with the property that n is divisible by all positive integers that are less than $\sqrt[3]{n}$.

144. (1997 Ukrainian Math Olympiad) Find the smallest integer n such that among any n integers (with possible repetitions), there exist 18 integers whose sum is divisible by 18.

Perfect Squares, Perfect Cubes

145. Let a, b, c be positive integers such that $\dfrac{1}{a} + \dfrac{1}{b} = \dfrac{1}{c}$. If the greatest common divisor of a, b, c is 1, then prove that $a + b$ must be a perfect square.

146. (1969 Eötvös-Kürschák Math Competition) Let n be a positive integer. Show that if $2 + 2\sqrt{28n^2+1}$ is an integer, then it is a square.

147. (1998 Putnam Exam) Prove that, for any integers a, b, c, there exists a positive integer n such that $\sqrt{n^3 + an^2 + bn + c}$ is not an integer.

148. (1995 IMO shortlisted problem) Let k be a positive integer. Prove that there are infinitely many perfect squares of the form $n2^k - 7$, where n is a positive integer.

149. Let a, b, c be integers such that $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = 3$. Prove that abc is the cube of an integer.

Diophantine Equations

150. Find all sets of positive integers x, y and z such that $x \le y \le z$ and $x^y + y^z = z^x$.

151. (Due to W. Sierpinski in 1955) Find all positive integral solutions of $3^x + 4^y = 5^z$.

152. (Due to Euler, also 1985 Moscow Math Olympiad) If $n \ge 3$, then prove that 2^n can be represented in the form $2^n = 7x^2 + y^2$ with x, y odd positive integers.

153. (1995 IMO shortlisted problem) Find all positive integers x and y such that $x + y^2 + z^3 = xyz$, where z is the greatest common divisor of x and y.

154. Find all positive integral solutions to the equation $xy + yz + zx = xyz + 2$.

155. Show that if the equation $x^2 + y^2 + 1 = xyz$ has positive integral solutions x, y, z, then $z = 3$.

156. (1995 Czech-Slovak Match) Find all pairs of nonnegative integers x and y which solve the equation $p^x - y^p = 1$, where p is a given odd prime.

157. Find all integer solutions of the system of equations

$$x + y + z = 3 \quad \text{and} \quad x^3 + y^3 + z^3 = 3.$$

Combinatorics Problems

Counting Methods

158. (1996 Italian Mathematical Olympiad) Given an alphabet with three letters a, b, c, find the number of words of n letters which contain an even number of a's.

159. Find the number of n-words from the alphabet $A = \{0, 1, 2\}$, if any two neighbors can differ by at most 1.

160. (1995 Romanian Math Olympiad) Let $A_1, A_2, \ldots, A_n$ be points on a circle. Find the number of possible colorings of these points with p colors, $p \ge 2$, such that any two neighboring points have distinct colors.

Pigeonhole Principle

161. (1987 Austrian-Polish Math Competition) Does the set $\{1, 2, \ldots, 3000\}$ contain a subset A consisting of 2000 numbers such that $x \in A$ implies $2x \notin A$?

162. (1989 Polish Math Olympiad) Suppose a triangle can be placed inside a square of unit area in such a way that the center of the square is not inside the triangle. Show that one side of the triangle has length less than 1.

163. The cells of a 7×7 square are colored with two colors. Prove that there exist at least 21 rectangles with vertices of the same color and with sides parallel to the sides of the square.

164. For $n > 1$, let $2n$ chess pieces be placed at the centers of $2n$ squares of an $n \times n$ chessboard. Show that there are four pieces among them that formed the vertices of a parallelogram. If $2n$ is replaced by $2n - 1$, is the statement still true in general?

165. The set $\{1, 2, \ldots, 49\}$ is partitioned into three subsets. Show that at least one of the subsets contains three different numbers a, b, c such that $a + b = c$.

Inclusion-Exclusion Principle

166. Let $m \geq n > 0$. Find the number of surjective functions from $B_m = \{1, 2, \ldots, m\}$ to $B_n = \{1, 2, \ldots, n\}$.

167. Let A be a set with 8 elements. Find the maximal number of 3-element subsets of A, such that the intersection of any two of them is not a 2-element set.

168. (a) (1999 China Hong Kong Math Olympiad) Students have taken a test paper in each of n ($n \geq 3$) subjects. It is known that for any subject exactly three students get the best score in the subject, and for any two subjects excatly one student gets the best score in every one of these two subjects. Determine the smallest n so that the above conditions imply that exactly one student gets the best score in every one of the n subjects.

(b) (1978 Austrian-Polish Math Competition) There are 1978 clubs. Each has 40 members. If every two clubs have exactly one common member, then prove that all 1978 clubs have a common member.

Combinatorial Designs

169. (1995 Byelorussian Math Olympiad) In the begining, 65 beetles are placed at different squares of a 9×9 square board. In each move, every beetle creeps to a horizontal or vertical adjacent square. If no beetle makes either two horizontal moves or two vertical moves in succession, show that after some moves, there will be at least two beetles in the same square.

170. (1995 Greek Math Olympiad) Lines $l_1, l_2, \ldots, l_k$ are on a plane such that no two are parallel and no three are concurrent. Show that we can label the C_2^k intersection points of these lines by the numbers $1, 2, \ldots, k-1$ so that in each of the lines $l_1, l_2, \ldots, l_k$ the numbers $1, 2, \ldots, k-1$ appear exactly once if and only if k is even.

171. (1996 Tournaments of the Towns) In a lottery game, a person must select six distinct numbers from $1, 2, 3, \ldots, 36$ to put on a ticket. The lottery commitee will then draw six distinct numbers randomly from $1, 2, 3, \ldots, 36$. Any ticket with numbers *not* containing any of these six numbers is a winning ticket. Show that there is a scheme of buying 9 tickets guaranteeing at least a winning ticket, but 8 tickets is not enough to guarantee a winning ticket in general.

172. (1995 Byelorussian Math Olympiad) By dividing each side of an equilateral triangle into 6 equal parts, the triangle can be divided into 36 smaller equilateral triangles. A beetle is placed on each vertex of these triangles at the same time. Then the beetles move along different edges with the same speed. When they get to a vertex, they must make a 60° or 120° turn. Prove that at some moment two beetles must meet at some vertex. Is the statement true if 6 is replaced by 5?

Covering, Convex Hull

173. (1991 Australian Math Olympiad) There are n points given on a plane such that the area of the triangle formed by every 3 of them is at most 1. Show that the n points lie on or inside some triangle of area at most 4.

174. (1969 Putnam Exam) Show that any continuous curve of unit length can be covered by a closed rectangles of area $1/4$.

175. (1998 Putnam Exam) Let $\mathcal{F}$ be a finite collection of open discs in the plane whose union covers a set E. Show that there is a pairwise disjoint subcollection $D_1, \ldots, D_n$ in $\mathcal{F}$ such that the union of $3D_1, \ldots, 3D_n$ covers E, where $3D$ is the disc with the same center as D but having three times the radius.

176. (1995 IMO) Determine all integers $n > 3$ for which there exist n points $A_1, A_2, \ldots, A_n$ in the plane, and real numbers $r_1, r_2, \ldots, r_n$ satisfying the following two conditions:

(a) no three of the points $A_1, A_2, \ldots, A_n$ lie on a line;
(b) for each triple i, j, k ($1 \leq i < j < k \leq n$) the triangle $A_iA_jA_k$ has area equal to $r_i + r_j + r_k$.

177. (1999 IMO) Determine all finite sets S of at least three points in the plane which satisfy the following condition: for any two distinct points A and B in S, the perpendicular bisector of the line segment AB is an axis of symmetry of S.

Miscellaneous Problems

178. (1995 Russian Math Olympiad) There are n seats at a merry-go-around. A boy takes n rides. Between each ride, he moves clockwise a certain number (less than n) of places to a new horse. Each time he moves a different number of places. Find all n for which the boy ends up riding each horse.

179. (1995 Israeli Math Olympiad) Two players play a game on an infinite board that consists of 1×1 squares. Player I chooses a square and marks it with an O. Then, player II chooses another square and marks it with X. They play until one of the players marks a row or a column of 5 consecutive squares, and this player wins the game. If no player can achieve this, the game is a tie. Show that player II can prevent player I from winning.

180. (1995 USAMO) A calculator is broken so that the only keys that still work are the sin, cos, tan, $\sin^{-1}, \cos^{-1}$, and $\tan^{-1}$ buttons. The display initially shows 0. Given any positive rational number q, show that pressing some finite sequence of buttons will yield q. Assume that the calculator does real number calculations with infinite precision. All functions are in terms of radians.

181. (1977 Eötvös-Kürschák Math Competition) Each of three schools is attended by exactly n students. Each student has exactly $n+1$ acquaintances in the other two schools. Prove that one can pick three students, one from each school, who know one another. It is assumed that acquaintance is mutual.

182. Is there a way to pack 250 $1 \times 1 \times 4$ bricks into a $10 \times 10 \times 10$ box?

183. Is it possible to write a positive integer into each square of the first quadrant such that each column and each row contains every positive integer exactly once?

184. There are n identical cars on a circular track. Among all of them, they have just enough gas for one car to complete a lap. Show that there is

a car which can complete a lap by collecting gas from the other cars on its way around the track in the clockwise direction.

185. (1996 Russian Math Olympiad) At the vertices of a cube are written eight pairwise distinct natural numbers, and on each of its edges is written the greatest common divisor of the numbers at the endpoints of the edge. Can the sum of the numbers written at the vertices be the same as the sum of the numbers written at the edges?

186. Can the positive integers be partitioned into infinitely many subsets such that each subset is obtained from any other subset by adding the same integer to each element of the other subset?

187. (1995 Russian Math Olympiad) Is it possible to fill in the cells of a 9×9 table with positive integers ranging from 1 to 81 in such a way that the sum of the elements of every 3×3 square is the same?

188. (1991 German Mathematical Olympiad) Show that for every positive integer $n \geq 2$, there exists a permutation $p_1, p_2, \ldots, p_n$ of $1, 2, \ldots, n$ such that p_{k+1} divides $p_1 + p_2 + \cdots + p_k$ for $k = 1, 2, \ldots, n-1$.

189. Each lattice point of the plane is labeled by a positive integer. Each of these numbers is the arithmetic mean of its four neighbors (above, below, left, right). Show that all the numbers are equal.

190. (1984 Tournament of the Towns) In a party, n boys and n girls are paired. It is observed that in each pair, the difference in height is less than 10 cm. Show that the difference in height of the k-th tallest boy and the k-th tallest girl is also less than 10 cm for $k = 1, 2, \ldots, n$.

191. (1991 Leningrad Math Olympiad) One may perform the following two operations on a positive integer:

(a) multiply it by any positive integer and
(b) delete zeros in its decimal representation.

Prove that for every positive integer X, one can perform a sequence of these operations that will transform X to a one-digit number.

192. (1996 IMO shortlisted problem) Four integers are marked on a circle. On each step we simultaneously replace each number by the difference between this number and next number on the circle in a given direction (that is, the numbers a, b, c, d are replaced by $a-b, b-c, c-d, d-a$). Is it possible after 1996 such steps to have numbers a, b, c, d such that the numbers $|bc-ad|, |ac-bd|, |ab-cd|$ are primes?

193. (1989 Nanchang City Math Competition) There are 1989 coins on a table. Some are placed with the head sides up and some the tail sides up. A group of 1989 persons will perform the following operations: the first person is allowed turn over any one coin, the second person is allowed turn over any two coins, ..., the k-th person is allowed turn over any k coins, ..., the 1989th person is allowed to turn over every coin. Prove that

(1) no matter which sides of the coins are up initially, the 1989 persons can come up with a procedure turning all coins the same sides up at the end of the operations,

(2) in the above procedure, whether the head or the tail sides turned up at the end will depend on the initial placement of the coins.

194. (Proposed by India for 1992 IMO) Show that there exists a convex polygon of 1992 sides satisfying the following conditions:

(a) its sides are $1, 2, 3, \ldots, 1992$ in some order;
(b) the polygon is circumscribable about a circle.

195. There are 13 white, 15 black, 17 red chips on a table. In one step, you may choose 2 chips of different colors and replace each one by a chip of the third color. Can all chips become the same color after some steps?

196. The following operations are permitted with the quadratic polynomial $ax^2 + bx + c$:

(a) switch a and c,
(b) replace x by $x + t$, where t is a real number.

By repeating these operations, can you transform $x^2 - x - 2$ into $x^2 - x - 1$?

197. Five numbers $1, 2, 3, 4, 5$ are written on a blackboard. A student may erase any two of the numbers a and b on the board and write the numbers $a+b$ and ab replacing them. If this operation is performed repeatedly, can the numbers $21, 27, 64, 180, 540$ ever appear on the board?

198. Nine 1×1 cells of a 10×10 square are infected. In one unit time, the cells with at least 2 infected neighbors (having a common side) become infected. Can the infection spread to the whole square? What if nine is replaced by ten?

199. (1997 Colombian Math Olympiad) We play the following game with an equilateral triangle of $n(n+1)/2$ dollar coins (with n coins on each side). Initially, all of the coins are turned heads up. On each turn, we may turn over three coins which are mutually adjacent; the goal is to make all of the coins turned tails up. For which values of n can this be done?

200. (1990 Chinese Team Selection Test) Every integer is colored with one of 100 colors and all 100 colors are used. For intervals $[a, b]$, $[c, d]$ having integers endpoints and same lengths, if a, c have the same color and b, d have the same color, then the intervals are colored the same way, which means $a+x$ and $c+x$ have the same color for $x = 0, 1, \ldots, b-a$. Prove that -1990 and 1990 have different colors.

Solutions

Solutions to Algebra Problems

Polynomials

1. (Crux Mathematicorum, Problem 7) Find (without calculus) a fifth degree polynomial $p(x)$ such that $p(x)+1$ is divisible by $(x-1)^3$ and $p(x)-1$ is divisible by $(x+1)^3$.

   Solution. (Due to Law Ka Ho, Ng Ka Wing, Tam Siu Lung) Note $(x-1)^3$ divides $p(x)+1$ and $p(-x)-1$, so $(x-1)^3$ divides their sum $p(x)+p(-x)$. Also $(x+1)^3$ divides $p(x)-1$ and $p(-x)+1$, so $(x+1)^3$ divides $p(x)+p(-x)$. Then $(x-1)^3(x+1)^3$ divides $p(x)+p(-x)$, which is of degree at most 5. So $p(x)+p(-x)=0$ for all x. Then the even degree term coefficients of $p(x)$ are zero. Now $p(x)+1=(x-1)^3(Ax^2+Bx-1)$. Comparing the degree 2 and 4 coefficients, we get $B-3A=0$ and $3+3B-A=0$, which implies $A=-3/8$ and $B=-9/8$. This yields $p(x)=-3x^5/8+5x^3/4-15x/8$.

2. A polynomial $P(x)$ of the n-th degree satisfies $P(k)=2^k$ for $k=0,1,2,\ldots,n$. Find the value of $P(n+1)$.

   Solution. For $0\le r\le n$, the polynomial $\binom{x}{r}=\dfrac{x(x-1)\cdots(x-r+1)}{r!}$ is of degree r. Consider the degree n polynomial

$$Q(x)=\binom{x}{0}+\binom{x}{1}+\cdots+\binom{x}{n}.$$

   By the binomial theorem, $Q(k)=(1+1)^k=2^k$ for $k=0,1,2,\ldots,n$. So $P(x)=Q(x)$ for all x. Then

$$P(n+1)=Q(n+1)=\binom{n+1}{0}+\binom{n+1}{1}+\cdots+\binom{n+1}{n}=2^{n+1}-1.$$

3. (1999 Putnam Exam) Let $P(x)$ be a polynomial with real coefficients such that $P(x)\ge 0$ for every real x. Prove that

$$P(x)=f_1(x)^2+f_2(x)^2+\cdots+f_n(x)^2$$

   for some polynomials $f_1(x),f_2(x),\ldots,f_n(x)$ with real coefficients.

   Solution. (Due to Cheung Pok Man) Write $P(x)=aR(x)C(x)$, where a is the coefficient of the highest degree term, $R(x)$ is the product of all real root factors $(x-r)$ repeated according to multiplicities and $C(x)$ is the product of all conjugate pairs of nonreal root factors $(x-z_k)(x-\overline{z_k})$. Then $a\ge 0$. Since $P(x)\ge 0$ for every real x and a factor $(x-r)^{2n+1}$ would change sign near a real root r of odd multiplicity, each real root of P must have even multiplicity. So $R(x)=f(x)^2$ for some polynomial $f(x)$ with real coefficients.

   Next pick one factor from each conjugate pair of nonreal factors and let the product of these factors $(x-z_k)$ be equal to $U(x)+iV(x)$, where $U(x),V(x)$ are polynomials with real coefficients. We have

$$\begin{aligned}P(x)&=af(x)^2(U(x)+iV(x))(U(x)-iV(x))\\&=(\sqrt{a}f(x)U(x))^2+(\sqrt{a}f(x)V(x))^2.\end{aligned}$$

4. (1995 Russian Math Olympiad) Is it possible to find three quadratic polynomials $f(x),g(x),h(x)$ such that the equation $f(g(h(x)))=0$ has the eight roots $1,2,3,4,5,6,7,8$?

   Solution. Suppose there are such f,g,h. Then $h(1),h(2),\ldots,h(8)$ will be the roots of the 4-th degree polynomial $f(g(x))$. Since $h(a)=h(b),a\ne b$ if and only if a,b are symmetric with respect to the axis of the parabola, it follows that $h(1)=h(8),h(2)=h(7),h(3)=h(6),h(4)=h(5)$ and the parabola $y=h(x)$ is symmetric with respect to $x=9/2$. Also, we have either $h(1)<h(2)<h(3)<h(4)$ or $h(1)>h(2)>h(3)>h(4)$.

   Now $g(h(1)),g(h(2)),g(h(3)),g(h(4))$ are the roots of the quadratic polynomial $f(x)$, so $g(h(1))=g(h(4))$ and $g(h(2))=g(h(3))$, which implies $h(1)+h(4)=h(2)+h(3)$. For $h(x)=Ax^2+Bx+C$, this would force $A=0$, a contradiction.

5. (1968 Putnam Exam) Determine all polynomials whose coefficients are all ±1 that have only real roots.

Solution. If a polynomial $a_0x^n+a_1x^{n-1}+\cdots+a_n$ is such a polynomial, then so is its negative. Hence we may assume $a_0=1$. Let $r_1,\ldots,r_n$ be the roots. Then $r_1^2+\cdots+r_n^2=a_1^2-2a_2$ and $r_1^2\cdots r_n^2=a_n^2$. If the roots are all real, then by the AM-GM inequality, we get $(a_1^2-2a_2)/n\ge a_n^{2/n}$. Since $a_1,a_2=\pm1$, we must have $a_2=-1$ and $n\le 3$. By simple checking, we get the list

$$\pm(x-1),\quad \pm(x+1),\quad \pm(x^2+x-1),\quad \pm(x^2-x-1),$$

$$\pm(x^3+x^2-x-1),\quad \pm(x^3-x^2-x+1).$$

6. (1990 Putnam Exam) Is there an infinite sequence $a_0,a_1,a_2,\ldots$ of nonzero real numbers such that for $n=1,2,3,\ldots,$ the polynomial $P_n(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n$ has exactly n distinct real roots?

Solution. Yes. Take $a_0=1,a_1=-1$ and proceed by induction. Suppose $a_0,\ldots,a_n$ have been chosen so that $P_n(x)$ has n distinct real roots *and $P_n(x)\to\infty$ or $-\infty$ as $x\to\infty$ depending upon whether n is even or odd.* Suppose the roots of $P_n(x)$ is in the interval $(-T,T)$. Let $a_{n+1}=(-1)^{n+1}/M$, where M is chosen to be very large so that T^{n+1}/M is very small. Then $P_{n+1}(x)=P_n(x)+(-x)^{n+1}/M$ is very close to $P_n(x)$ on $[-T,T]$ because $|P_{n+1}(x)-P_n(x)|\le T^{n+1}/M$ for every x on $[-T,T]$. So, $P_{n+1}(x)$ has a sign change very close to every root of $P_n(x)$ and has the same sign as $P_n(x)$ at T. Since $P_n(x)$ and $P_{n+1}(x)$ take on different sign when $x\to\infty$, there must be another sign change beyond T. So $P_{n+1}(x)$ must have $n+1$ real roots.

7. (1991 Austrian-Polish Math Competition) Let $P(x)$ be a polynomial with real coefficients such that $P(x)\ge 0$ for $0\le x\le 1$. Show that there are polynomials $A(x),B(x),C(x)$ with real coefficients such that

   (a) $A(x)\ge 0, B(x)\ge 0, C(x)\ge 0$ for all real x and
   (b) $P(x)=A(x)+xB(x)+(1-x)C(x)$ for all real x.

   (For example, if $P(x)=x(1-x)$, then $P(x)=0+x(1-x)^2+(1-x)x^2$.)

Solution. (Below all polynomials have real coefficients.) We induct on the degree of $P(x)$. If $P(x)$ is a constant polynomial c, then $c\ge 0$

and we can take $A(x)=c, B(x)=C(x)=0$. Next suppose the degree n case is true. For the case $P(x)$ is of degree $n+1$. If $P(x)\ge 0$ for all real x, then simply let $A(x)=P(x), B(x)=C(x)=0$. Otherwise, $P(x)$ has a root x_0 in $(-\infty,0]$ or $[1,+\infty)$.

<u>Case x_0 in $(-\infty,0]$.</u> Then $P(x)=(x-x_0)Q(x)$ and $Q(x)$ is of degree n with $Q(x)\ge 0$ for all x n $[0,1]$. So $Q(x)=A_0(x)+xB_0(x)+(1-x)C_0(x)$, where $A_0(x),B_0(x),C_0(x)\ge 0$ for all x in $[0,1]$. Using $x(1-x)=x(1-x)^2+(1-x)x^2$, we have

$$\begin{aligned}P(x)&=(x-x_0)(A_0(x)+xB_0(x)+(1-x)C_0(x))\\&=\underbrace{(-x_0A_0(x)+x^2B_0(x))}_{A(x)}+x\underbrace{(A_0(x)-x_0B_0(x)+(1-x)^2C_0(x))}_{B(x)}\\&\quad+(1-x)\underbrace{(-x_0C_0(x)+x^2B_0(x))}_{C(x)},\end{aligned}$$

where the polynomials $A(x),B(x),C(x)\ge 0$ for all x in $[0,1]$.

<u>Case x_0 in $[1,+\infty)$.</u> Consider $Q(x)=P(1-x)$. This reduces to the previous case. We have $Q(x)=A_1(x)+xB_1(x)+(1-x)C_1(x)$, where the polynomials $A_1(x),B_1(x),C_1(x)\ge 0$ for all x in $[0,1]$. Then

$$P(x)=Q(1-x)=\underbrace{A_1(1-x)}_{A(x)}+x\underbrace{C_1(1-x)}_{B(x)}+(1-x)\underbrace{B_1(1-x)}_{C(x)},$$

where the polynomials $A(x),B(x),C(x)\ge 0$ for all x in $[0,1]$.

8. (1993 IMO) Let $f(x)=x^n+5x^{n-1}+3$, where $n>1$ is an integer. Prove that $f(x)$ cannot be expressed as a product of two polynomials, each has integer coefficients and degree at least 1.

Solution. Suppose $f(x)=b(x)c(x)$ for nonconstant polynomials $b(x)$ and $c(x)$ with integer coefficients. Since $f(0)=3$, we may assume $b(0)=\pm1$ and $b(x)=x^r+\cdots\pm1$. Since $f(\pm1)\ne0$, $r>1$. Let $z_1,\ldots,z_r$ be the roots of $b(x)$. Then $|z_1\cdots z_r|=|b(0)|=1$ and

$$|b(-5)|=|(-5-z_1)\cdots(-5-z_r)|=\prod_{i=1}^{r}|z_i^{n-1}(z_i+5)|=3^r\ge 9.$$

However, $b(-5)$ also divides $f(-5) = 3$, a contradiction.

9. Prove that if the integer a is not divisible by 5, then $f(x) = x^5 - x + a$ cannot be factored as the product of two nonconstant polynomials with integer coefficients.

Solution. Suppose f can be factored, then $f(x) = (x - b)g(x)$ or $f(x) = (x^2 - bx + c)g(x)$. In the fomer case, $b^5 - b + a = f(b) = 0$. Now $b^5 \equiv b \pmod 5$ by Fermat's little theorem or simply checking the cases $b \equiv 0, 1, 2, 3, 4 \pmod 5$. Then 5 divides $b - b^5 = a$, a contradiction. In the latter case, divding $f(x) = x^5 - x + a$ by $x^2 - bx + c$, we get the remainder $(b^4 + 3b^2c + c^2 - 1)x + (b^3c + 2bc^2 + a)$. Since $x^2 - bx + c$ is a factor of $f(x)$, both coefficients equal 0. Finally,

$$0 = b(b^4 + 3b^2c + c^2 - 1) - 3(b^3c + 2bc^2 + a) = b^5 - b - 5bc^2 - 3a$$

implies $3a = b^5 - b - 5bc^2$ is divisible by 5. Then a would be divisible by 5, a contradiction.

10. (1991 Soviet Math Olympiad) Given $2n$ distinct numbers $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n$, an $n \times n$ table is filled as follows: into the cell in the i-th row and j-th column is written the number $a_i + b_j$. Prove that if the product of each column is the same, then also the product of each row is the same.

Solution. Let

$$P(x) = (x + a_1)(x + a_2)\cdots(x + a_n) - (x - b_1)(x - b_2)\cdots(x - b_n),$$

then $\deg P < n$. Now $P(b_j) = (b_j + a_1)(b_j + a_2)\cdots(b_j + a_n) = c$, some constant, for $j = 1, 2, \ldots, n$. So $P(x) - c$ has distinct roots $b_1, b_2, \cdots, b_n$. Therefore, $P(x) = c$ for all x and so

$$c = P(-a_i) = (-1)^{n+1}(a_i + b_1)(a_i + b_2)\cdots(a_i + b_n)$$

for $i = 1, 2, \ldots, n$. Then the product of each row is $(-1)^{n+1}c$.

11. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be two distinct collections of n positive integers, where each collection may contain repetitions. If the two collections of integers $a_i + a_j (1 \le i < j \le n)$ and $b_i + b_j$ $(1 \le i < j \le n)$ are the same, then show that n is a power of 2.

Solution. (Due to Law Siu Lung) Consider the functions $f(x) = \sum_{i+1}^{n} x^{a_i}$ and $g(x) = \sum_{i=1}^{n} x^{b_i}$. Since the a_i's and b_i's are distinct, f and g are distinct polynomials. Now

$$f(x)^2 = \sum_{i=1}^{n} x^{2a_i} + 2\sum_{1 \le i < j \le n} x^{a_i + a_j} = f(x^2) + 2\sum_{1 \le i < j \le n} x^{a_i + a_j}.$$

Since the $a_i + a_j$'s and the $b_i + b_j$'s are the same, so $f(x)^2 - f(x^2) = g(x)^2 - g(x^2)$. Since $f(1) - g(1) = n - n = 0$, so $f(x) - g(x) = (x - 1)^k Q(x)$ for some $k \ge 1$ and polynomial Q such that $Q(1) \ne 0$. Then

$$f(x) + g(x) = \frac{f(x^2) - g(x^2)}{f(x) - g(x)} = \frac{(x^2 - 1)^k Q(x^2)}{(x - 1)^k Q(x)} = (x + 1)^k \frac{Q(x^2)}{Q(x)}.$$

Setting $x = 1$, we have $n = 2^{k-1}$.

<u>*Recurrence Relations*</u>

12. The sequence x_n is defined by

$$x_1 = 2, \quad x_{n+1} = \frac{2 + x_n}{1 - 2x_n}, \quad n = 1, 2, 3, \ldots.$$

Prove that $x_n \ne \frac{1}{2}$ or 0 for all n and the terms of the sequence are all distinct.

Solution. (Due to Wong Chun Wai) The terms x_n's are clearly rational by induction. Write $x_n = p_n/q_n$, where p_n, q_n are relatively prime integers and $q_n > 0$. Then $q_1 = 1$ and $p_{n+1}/q_{n+1} = (2q_n + p_n)/(q_n - 2p_n)$. So q_{n+1} divides $q_n - 2p_n$, which implies every q_n is odd by induction. Hence, every $x_n \ne \frac{1}{2}$.

Next, to show every $x_n \ne 0$, let $\alpha = \arctan 2$, then $x_n = \tan n\alpha$ by induction. Suppose $x_n = 0$ and n is the least such index. If $n =$

$2m$ is even, then $0 = x_{2m} = \tan 2m\alpha = 2x_m/(1-x_m^2)$ would imply $x_m = 0$, a contradiction to n being least. If $n = 2m+1$ is odd, then $0 = x_{2m+1} = \tan(\alpha + 2m\alpha) = (2+x_{2m})/(1-2x_{2m})$ would imply $x_{2m} = -2$. Then $-2 = 2x_m/(1-x_m^2)$ would imply $x_m = (1\pm\sqrt{5})/2$ is irrational, a contradiction. Finally, if $x_m = x_n$ for some $m > n$, then $x_{m-n} = \tan(m\alpha - n\alpha) = (x_m - x_n)/(1+x_m x_n) = 0$, a contradiction. Therefore the terms are nonzero and distinct.

13. (1988 Nanchang City Math Competition) Define $a_1 = 1,\ a_2 = 7$ and $a_{n+2} = \dfrac{a_{n+1}^2 - 1}{a_n}$ for positive integer n. Prove that $9a_na_{n+1}+1$ is a perfect square for every positive integer n.

Solution. (Due to Chan Kin Hang) (Since a_{n+2} depends on a_{n+1} and a_n, it is plausible that the sequence satisfies a linear recurrence relation $a_{n+2} = ca_{n+1} + c'a_n$. If this is so, then using the first 4 terms, we find $c = 7, c' = -1$.) Define $b_1 = a_1, b_2 = a_2, b_{n+2} = 7b_{n+1} - b_n$ for $n \geq 1$. Then $b_3 = 48 = a_3$. Suppose $a_k = b_k$ for $k \leq n+1$, then

$$\begin{aligned} a_{n+2} &= \frac{b_{n+1}^2 - 1}{b_n} = \frac{(7b_n - b_{n-1})^2 - 1}{b_n} \\ &= 49b_n - 14b_{n-1} + b_{n-2} \\ &= 7b_{n+1} - b_n = b_{n+2}. \end{aligned}$$

So $a_k = b_k$ for all k.

Next, writing out the first few terms of $9a_na_{n+1}+1$ will suggest that $9a_na_{n+1}+1 = (a_n + a_{n+1})^2$. The case $n=1$ is true as $9\cdot 7+1 = (1+7)^2$. Suppose this is true for $n = k$. Using the recurrence relations and (*) $2a_{k+1}^2 - 2 = 2a_ka_{k+2} = 14a_ka_{k+1} - 2a_k^2$, we get the case $n = k+1$ as follow:

$$\begin{aligned} 9a_{k+1}a_{k+2}+1 &= 9a_{k+1}(7a_{k+1} - a_k) + 1 \\ &= 63a_{k+1}^2 - (a_k + a_{k+1})^2 + 2 \\ &= 62a_{k+1}^2 - 2a_ka_{k+1} - a_k^2 + 2 \\ &= 64a_{k+1}^2 - 16a_{k+1}a_k + a_k^2 \quad \text{by (*)} \\ &= (8a_{k+1} - a_k)^2 = (a_{k+1} + a_{k+2})^2. \end{aligned}$$

14. (Proposed by Bulgaria for 1988 IMO) Define $a_0 = 0, a_1 = 1$ and $a_n = 2a_{n-1} + a_{n-2}$ for $n > 1$. Show that for positive integer k, a_n is divisible by 2^k if and only if n is divisible by 2^k.

Solution. By the binomial theorem, if $(1+\sqrt{2})^n = A_n + B_n\sqrt{2}$, then $(1-\sqrt{2})^n = A_n - B_n\sqrt{2}$. Multiplying these 2 equations, we get $A_n^2 - 2B_n^2 = (-1)^n$. This implies A_n is always odd. Using characteristic equation method to solve the given recurrence relations on a_n, we find that $a_n = B_n$. Now write $n = 2^km$, where m is odd. We have $k = 0$ (i.e. n is odd) if and only if $2B_n^2 = A_n^2 + 1 \equiv 2 \pmod 4$, (i.e. B_n is odd). Next suppose case k is true. Since $(1+\sqrt{2})^{2n} = (A_n + B_n\sqrt{2})^2 = A_{2n} + B_{2n}\sqrt{2}$, so $B_{2n} = 2A_nB_n$. Then it follows case k implies case $k+1$.

15. (American Mathematical Monthly, Problem E2998) Let x and y be distinct complex numbers such that $\dfrac{x^n - y^n}{x-y}$ is an integer for some four consecutive positive integers n. Show that $\dfrac{x^n - y^n}{x-y}$ is an integer for all positive integers n.

Solution. For nonnegative integer n, let $t_n = (x^n - y^n)/(x-y)$. So $t_0 = 0, t_1 = 1$ and we have a recurrence relation

$$t_{n+2} + bt_{n+1} + ct_n = 0, \quad \text{where } b = -(x+y),\ c = xy.$$

Suppose t_n is an integer for $m, m+1, m+2, m+3$. Since $c^n = (xy)^n = t_{n+2}^2 - t_nt_{n+2}$ is an integer for $n = m, m+1$, so c is rational. Since c^{m+1} is integer, c must, in fact, be an integer. Next

$$b = \frac{t_mt_{m+3} - t_{m+1}t_{m+2}}{c^m}.$$

So b is rational. From the recurrence relation, it follows by induction that $t_n = f_{n-1}(b)$ for some polynomial f_{n-1} of degree $n-1$ with integer coefficients. Note the coefficient of x^{n-1} in f_{n-1} is 1, i.e. f_{n-1} is monic. Since b is a root of the integer coefficient polynomial $f_m(z) - t_{m+1} = 0$, b must be an integer. So the recurrence relation implies all t_n's are integers.

16. For real numbers $a_1, a_2, a_3, \ldots$, if $a_{n-1} + a_{n+1} \geq 2a_n$ for $n = 2, 3, \ldots$, then prove that

$$A_{n-1} + A_{n+1} \geq 2A_n \quad \text{for} \ \ n = 2, 3, \ldots,$$

where A_n is the average of $a_1, a_2, \ldots, a_n$.

Solution. Expressing in a_k, the required inequality is equivalent to

$$a_1 + \cdots + a_{n-1} - \frac{n^2+n-2}{2}a_n + \frac{n(n-1)}{2}a_{n+1} \geq 0.$$

(From the cases $n = 2, 3$, we easily see the pattern.) We have

$$\begin{aligned} & a_1 + \cdots + a_{n-1} - \frac{n^2+n-2}{2}a_n + \frac{n(n-1)}{2}a_{n+1} \\ =& \sum_{k=2}^{n} \frac{k(k-1)}{2}(a_{k-1} - 2a_k + a_{k+2}) \geq 0. \end{aligned}$$

17. Let $a, b, c > 0$ and $abc \leq 1$. Prove that

$$\frac{a}{c} + \frac{b}{a} + \frac{c}{b} \geq a + b + c.$$

Solution. (Due to Leung Wai Ying) Since $abc \leq 1$, we get $1/(bc) \geq a$, $1/(ac) \geq b$ and $1/(ab) \geq c$. By the AM-GM inequality,

$$\frac{2a}{c} + \frac{c}{b} = \frac{a}{c} + \frac{a}{c} + \frac{c}{b} \geq 3\sqrt[3]{\frac{a^2}{bc}} \geq 3a.$$

Similarly, $2b/a + a/c \geq 3b$ and $2c/b + b/a \geq 3c$. Adding these and dividing by 3, we get the desired inequality.

Alternatively, let $x = \sqrt[9]{a^4b/c^2}$, $y = \sqrt[9]{c^4a/b^2}$ and $z = \sqrt[9]{b^4c/a^2}$. We have $a = x^2y, b = z^2x, c = y^2z$ and $xyz = \sqrt[3]{abc} \leq 1$. Using this and the rearrangement inequality, we get

$$\begin{aligned} & \frac{a}{c} + \frac{b}{a} + \frac{c}{b} = \frac{x^2}{yz} + \frac{z^2}{xy} + \frac{y^2}{zx} \\ \geq & xyz\left(\frac{x^2}{yz} + \frac{z^2}{xy} + \frac{y^2}{zx}\right) = x^3 + y^3 + z^3 \\ \geq & x^2y + y^2z + z^2x = a + b + c. \end{aligned}$$

18. (1982 Moscow Math Olympiad) Use the identity $1^3 + 2^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$ to prove that for distinct positive integers $a_1, a_2, \ldots, a_n$,

$$(a_1^7 + a_2^7 + \cdots + a_n^7) + (a_1^5 + a_2^5 + \cdots + a_n^5) \geq 2(a_1^3 + a_2^3 + \cdots + a_n^3)^2.$$

Can equality occur?

Solution. For $n = 1$, $a_1^7 + a_1^5 - 2(a_1^3)^2 = a_1^5(a_1 - 1)^2 \geq 0$ and so case $n = 1$ is true. Suppose the case $n = k$ is true. For the case $n = k + 1$, without loss of generality, we may assume $a_1 < a_2 < \ldots < a_{k+1}$. Now

$$\begin{aligned} & 2(a_1^3 + \cdots + a_{k+1}^3)^2 - 2(a_1^3 + \cdots + a_k^3)^2 \\ =& 2a_{k+1}^6 + 4a_{k+1}^3(a_1^3 + \cdots + a_k^3) \\ \leq& 2a_{k+1}^6 + 4a_{k+1}^3(1^3 + 2^3 + \cdots + (a_{k+1} - 1)^3) \\ =& 2a_{k+1}^6 + 4a_{k+1}^3\frac{(a_{k+1} - 1)^2a_k^2}{4} = a_{k+1}^7 + a_{k+1}^5. \end{aligned}$$

So $(a_1^7 + \cdots + a_{k+1}^7) + (a_1^5 + \cdots + a_{k+1}^5) \geq 2(a_1^3 + \cdots + a_{k+1}^3)^2$ follows. Equality occurs if and only if $a_1, a_2, \ldots, a_n$ are $1, 2, \ldots, n$.

19. (1997 IMO shortlisted problem) Let $a_1 \geq \cdots \geq a_n \geq a_{n+1} = 0$ be a sequence of real numbers. Prove that

$$\sqrt{\sum_{k=1}^{n} a_k} \leq \sum_{k=1}^{n} \sqrt{k}(\sqrt{a_k} - \sqrt{a_{k+1}}).$$

Solution. (Due to Lee Tak Wing) Let $x_k = \sqrt{a_k} - \sqrt{a_{k+1}}$. Then $a_k = (x_k + x_{k+1} + \cdots + x_n)^2$. So,

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n}(x_k + x_{k+1} + \cdots + x_n)^2 = \sum_{k=1}^{n} kx_k^2 + 2\sum_{1\le i<j\le n} ix_ix_j$$
$$\le \sum_{k=1}^{n} kx_k^2 + 2\sum_{1\le i<j\le n}\sqrt{ij}x_ix_j = \Big(\sum_{k=1}^{n}\sqrt{k}x_k\Big)^2.$$

Taking square root of both sides, we get the desired inequality.

20. (1994 Chinese Team Selection Test) For $0 \le a \le b \le c \le d \le e$ and $a+b+c+d+e = 1$, show that

$$ad + dc + cb + be + ea \le \frac{1}{5}.$$

Solution. (Due to Lau Lap Ming) Since $a \le b \le c \le d \le e$, so $d+e \ge c+e \ge b+d \ge a+c \ge a+b$. By Chebysev's inequality,

$$\begin{aligned} &ad + dc + cb + be + ea \\ =&\frac{a(d+e)+b(c+e)+c(b+d)+d(a+c)+e(a+b)}{2} \\ \le&\frac{(a+b+c+d+e)((d+e)+(c+e)+(b+d)+(a+c)+(a+b))}{10} \\ =&\frac{2}{5}. \end{aligned}$$

21. (1985 Wuhu City Math Competition) Let x, y, z be real numbers such that $x+y+z = 0$. Show that

$$6(x^3+y^3+z^3)^2 \le (x^2+y^2+z^2)^3.$$

Solution. (Due to Ng Ka Wing) We have $z = -(x+y)$ and so

$$\begin{aligned}(x^2+y^2+z^2)^3 &= (x^2+y^2+(x+y)^2)^3 \\ &\ge \Big(\frac{3}{2}(x+y)^2\Big)^3 = \frac{27}{8}(x+y)^4(x+y)^2 \\ &\ge \frac{27}{8}(2\sqrt{xy})^4(x+y)^2 = 6\big(3xy(x+y)\big)^2 \\ &= 6\big(x^3+y^3-(x+y)^3\big)^2 = 6(x^3+y^3+z^3)^2.\end{aligned}$$

Comments. Let $f(w) = (w-x)(w-y)(w-z) = w^3 + bw + c$. Then $x^2+y^2+z^2 = (x+y+z)^2 - 2(xy+yz+zx) = -2b$ and $0 = f(x)+f(y)+f(z) = (x^3+y^3+z^3) + b(x+y+z) + 3c$ implies $x^3+y^3+z^3 = -3c$. So the inequality is the same as $2(-4b^3 - 27c^2) \ge 0$. For the cubic polynomial $f(w) = w^3 + bw + c$, it is well-known that the *discriminant* $\triangle = (x-y)^2(y-z)^2(z-x)^2$ equals $-4b^3 - 27c^2$. The inequality follows easily from this.

22. (1999 IMO) Let n be a fixed integer, with $n \ge 2$.

(a) Determine the least constant C such that the inequality

$$\sum_{1\le i<j\le n} x_ix_j(x_i^2 + x_j^2) \le C\Big(\sum_{1\le i\le n} x_i\Big)^4$$

holds for all nonnegative real numbers $x_1, x_2, \ldots, x_n$.

(b) For this constant C, determine when equality holds.

Solution. (Due to Law Ka Ho and Ng Ka Wing) We will show the least C is $1/8$. By the AM-GM inequality,

$$\begin{aligned}\Big(\sum_{1\le i\le n} x_i\Big)^4 &= \Big(\sum_{1\le i\le n} x_i^2 + 2\sum_{1\le i<j\le n} x_ix_j\Big)^2 \\ &\ge \Big(2\sqrt{2\sum_{1\le i<j\le n} x_ix_j \sum_{1\le i\le n} x_i^2}\Big)^2 \\ &= 8\sum_{1\le i<j\le n} x_ix_j(x_1^2 + \cdots + x_n^2) \\ &\ge 8\sum_{1\le i<j\le n} x_ix_j(x_i^2 + x_j^2).\end{aligned}$$

(Equality holds in the second inequality if and only if at least $n-2$ of the x_i's are zeros. Then equality holds in the first inequality if and only if the remaining pair of x_i's are equal.) Overall, equality holds if and only if two of the x_i's are equal and the others are zeros.

23. (1995 Bulgarian Math Competition) Let $n \geq 2$ and $0 \leq x_i \leq 1$ for $i = 1, 2, \ldots, n$. Prove that

$$(x_1 + x_2 + \cdots + x_n) - (x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1) \leq \left[\frac{n}{2}\right],$$

where $[x]$ is the greatest integer less than or equal to x.

Solution. When $x_2, \ldots, x_n$ are fixed, the left side is a degree one polynomial in x_1, so the maximum value is attained when $x_1 = 0$ or 1. The situation is similar for the other x_i's. So when the left side is maximum, every x_i's is 0 or 1 and the value is an integer. Now

$$\begin{aligned} &2\big((x_1 + \cdots + x_n) - (x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1)\big) \\ =&n - (1 - x_1)(1 - x_2) - (1 - x_2)(1 - x_3) - \ldots - (1 - x_n)(1 - x_1) \\ &- x_1x_2 - x_2x_3 - \ldots - x_nx_1. \end{aligned}$$

Since $0 \leq x_i \leq 1$, the expression above is at most n. So

$$\max\big((x_1 + \cdots + x_n) - (x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1)\big) \leq \left[\frac{n}{2}\right].$$

24. For every triplet of functions $f, g, h : [0, 1] \to R$, prove that there are numbers x, y, z in $[0, 1]$ such that

$$|f(x) + g(y) + h(z) - xyz| \geq \frac{1}{3}.$$

Solution. Suppose for all x, y, z in $[0, 1]$, $|f(x) + g(y) + h(z) - xyz| < 1/3$. Then

$$|f(0) + g(0) + h(0)| < \frac{1}{3}, \quad |f(0) + g(y) + h(z)| < \frac{1}{3},$$

$$|f(x) + g(0) + h(z)| < \frac{1}{3}, \quad |f(x) + g(y) + h(0)| < \frac{1}{3}.$$

Since

$$\begin{aligned} f(x) + g(y) + h(z) = &\frac{f(0) + g(y) + h(z)}{2} + \frac{f(x) + g(0) + h(z)}{2} \\ &+ \frac{f(x) + g(y) + h(0)}{2} + \frac{-f(0) - g(0) - h(0)}{2}, \end{aligned}$$

by the triangle inequality, $|f(x) + g(y) + h(z)| < 2/3$. In particular, $|f(1) + g(1) + h(1)| < 2/3$. However, $|1 - f(1) - g(1) - h(1)| < 1/3$. Adding these two inequality and applying the triangle inequality to the left side, we get $1 < 1$, a contradiction.

There is a simpler proof. By the triangle inequality, the sum of $|-(f(0) + g(0) + h(0))|$, $|f(0) + g(1) + h(1)|$, $|f(1) + g(0) + h(1)|$, $|f(1)+g(1)+h(0)|$, $|-(f(1)+g(1)+h(1)-1)|$, $|-(f(1)+g(1)+h(1)-1)|$ is at least 2. So, one of them is at least 2/6.

25. (Proposed by Great Britain for 1987 IMO) If x, y, z are real numbers such that $x^2 + y^2 + z^2 = 2$, then show that $x + y + z \leq xyz + 2$.

Solution. (Due to Chan Ming Chiu) If one of x, y, z is nonpositive, say z, then

$$2 + xyz - x - y - z = (2 - x - y) - z(1 - xy) \geq 0$$

because $x + y \leq \sqrt{2(x^2 + y^2)} \leq 2$ and $xy \leq (x^2 + y^2)/2 \leq 1$. So we may assume x, y, z are positive, say $0 < x \leq y \leq z$. If $z \leq 1$, then

$$2 + xyz - x - y - z = (1 - x)(1 - y) + (1 - z)(1 - xy) \geq 0.$$

If $z > 1$, then

$$(x + y) + z \leq \sqrt{2((x + y)^2 + z^2)} = 2\sqrt{xy + 1} \leq xy + 2 \leq xyz + 2.$$

26. (Proposed by USA for 1993 IMO) Prove that for positive real numbers a, b, c, d,

$$\frac{a}{b + 2c + 3d} + \frac{b}{c + 2d + 3a} + \frac{c}{d + 2a + 3b} + \frac{d}{a + 2b + 3c} \geq \frac{2}{3}.$$

Solution. Let

$$x_1 = \sqrt{\frac{a}{b+2c+3d}}, \qquad y_1 = \sqrt{a(b+2c+3d)},$$

$$x_2 = \sqrt{\frac{b}{c+2d+3a}}, \qquad y_2 = \sqrt{b(c+2d+3a)},$$

$$x_3 = \sqrt{\frac{c}{d+2a+3b}}, \qquad y_3 = \sqrt{c(d+2a+3b)},$$

$$x_4 = \sqrt{\frac{d}{a+2b+3c}}, \qquad y_4 = \sqrt{d(a+2b+3c)}.$$

The inequality to be proved is $x_1^2+x_2^2+x_3^2+x_4^2 \geq 2/3$. By the Cauchy-Schwarz inequality, $(x_1^2+\cdots+x_4^2)(y_1^2+\cdots+y_4^2) \geq (a+b+c+d)^2$. To finish, it suffices to show $(a+b+c+d)^2/(y_1^2+y_2^2+y_3^2+y_4^2) \geq 2/3$. This follows from

$$\begin{aligned}&3(a+b+c+d)^2 - 2(y_1^2+y_2^2+y_3^2+y_4^2)\\ =&3(a+b+c+d)^2 - 8(ab+ac+ad+bc+bd+cd)\\ =&(a-b)^2+(a-c)^2+(a-d)^2+(b-c)^2+(b-d)^2+(c-d)^2 \geq 0.\end{aligned}$$

27. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be $2n$ positive real numbers such that

 (a) $a_1 \geq a_2 \geq \cdots \geq a_n$ and
 (b) $b_1 b_2 \cdots b_k \geq a_1 a_2 \cdots a_k$ for all $k, 1 \leq k \leq n$.

 Show that $b_1 + b_2 + \cdots + b_n \geq a_1 + a_2 + \cdots + a_n$.

Solution. Let $c_k = b_k/a_k$ and $d_k = (c_1-1)+(c_2-1)+\cdots+(c_k-1)$ for $1 \leq k \leq n$. By the AM-GM inequality and (b), $(c_1+c_2+\cdots+c_k)/k \geq \sqrt[k]{c_1c_2\cdots c_k} \geq 1$, which implies $d_k \geq 0$. Finally,

$$\begin{aligned}&(b_1+b_2+\cdots+b_n)-(a_1+a_2+\cdots+a_n)\\ =&(c_1-1)a_1+(c_2-1)a_2+\cdots+(c_n-1)a_n\\ =&d_1a_1+(d_2-d_1)a_2+\cdots+(d_n-d_{n-1})a_n\\ =&d_1(a_1-a_2)+d_2(a_2-a_3)+\cdots+d_na_n \geq 0.\end{aligned}$$

28. (Proposed by Greece for 1987 IMO) Let $a, b, c > 0$ and m be a positive integer, prove that

$$\frac{a^m}{b+c}+\frac{b^m}{c+a}+\frac{c^m}{a+b} \geq \frac{3}{2}\left(\frac{a+b+c}{3}\right)^{m-1}.$$

Solution. Without loss of generality, assume $a \geq b \geq c$. So $a+b \geq c+a \geq b+c$, which implies $\frac{1}{b+c} \geq \frac{1}{c+a} \geq \frac{1}{a+b}$. By the AM-HM inequality,

$$\frac{(b+c)+(c+a)+(a+b)}{3} \geq \frac{3}{\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}}.$$

This yields $\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b} \geq \frac{9}{2(a+b+c)}$. By the Chebysev inequality and the power mean inequality respectively, we have

$$\begin{aligned}\frac{a^m}{b+c}+\frac{b^m}{c+a}+\frac{c^m}{a+b} &\geq \frac{1}{3}(a^m+b^m+c^m)\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)\\ &\geq \left(\frac{a+b+c}{3}\right)^m \frac{9}{2(a+b+c)}\\ &= \frac{3}{2}\left(\frac{a+b+c}{3}\right)^{m-1}.\end{aligned}$$

29. Let $a_1, a_2, \ldots, a_n$ be distinct positive integers, show that

$$\frac{a_1}{2}+\frac{a_2}{8}+\cdots+\frac{a_n}{n2^n} \geq 1-\frac{1}{2^n}.$$

Solution. (Due to Chan Kin Hang) Arrange $a_1, a_2, \ldots, a_n$ into increasing order as $b_1, b_2, \ldots, b_n$. Then $b_n \geq n$ because they are distinct positive integers. Since $\frac{1}{2}, \frac{1}{8}, \ldots, \frac{1}{n2^n}$, by the rearrangement inequality,

$$\begin{aligned}\frac{a_1}{2}+\frac{a_2}{8}+\cdots+\frac{a_n}{n2^n} &\geq \frac{b_1}{2}+\frac{b_2}{8}+\cdots+\frac{b_n}{n2^n}\\ &\geq \frac{1}{2}+\frac{2}{8}+\cdots+\frac{n}{n2^n} = 1-\frac{1}{2^n}.\end{aligned}$$

30. (1982 West German Math Olympiad) If $a_1, a_2, \ldots, a_n > 0$ and $a = a_1 + a_2 + \cdots + a_n$, then show that

$$\sum_{i=1}^{n} \frac{a_i}{2a - a_i} \geq \frac{n}{2n-1}.$$

Solution. By symmetry, we may assume $a_1 \geq a_2 \geq \ldots \geq a_n$. Then $\dfrac{1}{2a - a_n} \leq \cdots \leq \dfrac{1}{2a - a_1}.$ For convenience, let $a_i = a_j$ if $i \equiv j \pmod n$. For $m = 0, 1, \ldots, n-1$, by the rearrangement inequality, we get

$$\sum_{i=1}^{n} \frac{a_{m+i}}{2a - a_i} \leq \sum_{i=1}^{n} \frac{a_i}{2a - a_i}.$$

Adding these n inequalities, we get $\displaystyle\sum_{i=1}^{n} \frac{a}{2a - a_i} \leq \sum_{i=1}^{n} \frac{na_i}{2a - a_i}$. Since $\dfrac{a}{2a - a_i} = \dfrac{1}{2} + \dfrac{1}{2} \dfrac{a_i}{2a - a_i}$, we get

$$\frac{n}{2} + \frac{1}{2} \sum_{i=1}^{n} \frac{a_i}{2a - a_i} \leq n \sum_{i=1}^{n} \frac{a_i}{2a - a_i}.$$

Solving for the sum, we get the desired inequality.

31. Prove that if $a, b, c > 0$, then $\dfrac{a^3}{b+c} + \dfrac{b^3}{c+a} + \dfrac{c^3}{a+b} \geq \dfrac{a^2 + b^2 + c^2}{2}$.

Solution. (Due to Ho Wing Yip) By symmetry, we may assume $a \leq b \leq c$. Then $a + b \leq c + a \leq b + c$. So $\dfrac{1}{b+c} \leq \dfrac{1}{c+a} \leq \dfrac{1}{a+b}$. By the rearrangement inequality, we have

$$\frac{a^3}{a+b} + \frac{b^3}{b+c} + \frac{c^3}{c+a} \leq \frac{a^3}{b+c} + \frac{b^3}{c+a} + \frac{c^3}{a+b},$$

$$\frac{a^3}{c+a} + \frac{b^3}{a+b} + \frac{c^3}{b+c} \leq \frac{a^3}{b+c} + \frac{b^3}{c+a} + \frac{c^3}{a+b}.$$

Adding these, then dividing by 2, we get

$$\frac{1}{2}\left(\frac{a^3+b^3}{a+b} + \frac{b^3+c^3}{b+c} + \frac{c^3+a^3}{c+a}\right) \leq \frac{a^3}{b+c} + \frac{b^3}{c+a} + \frac{c^3}{a+b}.$$

Finally, since $(x^3 + y^3)/(x+y) = x^2 - xy + y^2 \geq (x^2 + y^2)/2$, we have

$$\begin{aligned} \frac{a^2+b^2+c^2}{2} &= \frac{1}{2}\left(\frac{a^2+b^2}{2} + \frac{b^2+c^2}{2} + \frac{c^2+a^2}{2}\right) \\ &\leq \frac{1}{2}\left(\frac{a^3+b^3}{a+b} + \frac{b^3+c^3}{b+c} + \frac{c^3+a^3}{c+a}\right) \\ &\leq \frac{a^3}{b+c} + \frac{b^3}{c+a} + \frac{c^3}{a+b}. \end{aligned}$$

32. Let $a, b, c, d > 0$ and

$$\frac{1}{1+a^4} + \frac{1}{1+b^4} + \frac{1}{1+c^4} + \frac{1}{1+d^4} = 1.$$

Prove that $abcd \geq 3$.

Solution. Let $a^2 = \tan\alpha,\ b^2 = \tan\beta,\ c^2 = \tan\gamma,\ d^2 = \tan\delta$. Then $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = 1$. By the AM-GM inequality,

$$\sin^2\alpha = \cos^2\beta + \cos^2\gamma + \cos^2\delta \geq 3(\cos\beta\cos\gamma\cos\delta)^{2/3}.$$

Multiplying this and three other similar inequalities, we have

$$\sin^2\alpha\sin^2\beta\sin^2\gamma\sin^2\delta \geq 81\cos^2\alpha\cos^2\beta\cos^2\gamma\cos^2\delta.$$

Then $abcd = \sqrt{\tan\alpha\tan\beta\tan\gamma\tan\delta} \geq 3$.

33. (Due to Paul Erdös) Each of the positive integers $a_1, \ldots, a_n$ is less than 1951. The least common multiple of any two of these is greater than 1951. Show that

$$\frac{1}{a_1} + \cdots + \frac{1}{a_n} < 1 + \frac{n}{1951}.$$

Solution. Observe that none of the numbers $1, 2, \ldots, 1951$ is a common multiple of more than one a_i's. The number of multiples of a_i among $1, 2, \ldots, 1951$ is $[1951/a_i]$. So we have $[1951/a_1]+\cdots+[1951/a_n] \le 1951$. Since $x - 1 < [x]$, so

$$\Big(\frac{1951}{a_1} - 1\Big) + \cdots + \Big(\frac{1951}{a_n} - 1\Big) < 1951.$$

Dividing by 1951 and moving the negative terms to the right, we get the desired inequality.

34. A sequence (P_n) of polynomials is defined recursively as follows:

$$P_0(x) = 0 \quad \text{and for } n \ge 0, \quad P_{n+1}(x) = P_n(x) + \frac{x - P_n(x)^2}{2}.$$

Prove that

$$0 \le \sqrt{x} - P_n(x) \le \frac{2}{n+1}$$

for every nonnegative integer n and all x in $[0, 1]$.

Solution. (Due to Wong Chun Wai) For x in $[0, 1]$,

$$\sqrt{x} - P_{n+1}(x) = (\sqrt{x} - P_n(x))\Big(1 - \frac{\sqrt{x} + P_n(x)}{2}\Big).$$

By induction, we can show that $0 \le P_n(x) \le \sqrt{x} \le 1$ for all x in $[0, 1]$. Then

$$\frac{\sqrt{x} - P_n(x)}{\sqrt{x}} = \prod_{k=0}^{n-1} \frac{\sqrt{x} - P_{k+1}(x)}{\sqrt{x} - P_k(x)} = \prod_{k=0}^{n-1} \Big(1 - \frac{\sqrt{x} + P_k(x)}{2}\Big) \le \Big(1 - \frac{\sqrt{x}}{2}\Big)^n.$$

Multiplying both sides by $\sqrt{x}$ and applying the AM-GM inequality, we have

$$\begin{aligned} 0 \le \sqrt{x} - P_n(x) &\le \sqrt{x}\Big(1 - \frac{\sqrt{x}}{2}\Big)^n \\ &\le \frac{2}{n}\left(\frac{\frac{n}{2}\sqrt{x} + (1 - \frac{\sqrt{x}}{2}) + \cdots + (1 - \frac{\sqrt{x}}{2})}{n+1}\right)^{n+1} \\ &= \frac{2}{n}\Big(\frac{n}{n+1}\Big)^{n+1} \le \frac{2}{n+1}. \end{aligned}$$

35. (1996 IMO shortlisted problem) Let $P(x)$ be the real polynomial function, $P(x) = ax^3 + bx^2 + cx + d$. Prove that if $|P(x)| \le 1$ for all x such that $|x| \le 1$, then

$$|a| + |b| + |c| + |d| \le 7.$$

Solution. Note the four polynomials $\pm P(\pm x)$ satisfy the same conditions as $P(x)$. One of these have $a, b \ge 0$. The problem stays the same if $P(x)$ is replaced by this polynomial. So we may assume $a, b \ge 0$.

<u>Case c in $[0, +\infty)$.</u> If $d \ge 0$, then $|a| + |b| + |c| + |d| = a + b + c + d = P(1) \le 1$. If $d < 0$, then

$$|a| + |b| + |c| + |d| = a + b + c + d + 2(-d) = P(1) - 2P(0) \le 3.$$

<u>Case c in $(-\infty, 0)$.</u> If $d \ge 0$, then

$$\begin{aligned} |a| + |b| + |c| + |d| &= a + b - c + d \\ &= \frac{4}{3}P(1) - \frac{1}{3}P(-1) - \frac{8}{3}P(\frac{1}{2}) + \frac{8}{3}P(-\frac{1}{2}) \\ &\le \frac{4}{3} + \frac{1}{3} + \frac{8}{3} + \frac{8}{3} = 7. \end{aligned}$$

If $d < 0$, then

$$\begin{aligned} |a| + |b| + |c| + |d| &= a + b - c - d \\ &= \frac{5}{3}P(1) - 4P(\frac{1}{2}) + \frac{4}{3}P(-\frac{1}{2}) \\ &\le \frac{5}{3} + 4 + \frac{4}{3} = 7. \end{aligned}$$

Comments. Tracing the equality cases, we see that the maximum 7 is obtained by $P(x) = \pm(4x^3 - 3x)$ only.

36. (American Mathematical Monthly, Problem 4426) Let $P(z) = az^3 + bz^2 + cz + d$, where a, b, c, d are complex numbers with $|a| = |b| = |c| = |d| = 1$. Show that $|P(z)| \ge \sqrt{6}$ for at least one complex number z satisfying $|z| = 1$.

Solution. (Due to Yung Fai) We have $a\overline{a} = |a|^2 = 1$ and similarly for b, c, d. Using $w + \overline{w} = 2\text{Re } w$, we get

$$\begin{aligned} |P(z)|^2 &= (az^3 + bz^2 + cz + d)(\overline{az}^3 + \overline{bz}^2 + \overline{cz} + \overline{d}) \\ &= 4 + 2\text{Re } \big(a\overline{d}z^3 + (a\overline{c} + b\overline{d})z^2 + (a\overline{b} + b\overline{c} + c\overline{d})z\big). \end{aligned}$$

Let $Q(z) = a\overline{d}z^3 + (a\overline{c} + b\overline{d})z^2 + (a\overline{b} + b\overline{c} + c\overline{d})z$, then $|P(z)|^2 = 4 + 2\text{Re } Q(z)$. Now we use the roots of unity trick! Let ω be a cube root of unity not equal to 1. Since $1 + \omega + \omega^2 = 0$ and $1 + \omega^2 + \omega^4 = 0$, so

$$\begin{aligned} &Q(z) + Q(\omega z) + Q(\omega^2 z) \\ =&3a\overline{d}z^3 + (a\overline{c} + b\overline{d})(1 + \omega + \omega^2) + (a\overline{d} + b\overline{c} + c\overline{d})(1 + \omega^2 + \omega^4)z \\ =&3a\overline{d}z^3. \end{aligned}$$

If we now choose z to be a cube root of $\overline{a}d$, then $|z| = 1$ and $\text{Re } Q(z) + \text{Re } Q(\omega z) + \text{Re } Q(\omega^2 z) = 3$. So $|P(z)|^2 + |P(\omega z)|^2 + |P(\omega^2 z)|^2 = 18$. Then one of $|P(z)|, |P(\omega z)|, |P(\omega^2 z)|$ is at least $\sqrt{6}$.

37. (1997 Hungarian-Israeli Math Competition) Find all real numbers α with the following property: for any positive integer n, there exists an integer m such that $\left|\alpha - \dfrac{m}{n}\right| < \dfrac{1}{3n}$?

Solution. The condition holds if and only if x is an integer. If x is an integer, then for any n, take $m = nx$. Conversely, suppose the condition holds for x. Let m_k be the integer corresponding to $n = 2^k,\ k = 0, 1, 2, \ldots$. By the triangle inequality,

$$\left|\frac{m_k}{2^k} - \frac{m_{k+1}}{2^{k+1}}\right| \le \left|\frac{m_k}{2^k} - x\right| + \left|x - \frac{m_{k+1}}{2^{k+1}}\right| < \frac{1}{3 \cdot 2^k} + \frac{1}{3 \cdot 2^{k+1}} = \frac{1}{2^{k+1}}.$$

Since the leftmost expression is $|2m_k - m_{k+1}|/2^{k+1}$, the inequalities imply it is 0, that is $m_k/2^k = m_{k+1}/2^{k+1}$ for every k. Then $|x - m_0| = |x - (m_k/2^k)| \le 1/(3 \cdot 2^k)$ for every k. Therefore, $x = m_0$ is an integer.

38. (1979 British Math Olympiad) If n is a positive integer, denote by $p(n)$ the number of ways of expressing n as the sum of one or more positive integers. Thus $p(4) = 5$, as there are five different ways of expressing 4 in terms of positive integers; namely

$$1 + 1 + 1 + 1,\ \ 1 + 1 + 2,\ \ 1 + 3,\ \ 2 + 2,\ \ \text{and}\ \ 4.$$

Prove that $p(n + 1) - 2p(n) + p(n - 1) \ge 0$ for each $n > 1$.

Solution. The required inequality can be written as $p(n+1) - p(n) \ge p(n) - p(n-1)$. Note that adding a 1 to each $p(n-1)$ sums of $n - 1$ will yield $p(n-1)$ sums of n. Conversely, for each sum of n whose least summand is 1, removing that 1 will results in a sum of $n - 1$. So $p(n) - p(n-1)$ is the number of sums of n whose least summands are at least 2. For every one of these $p(n) - p(n-1)$ sums of n, increasing the largest summand by 1 will give a sum of $n+1$ with least summand at least 2. So $p(n+1) - p(n) \ge p(n) - p(n-1)$.

<u>*Functional Equations*</u>

39. Find all polynomials f satisfying $f(x^2) + f(x)f(x+1) = 0$.

Solution. If f is constant, then f is 0 or -1. If f is not constant, then let z be a root of f. Setting $x = z$ and $x = z - 1$, respectively, we see that z^2 and $(z-1)^2$ are also roots, respectively. Since f has finitely many roots and z^{2^n} are all roots, so we must have $|z| = 0$ or 1. Since z is a root implies $(z-1)^2$ is a root, $|z - 1|$ also equals 0 or 1. It follows that $z = 0$ or 1. Then $f(x) = cx^m(x-1)^n$ for some real c and nonnegative integers m, n. If $c \ne 0$, then after simplifying the functional equation, we will see that $n = m$ and $c = 1$. Therefore, $f(x) = 0$ or $-x^n(1-x)^n$ for nonnegative integer n.

40. (1997 Greek Math Olympiad) Let $f : (0, \infty) \to R$ be a function such that
   (a) f is strictly increasing,
   (b) $f(x) > -\frac{1}{x}$ for all $x > 0$ and
   (c) $f(x)f(f(x) + \frac{1}{x}) = 1$ for all $x > 0$.

   Find $f(1)$.

Solution. Let $t = f(1)$. Setting $x = 1$ in (c), we get $tf(t+1) = 1$. So $t \neq 0$ and $f(t+1) = 1/t$. Setting $x = t+1$ in (c), we get $f(t+1)f(f(t+1)+\frac{1}{t+1}) = 1$. Then $f(\frac{1}{t}+\frac{1}{t+1}) = t = f(1)$. Since f is strictly increasing, $\frac{1}{t} + \frac{1}{t+1} = 1$. Solving, we get $t = (1 \pm \sqrt{5})/2$. If $t = (1+\sqrt{5})/2 > 0$, then $1 < t = f(1) < f(1+t) = \frac{1}{t} < 1$, a contradiction. Therefore, $f(1) = t = (1-\sqrt{5})/2$. (Note $f(x) = (1-\sqrt{5})/(2x)$ is such a function.)

41. (1979 Eötvös-Kürschák Math Competition) The function f is defined for all real numbers and satisfies $f(x) \leq x$ and $f(x+y) \leq f(x) + f(y)$ for all real x, y. Prove that $f(x) = x$ for every real number x.

Solution. (Due to Ng Ka Wing) Since $f(0+0) \leq f(0) + f(0)$, so $0 \leq f(0)$. Since $f(0) \leq 0$ also, we get $f(0) = 0$. For all real x,

$$0 = f(x + (-x)) \leq f(x) + f(-x) \leq x + (-x) = 0.$$

So $f(x) + f(-x) = 0$, hence $-f(-x) = f(x)$ for all real x. Since $f(-x) \leq -x$, so $x \leq -f(-x) = f(x) \leq x$. Therefore, $f(x) = x$ for all real x.

42. (Proposed by Ireland for 1989 IMO) Suppose $f : R \to R$ satisfies $f(1) = 1, f(a+b) = f(a) + f(b)$ for all $a, b \in R$ and $f(x)f(\frac{1}{x}) = 1$ for $x \neq 0$. Show that $f(x) = x$ for all x.

Solution. (Due to Yung Fai) From $f(0+0) = f(0) + f(0)$, we get $f(0) = 0$. From $0 = f(x + (-x)) = f(x) + f(-x)$, we get $f(-x) = -f(x)$. By induction, $f(nx) = nf(x)$ for positive integer n. For $x = \frac{1}{n}$, $1 = f(1) = f(n\frac{1}{n}) = nf(\frac{1}{n})$. Then $f(\frac{1}{n}) = \frac{1}{n}$ and $f(\frac{m}{n}) = f(m\frac{1}{n}) = mf(\frac{1}{n}) = \frac{m}{n}$. So $f(x) = x$ for rational x. (The argument up to this point is well-known. The so-called *Cauchy's equation* $f(a+b) = f(a) + f(b)$ implies $f(x) = f(1)x$ for rational x.)

Next we will show f is continuous at 0. For $0 < |x| < \frac{1}{2n}$, we have $|\frac{1}{nx}| > 2$. So there is w such that $w + \frac{1}{w} = \frac{1}{nx}$. We have $|f(\frac{1}{nx})| = |f(w) + f(\frac{1}{w})| \leq 2\sqrt{f(w)f(\frac{1}{w})} = 2$. So $|f(x)| = \dfrac{1}{n|f(\frac{1}{nx})|} \leq \dfrac{1}{2n}$. Then $\lim_{x\to 0} f(x) = 0 = f(0)$.

Now for every real x, let r_n be a rational number agreeing with x to n places after the decimal point. Then $\lim_{n\to\infty}(x - r_n) = 0$. By continuity at 0, $f(x) = \lim_{n\to\infty}(f(x-r_n) + f(r_n)) = \lim_{n\to\infty} r_n = x$. Therefore, $f(x) = x$ for all x. (This first and third paragraphs show the Cauchy equation with continuity at a point has the unique solution $f(x) = f(1)x$.)

43. (1992 Polish Math Olympiad) Let Q^+ be the positive rational numbers. Determine all functions $f : Q^+ \to Q^+$ such that $f(x+1) = f(x) + 1$ and $f(x^3) = f(x)^3$ for every $x \in Q^+$.

Solution. From $f(x+1) = f(x) + 1$, we get $f(x+n) = f(x) + n$ for all positive integer n. For $\frac{p}{q} \in Q^+$, let $t = f(\frac{p}{q})$. On one hand,

$$f((\frac{p}{q} + q^2)^3) = f(\frac{p^3}{q^3} + 3p^2 + 3pq^3 + q^6) = t^3 + 3p^2 + 3pq^3 + q^6$$

and on the other hand,

$$f((\frac{p}{q} + q^2)^3) = (f(\frac{p}{q}) + q^2)^3 = t^3 + 3t^2q^2 + 3tq^4 + q^6.$$

Equating the right sides and simplifying the equation to a quadratic in t, we get the only positive root $t = \frac{p}{q}$. So $f(x) = x$ for all $x \in Q^+$.

44. (1996 IMO shortlisted problem) Let R denote the real numbers and $f : R \to [-1, 1]$ satisfy

$$f\left(x + \frac{13}{42}\right) + f(x) = f\left(x + \frac{1}{6}\right) + f\left(x + \frac{1}{7}\right)$$

for every $x \in R$. Show that f is a periodic function, i.e. there is a nonzero real number T such that $f(x+T) = f(x)$ for every $x \in R$.

Solution. Setting $x = w + \frac{k}{6}$ for $k = 0, 1, \ldots, 5$, we get 6 equations. Adding these and cancelling terms, we will get $f(w + \frac{8}{7}) + f(w) = f(w+1) + f(w + \frac{1}{7})$ for all w. Setting $w = z + \frac{k}{7}$ for $k = 0, 1, \ldots, 6$

in this new equation, we get 7 equations. Adding these and cancelling terms, we will get $f(z+2)+f(z) = 2f(z+1)$ for all z. Rewriting this as $f(z+2)-f(z+1) = f(z+1)-f(z)$, we see that $f(z+n)-f(z+(n-1))$ is a constant, say c. If $c \neq 0$, then

$$\begin{aligned} f(z+k) &= \sum_{n=1}^{k}(f(z+n) - f(z+(n-1))) + f(z) \\ &= kc + f(z) \notin [-1, 1] \end{aligned}$$

for large k, a contradiction. So $c = 0$ and $f(z+1) = f(z)$ for all z.

45. Let N denote the positive integers. Suppose $s : N \to N$ is an increasing function such that $s(s(n)) = 3n$ for all $n \in N$. Find all possible values of $s(1997)$.

Solution. (Due to Chan Kin Hang) Note that if $s(m) = s(n)$, then $3m = s(s(m)) = s(s(n)) = 3n$ implies $m = n$. From this, we see that s is strictly increasing. Next we have $n < s(n)$ for all n (otherwise $s(n) \leq n$ for some n, which yields the contradiction that $3n = s(s(n)) \leq s(n) \leq n$.) Then $s(n) < s(s(n)) = 3n$. In particular, $1 < s(1) < 3$ implies $s(1) = 2$ and $s(2) = s(s(1)) = 3$. With the help of $s(3n) = s(s(s(n))) = 3s(n)$, we get $s(3^k) = 2{\cdot}3^k$ and $s(2{\cdot}3^k) = s(s(3^k)) = 3^{k+1}$.

Now there are 3^k-1 integers in each of the open intervals $(3^k, 2{\cdot}3^k)$ and $(2{\cdot}3^k, 3^{k+1})$. Since f is strictly increasing, we must have $s(3^k+j) = 2\cdot 3^k + j$ for $j = 1, 2, \ldots, 3^k - 1$. Then $s(2\cdot 3^k + j) = s(s(3^k + j)) = 3(3^k+j)$. Since $1997 = 2{\cdot}3^6+539 < 3^7$, so $s(1997) = 3(3^6+539) = 3804$.

46. Let N be the positive integers. Is there a function $f : N \to N$ such that $f^{(1996)}(n) = 2n$ for all $n \in N$, where $f^{(1)}(x) = f(x)$ and $f^{(k+1)}(x) = f(f^{(k)}(x))$?

Solution. For such a function $f(2n) = f^{(1997)}(n) = f^{(1996)}(f(n)) = 2f(n)$. So if $n = 2^e q$, where e, q are nonnegative integers and q odd, then $f(n) = 2^e f(q)$. To define such a function, we need to define it at odd integer q. Now define

$$f(q) = \begin{cases} q+2 & \text{if } q \equiv 1, 3, \ldots, 3989 \pmod{3992} \\ 2(q-3990) & \text{if } q \equiv 3991 \pmod{3992} \end{cases}$$

and $f(2n) = 2f(n)$ for positive integer n. If $q = 3992m + (2j-1)$, $j = 1, 2, \ldots, 1995$, then $f^{(1996-j)}(q) = q + 2(1996-j) = 3992m + 3991$, $f^{(1997-j)}(q) = 2(3992m+1)$ and

$$f^{(1996)}(q) = 2f^{(j-1)}(3992m+1) = 2(3992m+1+(2j-1)) = 2q.$$

If $q = 3992m + 3991$, then $f(q) = 2(3992m+1)$ and

$$f^{(1996)}(q) = 2f^{(1995)}(3992m+1) = 2(3992m+1+2\times 1995) = 2q.$$

So $f^{(1996)}(q) = 2q$ for odd q. If $n = 2^e q$, then

$$f^{(1996)}(n) = 2^e f^{(1996)}(q) = 2^e(2q) = 2n.$$

47. (American Mathematical Monthly, Problem E984) Let R denote the real numbers. Find all functions $f : R \to R$ such that $f(f(x)) = x^2 - 2$ or show no such function can exist.

Solution. Let $g(x) = x^2 - 2$ and suppose $f(f(x)) = g(x)$. Put $h(x) = g(g(x)) = x^4 - 4x^2 + 2$. The fixed points of g (i.e. the solutions of the equation $g(x) = x$) are -1 and 2. The set of fixed points of h contains the fixed points of g and is $S = \{-1, 2, (-1\pm\sqrt{5})/2\}$. Now observe that $x \in S$ implies $h(f(x)) = f(h(x)) = f(x)$, i.e. $f(x) \in S$. Also, $x, y \in S$ and $f(x) = f(y)$ imply $x = h(x) = h(y) = y$. So f is a bijection $S \to S$.

If $c = -1$ or 2, then $g(f(c)) = f(f(f(c))) = f(g(c)) = f(c)$ and consequently $\{f(-1), f(2)\} = \{-1, 2\}$. For $a = (-1+\sqrt{5})/2$, since f induces a bijection $S \to S$ and $g(a) = a^2 - 2 \neq a$ implies $f(a) \neq a$, we must have $f(a) = b = (-1-\sqrt{5})/2$. It follows that $f(b) = a$ and we have a contradiction $a = f(b) = f(f(a)) = g(a)$.

48. Let R be the real numbers. Find all functions $f : R \to R$ such that for all real numbers x and y,

$$f\big(xf(y) + x\big) = xy + f(x).$$

Solution 1. (Due to Leung Wai Ying) Putting $x = 1, y = -1 - f(1)$ and letting $a = f(y) + 1$, we get

$$f(a) = f\big(f(y) + 1\big) = y + f(1) = -1.$$

Putting $y = a$ and letting $b = f(0)$, we get

$$b = f\big(xf(a) + x\big) = ax + f(x),$$

so $f(x) = -ax + b$. Putting this into the equation, we have

$$a^2xy - abx - ax + b = xy - ax + b.$$

Equating coefficients, we get $a = \pm 1$ and $b = 0$, so $f(x) = x$ or $f(x) = -x$. We can easily check both are solutions.

Solution 2. Setting $x = 1$, we get

$$f\big(f(y) + 1\big) = y + f(1).$$

For every real number a, let $y = a - f(1)$, then $f\big(f(y) + 1\big) = a$ and f is surjective. In particular, there is b such that $f(b) = -1$. Also, if $f(c) = f(d)$, then

$$\begin{aligned} c + f(1) &= f\big(f(c) + 1\big) \\ &= f\big(f(d) + 1\big) \\ &= d + f(1). \end{aligned}$$

So $c = d$ and f is injective. Taking $x = 1, y = 0$, we get $f\big(f(0) + 1\big) = f(1)$. Since f is injective, we get $f(0) = 0$.

For $x \neq 0$, let $y = -f(x)/x$, then

$$f\big(xf(y) + x\big) = 0 = f(0).$$

By injectivity, we get $xf(y) + x = 0$. Then

$$f\big(-f(x)/x\big) = f(y) = -1 = f(b)$$

and so $-f(x)/x = b$ for every $x \neq 0$. That is, $f(x) = -bx$. Putting this into the given equation, we find $f(x) = x$ or $f(x) = -x$, which are easily checked to be solutions.

49. (1999 IMO) Determine all functions $f : R \to R$ such that

$$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$$

for all x, y in R.

Solution. Let A be the range of f and $c = f(0)$. Setting $x = y = 0$, we get $f(-c) = f(c) + c - 1$. So $c \neq 0$. For $x = f(y) \in A$, $f(x) = \frac{c+1}{2} - \frac{x^2}{2}$.

Next, if we set $y = 0$, we get

$$\{f(x - c) - f(x) : x \in R\} = \{cx + f(c) - 1 : x \in R\} = R$$

because $c \neq 0$. This means $A - A = \{y_1 - y_2 : y_1, y_2 \in A\} = R$.

Now for an arbitrary $x \in R$, let $y_1, y_2 \in A$ be such that $x = y_1 - y_2$. Then

$$\begin{aligned} f(x) &= f(y_1 - y_2) = f(y_2) + y_1y_2 + f(y_1) - 1 \\ &= \frac{c+1}{2} - \frac{y_2^2}{2} + y_1y_2 + \frac{c+1}{2} - \frac{y_1^2}{2} - 1 \\ &= c - \frac{(y_1 - y_2)^2}{2} = c - \frac{x^2}{2}. \end{aligned}$$

However, for $x \in A$, $f(x) = \frac{c+1}{2} - \frac{x^2}{2}$. So $c = 1$. Therefore, $f(x) = 1 - \frac{x^2}{2}$ for all $x \in R$.

50. (1995 Byelorussian Math Olympiad) Let R be the real numbers. Find all functions $f : R \to R$ such that

$$f(f(x + y)) = f(x + y) + f(x)f(y) - xy$$

for all $x, y \in R$.

Solution. (Due to Yung Fai) Clearly, from the equation, $f(x)$ is not constant. Putting $y = 0$, we get $f(f(x)) = (1 + f(0))f(x)$. Replacing x by $x + y$, we get

$$(1 + f(0))f(x + y) = f(f(x + y)) = f(x + y) + f(x)f(y) - xy,$$

which simplifies to (*) $f(0)f(x + y) = f(x)f(y) - xy$. Putting $y = 1$ in (*), we get $f(0)f(x + 1) = f(x)f(1) - x$. Putting $y = -1$ and replacing

x by $x+1$ in (*), we get $f(0)f(x) = f(x+1)f(-1)+x+1$. Eliminating $f(x+1)$ in the last two equations, we get

$$(f^2(0) - f(1)f(-1))f(x) = (f(0) - f(-1))x + f(0).$$

If $f^2(0) - f(1)f(-1) = 0$, then putting $x = 0$ in the last equation, we get $f(0) = 0$. By (*), $f(x)f(y) = xy$. Then $f(x)f(1) = x$ for all $x \in R$. So $f^2(0) - f(1)f(-1) = -1$, resulting in a contradiction. Therefore, $f^2(0) - f(1)f(-1) \neq 0$ and $f(x)$ is a degree 1 polynomial.

Finally, substituting $f(x) = ax + b$ into the original equation, we find $a = 1$ and $b = 0$, i.e. $f(x) = x$ for all $x \in R$.

51. (1993 Czechoslovak Math Olympiad) Let Z be the integers. Find all functions $f : Z \to Z$ such that

$$f(-1) = f(1) \text{ and } f(x) + f(y) = f(x + 2xy) + f(y - 2xy)$$

for all integers x, y.

Solution. We have (*) $f(1) + f(n) = f(1 + 2n) + f(-n)$ and $f(n) + f(-1) = f(-n)+f(-1+2n)$. Since $f(-1) = f(1)$, this gives $f(1+2n) = f(-1 + 2n)$ for every integer n. So $f(k)$ has the same value for every odd k. Then equation (*) implies $f(n) = f(-n)$ for every integer n. So we need to find $f(n)$ for nonnegative integers n only.

If we let $x = -(2k + 1), y = n$, then x and $x + 2xy$ are odd. The functional equation gives $f(n) = f(y) = f(y - 2xy) = f(n(4k + 3))$. If we let $x = n, y = -(2k + 1)$, then similarly, we get $f(n) = f(x) = f(x+2xy) = f(n(-4k-1)) = f(n(4k+1))$. So $f(n) = f(nm)$ for every odd m.

For a positive integer n, we can factor $n = 2^e m$, where e, m are nonnegative integers and m odd. Then $f(n) = f(2^e)$. So any such function f is determined by the values $f(0), f(1), f(2), f(4), f(8), f(16), \ldots$ (which may be arbitrary). All other values are given by $f(n) = f(2^e)$ as above. Finally, we check such functions satisfy the equations. Clearly, $f(-1) = f(1)$. If x or $y = 0$, then the functional equation is clearly satisfied. If $x = 2^e m, y = 2^d n$, where m, n are odd, then

$$f(x) + f(y) = f(2^e) + f(2^d) = f(x(1 + 2y)) + f(y(1 - 2x)).$$

52. (1995 South Korean Math Olympiad) Let A be the set of non-negative integers. Find all functions $f : A \to A$ satisfying the following two conditions:

(a) For any $m, n \in A$, $2f(m^2 + n^2) = (f(m))^2 + (f(n))^2$.

(b) For any $m, n \in A$ with $m \geq n$, $f(m^2) \geq f(n^2)$.

Solution. For $m = 0$, we get $2f(n^2) = f(0)^2 + f(n)^2$. Let $m > n$, then $f(m)^2 - f(n)^2 = 2(f(m^2) - f(n^2)) \geq 0$. So $f(m) \geq f(n)$. This means f is nondecreasing. Setting $m = 0 = n$, we get $2f(0) = f(0)^2 + f(0)^2$, which implies $f(0) = 0$ or 1.

<u>Case $f(0) = 1$.</u> Then $2f(n^2) = 1 + f(n)^2$. For $n = 1$, we get $f(1) = 1$. For $m = 1 = n$, we get $f(2) = 1$. Assume $f(2^{2^k}) = 1$. Then for $n = 2^{2^k}$, we get $2f(2^{2^{k+1}}) = 1 + f(2^{2^k})^2 = 2$. So $f(2^{2^{k+1}}) = 1$. Since $\lim_{k\to\infty} 2^{2^k} = \infty$ and f is nondecreasing, so $f(n) = 1$ for all n.

<u>Case $f(0) = 0$.</u> Then $2f(n^2) = f(n)^2$. So $f(n)$ is even for all n. For $m = 1 = n$, we get $2f(2) = f(1)^2 + f(1)^2$, which implies $f(2) = f(1)^2$. Using $2f(n^2) = f(n)^2$ repeatedly (or by induction), we get $2^{2^k-1} f(2^{2^k}) = f(1)^{2^{k+1}}$. Now $2f(1) = f(1)^2$ implies $f(1) = 0$ or 2. If $f(1) = 0$, then $\lim_{k\to\infty} 2^{2^k} = \infty$ and f nondecreasing imply $f(n) = 0$ for all n. If $f(1) = 2$, then $f(2^{2^k}) = 2^{2^{k+1}}$. Now

$$\begin{aligned} f(m+1)^2 &= 2f((m+1)^2) = 2f(m^2 + 2m + 1) \\ &\geq 2f(m^2 + 1) = f(m)^2 + f(1)^2 > f(m)^2. \end{aligned}$$

As $f(n)$ is always even, we get $f(m+1) \geq f(m) + 2$. By induction, we get $f(n) \geq 2n$. Since $f(2^{2^k}) = 2^{2^k+1} = 2 \cdot 2^{2^k}$ for all k, $\lim_{k\to\infty} 2^{2^k} = \infty$ and f is nondecreasing, so $f(n) = 2n$ for all n.

It is easy to check that $f(n) = 1$, $f(n) = 0$ and $f(n) = 2n$ are solutions. Therefore, they are the only solutions.

53. (American Mathematical Monthly, Problem E2176) Let Q denote the rational numbers. Find all functions $f : Q \to Q$ such that

$$f(2) = 2 \qquad \text{and} \qquad f\left(\frac{x + y}{x - y}\right) = \frac{f(x) + f(y)}{f(x) - f(y)} \quad \text{for } x \neq y.$$

Solution. We will show $f(x) = x$ is the only solution by a series of observations.

(1) Setting $y = 0$, we get $f(1) = (f(x) + f(0))/(f(x) - f(0))$, which yields $(f(1) - 1)f(x) = f(0)(1 + f(1))$. (Now f is not constant because the denominator in the equation cannot equal 0.) So, $f(1) = 1$ and then $f(0) = 0$.

(2) Setting $y = -x$, we get $0 = f(x) + f(-x)$, so $f(-x) = -f(x)$.

(3) Setting $y = cx, c \neq 1, x \neq 0$, we get

$$\frac{f(x) + f(cx)}{f(x) - f(cx)} = f\left(\frac{1+c}{1-c}\right) = \frac{1 + f(c)}{1 - f(c)},$$

which implies $f(cx) = f(c)f(x)$. Taking $c = q, x = p/q$, we get $f(p/q) = f(p)/f(q)$.

(4) Setting $y = x - 2$, we get $f(x-1) = (f(x) + f(x-2))/(f(x) - f(x-2))$. If $f(n-2) = n - 2 \neq 0$ and $f(n-1) = n - 1$, then this equation implies $f(n) = n$. Since $f(1) = 1$ and $f(2) = 2$, then $f(n) = n$ for all positive integers by induction and (2), (3) will imply $f(x) = x$ for all $x \in Q$.

Comments. The condition $f(2) = 2$ can also be deduced from the functional equation as shown below in (5). If rational numbers are replaced by real numbers, then again the only solution is still $f(x) = x$ as shown below in (6) and (7).

(5) We have

$$f(3) = \frac{f(2) + 1}{f(2) - 1}, \quad f(5) = \frac{f(3) + f(2)}{f(3) - f(2)} = \frac{f(2)^2 + 1}{1 + 2f(2) - f(2)^2}.$$

Also, $f(2)^2 = f(4) = (f(5) + f(3))/(f(5) - f(3))$. Substituting the equations for $f(3)$ and $f(5)$ in terms of $f(2)$ and simplifying, we get $f(2)^2 = 2f(2)$. (Now $f(2) \neq 0$, otherwise $f((x+2)/(x-2)) = (f(x) + 0)/(f(x) - 0) = 1$ will force f to be constant.) Therefore, $f(2) = 2$.

(6) Note $f(x) \neq 0$ for $x > 0$, otherwise $f(cx) = 0$ for $c \neq 1$ will force f to be constant. So, if $x > 0$, then $f(x) = f(\sqrt{x})^2 > 0$. If $x > y \geq 0$, then $f(x) - f(y) = (f(x) + f(y))/f((x+y)/(x-y)) > 0$. This implies f is strictly increasing for positive real numbers.

(7) For $x > 0$, if $x < f(x)$, then picking $r \in Q$ such that $x < r < f(x)$ will give the contradiction that $f(x) < f(r) = r < f(x)$. Similarly, $f(x) < x$ will also lead to a contradiction. So, $f(x) = x$ for all x.

54. (Mathematics Magazine, Problem 1552) Find all functions $f : R \to R$ such that

$$f(x + yf(x)) = f(x) + xf(y) \quad \text{for all } x, y \text{ in } R.$$

Solution. It is easy to check that $f(x) = 0$ and $f(x) = x$ are solutions. Suppose f is a solution that is not the zero function. (We will show $f(x) = x$ for all x.)

Step 1. Setting $y = 0, x = 1$, we get $f(0) = 0$. If $f(x) = 0$, then $0 = xf(y)$ for all y, which implies $x = 0$ as f is not the zero function. So $f(x) = 0$ if and only if $x = 0$.

Step 2. Setting $x = 1$, we get the equation (*) $f(1 + yf(1)) = f(1) + f(y)$ for all y. If $f(1) \neq 1$, then setting $y = 1/(1 - f(1))$ in (*), we get $f(y) = f(1) + f(y)$, resulting in $f(1) = 0$, contradicting step 1. So $f(1) = 1$ and (*) becomes $f(1 + y) = f(1) + f(y)$, which implies $f(n) = n$ for every integer n.

Step 3. For integer n, real z, setting $x = n, y = z - 1$ in the functional equation, we get

$$f(nz) = f(n + (z-1)f(n)) = n + nf(z-1) = nf(z).$$

Step 4. If $a = -b$, then $f(a) = f(-b) = -f(b)$ implies $f(a) + f(b) = 0 = f(a + b)$. If $a \neq -b$, then $a + b \neq 0$ and $f(a + b) \neq 0$ by step 1. Setting $x = (a + b)/2, y = \pm(a - b)/(2f(\frac{a+b}{2})$, we get

$$f(a) = f\big(\frac{a+b}{2} + \frac{a-b}{2f(\frac{a+b}{2})}f(\frac{a+b}{2})\big) = f(\frac{a+b}{2}) + \frac{a+b}{2}f\big(\frac{a-b}{2f(\frac{a+b}{2})}\big),$$

$$f(b) = f\big(\frac{a+b}{2} + \frac{b-a}{2f(\frac{a+b}{2})}f(\frac{a+b}{2})\big) = f(\frac{a+b}{2}) + \frac{a+b}{2}f\big(\frac{b-a}{2f(\frac{a+b}{2})}\big).$$

Adding these, we get $f(a)+f(b)=2f(\frac{a+b}{2})=f(a+b)$ by step 3.

Step 5. Applying step 4 to the functional equation, we get the equation (**) $f(yf(x))=xf(y)$. Setting $y=1$, we get $f(f(x)=x$. Then f is bijective. Setting $z=f(x)$ in (**), we get $f(yz)=f(y)f(z)$ for all y,z.

Step 6. Setting $z=y$ in the last equation, we get $f(y^2)=f(y)^2\geq 0$. Setting $z=-y$, we get $f(-y^2)=-f(y^2)=-f(y)^2\leq 0$. So $f(a)>0$ if and only if $a>0$.

Step 7. Setting $y=-1$ in the functional equation, we get $f(x-f(x))=f(x)-x$. Since $x-f(x)$ and $f(x)-x$ are of opposite signs, by step 6, we must have $x-f(x)=0$ for all x, i.e. $f(x)=x$ for all x.

Maximum/Minimum

55. (1985 Austrian Math Olympiad) For positive integers n, define

$$f(n)=1^n+2^{n-1}+3^{n-2}+\cdots+(n-2)^3+(n-1)^2+n.$$

What is the minimum of $f(n+1)/f(n)$?

Solution. For $n=1,2,3,4,5,6$, $f(n+1)/f(n)=3,\ 8/3,\ 22/8,\ 65/22,\ 209/65,\ 732/209$, respectively. The minimum of these is $8/3$. For $n>6$, we will show $f(n+1)/f(n)>3>8/3$. This follows from

$$\begin{aligned}&f(n+1)\\ >&1^{n+1}+2^n+3^{n-1}+4^{n-2}+5^{n-3}+6^{n-4}+\cdots+(n-1)^3+n^2\\ >&1^{n+1}+2^n+3^{n-1}+4^{n-2}+5^{n-3}+3(6^{n-5}+\cdots+(n-1)^2+n)\\ =&1^{n+1}+\cdots+5^{n-3}+3(f(n)-1^n-2^{n-1}+3^{n-2}+4^{n-3}+5^{n-4})\\ =&3f(n)+2(5^{n-4}-1)+2^{n-1}(2^{n-5}-1)>3f(n).\end{aligned}$$

Therefore, $8/3$ is the answer.

56. (1996 Putnam Exam) Given that $\{x_1,x_2,\ldots,x_n\}=\{1,2,\ldots,n\}$, find the largest possible value of $x_1x_2+x_2x_3+\cdots+x_{n-1}x_n+x_nx_1$ in terms of n (with $n\geq 2$).

Solution. Let M_n be the largest such cyclic sum for $x_1,x_2,\ldots,x_n$. In case $n=2$, we have $M_2=4$. Next suppose M_n is attained by some permutation of $1,2,\ldots,n$. Let x,y be the neighbors of n. Then removing n from the permutation, we get a permutation of $1,2,\ldots,n-1$. The difference of the cyclic sums before and after n is removed is $nx+ny-xy=n^2-(n-x)(n-y)\leq n^2-2$. (Equality holds if and only if x,y are $n-1,n-2$.) So $M_n-(n^2-2)\leq M_{n-1}$. Then

$$M_n\leq M_2+(3^2-2)+(4^2-2)+\cdots+(n^2-2)=\frac{2n^3+3n^2-11n+18}{6}.$$

Following the equality case above, we should consider the permutation constructed as follows: starting with $1,2$, we put 3 between 1 and 2 to get $1,3,2$, then put 4 between 3 and 2 to get $1,3,4,2$, then put 5 between 3 and 4 to get $1,3,5,4,2$ and so on. If n is odd, the permutation is $1,3,\ldots,n-2,n,n-1,\ldots,4,2$. If n is even, the permutation is $1,3,\ldots,n-1,n,n-2,\ldots,4,2$. The cyclic sum for each of these two permutations is $(2n^3+3n^2-11n+18)/6$ because of the equality case at each stage. Therefore, $M_n=(2n^3+3n^2-11n+18)/6$.

57. (1995 British Math Olympiad) Triangle ABC has a right angle at C. The internal bisectors of angles BAC and ABC meet BC and CA at P and Q respectively. The points M and N are the feet of the perpendiculars from P and Q to AB. Find angle MCN.

Solution. (Due to Poon Wai Hoi) Using protractor, the angle should be $45°$. To prove this, observe that since P is on the bisector of $\angle BAC$, we have $PC = PM$. Let L be the foot of the perpendicular from C to AB. Then $PM \parallel CL$. So $\angle PCM = \angle PMC = \angle MCL$. Similarly, $\angle QCN = \angle NCL$. So $\angle MCN = \frac{1}{2}\angle PCQ = 45°$.

58. (1988 Leningrad Math Olympiad) Squares $ABDE$ and $BCFG$ are drawn outside of triangle ABC. Prove that triangle ABC is isosceles if DG is parallel to AC.

Solution. (Due to Ng Ka Man, Ng Ka Wing, Yung Fai) From B, draw a perpendicular to AC (and hence also perpendicular to DG.) Let it intersect AC at X and DG at Y. Since $\angle ABX = 90° - \angle DBY = \angle BDY$ and $AB = BD$, the right triangles ABX and BDY are congruent and $AX = BY$. Similarly, the right triangles CBX and BGY are congruent and $BY = CX$. So $AX = CX$, which implies $AB = CB$.

59. AB is a chord of a circle, which is not a diameter. Chords A_1B_1 and A_2B_2 intersect at the midpoint P of AB. Let the tangents to the circle at A_1 and B_1 intersect at C_1. Similarly, let the tangents to the circle at A_2 and B_2 intersect at C_2. Prove that C_1C_2 is parallel to AB.

Solution. (Due to Poon Wai Hoi) Let OC_1 intersects A_1B_1 at M, OC_2 intersects A_2B_2 at N. and OC_1 intersect AB at K. Since OC_1 is a perpendicular bisector of A_1B_1, so $OM \perp A_1B_1$. Similarly, $ON \perp A_2B_2$. Then O, N, P, M are concyclic. So $\angle ONM = \angle OPM$. Since $\angle OKP = 90° - \angle KPM = \angle OPM$, we have $\angle OKP = \angle ONM$. From the right triangles OA_1C_1 and OB_2C_2, we get $OM \cdot OC_1 = OA_1^2 = OB_2^2 = ON \cdot OC_2$. By the converse of the intersecting chord theorem,

we get M, N, C_1, C_2 are concyclic. So $\angle OC_1C_2 = \angle ONM = \angle OKP$. Then $C_1C_2 \parallel KP$, that is C_1C_2 is parallel to AB.

60. (1991 Hunan Province Math Competition) Two circles with centers O_1 and O_2 intersect at points A and B. A line through A intersects the circles with centers O_1 and O_2 at points Y, Z, respectively. Let the tangents at Y and Z intersect at X and lines YO_1 and ZO_2 intersect at P. Let the circumcircle of $\triangle O_1O_2B$ have center at O and intersect line XB at B and Q. Prove that PQ is a diameter of the circumcircle of $\triangle O_1O_2B$.

Solution. (First we need to show P, O_1, O_2, B are concyclic. Then we will show $90° = \angle QBP = \angle XBP$. Since $\angle XYP$, $\angle PZX$ are both $90°$, it suffices to show X, Y, B, P, Z are concyclic.) Connect O_1A and O_2A. In $\triangle YPZ$,

$$\begin{aligned}\angle O_1PZ &= 180° - (\angle O_1YZ + \angle O_2ZA)\\ &= 180° - (\angle O_1AY + \angle O_2AZ)\\ &= \angle O_1AO_2 = \angle O_1BO_2.\end{aligned}$$

So B, P, O_1, O_2 are concyclic. Connect BY and BZ. Then

$$\begin{aligned}\angle YBZ &= 180° - (\angle AYB + \angle AZB)\\ &= 180° - (\frac{1}{2}\angle AO_1B + \frac{1}{2}\angle AO_2B)\\ &= 180° - (\angle BO_1O_2 + \angle BO_2O_1)\\ &= \angle O_1BO_2 = \angle O_1PZ = \angle YPZ.\end{aligned}$$

So Y, Z, P, B are concyclic. Since $\angle XYP = \angle XZP = 90°$, so the points Y, X, Z, P, B are concyclic. Then $\angle QBP = \angle XBP = 180° - \angle XZP = 90°$. Therefore, PQ is a diameter of the circumcircle of $\triangle O_1O_2B$.

61. (1981 Beijing City Math Competition) In a disk with center O, there are four points such that the distance between every pair of them is greater than the radius of the disk. Prove that there is a pair of perpendicular diameters such that exactly one of the four points lies inside each of the four quarter disks formed by the diameters.

Solution. (Due to Lee Tak Wing) By the distance condition on the four points, none of them equals O and no pair of them are on the same radius. Let us name the points A, B, C, D in the order a rotating radius encountered them. Since $AB > OA, OB$, so $\angle AOB > \angle OBA, \angle BAO$. Hence $\angle AOB > 60°$. Similarly, $\angle BOC, \angle COD$, $\angle DOA > 60°$. Let $\angle AOB$ be the largest among them, then $60° < \angle AOB < 360° - 3 \times 60° = 180°$. Let EF be the diameter bisecting $\angle AOB$ and with A, B, E on the same half disk. Now EF and its perpendicular through O divide the disk into four quarter disks. We have $90° = 30° + 60° < \angle EOB + \angle BOC$.

In the case $60° < \angle AOB < 120°$, we get $\angle EOB + \angle BOC < 60° + 120° = 180°$. In the case $120° \le \angle AOB < 180°$, we get $\angle AOB + \angle BOC < 360° - 2\times 60° = 240°$ and $\angle EOB + \angle BOC < 240° - \angle AOE \le 240° - 120°/2 = 180°$. So A, B, C each is on a different quarter disk. Similarly, $90° < \angle EOD = \angle EOA + \angle AOD < 180°$. Therefore, D will lie on the remaining quarter disk.

62. The lengths of the sides of a quadrilateral are positive integers. The length of each side divides the sum of the other three lengths. Prove that two of the sides have the same length.

Solution. (Due to Chao Khek Lun and Leung Wai Ying) Suppose the sides are a, b, c, d with $a < b < c < d$. Since $d < a + b + c < 3d$ and d divides $a + b + c$, we have $a + b + c = 2d$. Now each of a, b, c dvides $a + b + c + d = 3d$. Let $x = 3d/a, y = 3d/b$ and $z = 3d/c$. Then $a < b < c < d$ implies $x > y > z > 3$. So $z \ge 4, y \ge 5, x \ge 6$. Then

$$2d = a + b + c \le \frac{3d}{6} + \frac{3d}{5} + \frac{3d}{4} < 2d,$$

a contradiction. Therefore, two of the sides are equal.

63. (1988 Sichuan Province Math Competition) Suppose the lengths of the three sides of $\triangle ABC$ are integers and the inradius of the triangle is 1. Prove that the triangle is a right triangle.

Solution. (Due to Chan Kin Hang) Let $a = BC, b = CA, c = AB$ be the side lengths, r be the inradius and $s = (a+b+c)/2$. Since the area

of the triangle is rs, we get $\sqrt{s(s-a)(s-b)(s-c)} = 1 \cdot s = s$. Then

$$(s-a)(s-b)(s-c) = s = (s-a) + (s-b) + (s-c).$$

Now $4(a+b+c) = 8s = (2s-2a)(2s-2b)(2s-2c) = (b+c-a)(c+a-b)(a+b-c)$. In (mod 2), each of $b+c-a, c+a-b, a+b-c$ are the same. So either they are all odd or all even. Since their product is even, they are all even. Then $a+b+c$ is even and s is an integer.

The positive integers $x = s-a, y = s-b, z = s-c$ satisfy $xyz = x+y+z$. Suppose $x \ge y \ge z$. Then $yz \le 3$ for otherwise $xyz > 3x \ge x+y+z$. This implies $x = 3, y = 2, z = 1, s = 6, a = 3, b = 4, c = 5$. Therefore, the triangle is a right triangle.

<u>*Geometric Equations*</u>

64. (1985 IMO) A circle has center on the side AB of the cyclic quadrilateral $ABCD$. The other three sides are tangent to the circle. Prove that $AD + BC = AB$.

Solution. Let M be on AB such that $MB = BC$. Then

$$\angle CMB = \frac{180° - \angle ABC}{2} = \frac{\angle CDA}{2} = \angle CDO.$$

This implies C, D, M, O are concyclic. Then

$$\angle AMD = \angle OCD = \frac{\angle DCB}{2} = \frac{180° - \angle DAM}{2} = \frac{\angle AMD + \angle ADM}{2}.$$

So $\angle AMD = \angle ADM$. Therefore, $AM = AD$ and $AB = AM + MB = AD + BC$.

65. (1995 Russian Math Olympiad) Circles S_1 and S_2 with centers O_1, O_2 respectively intersect each other at points A and B. Ray O_1B intersects S_2 at point F and ray O_2B intersects S_1 at point E. The line parallel to EF and passing through B intersects S_1 and S_2 at points M and N, respectively. Prove that (B is the incenter of $\triangle EAF$ and) $MN = AE + AF$.

Solution. Since

$$\angle EAB = \frac{1}{2}\angle EO_1B = 90^\circ - \angle O_1BE = 90^\circ - \angle FBO_2 = \angle BAF,$$

AB bisects $\angle EAF$ and $\angle O_1BE = 90^\circ - \angle EAB = 90^\circ - \frac{1}{2}\angle EAF$. Now $\angle EBA + \angle FBA = \angle EBA + (180^\circ - \angle O_1BA) = 180^\circ + \angle O_1BE = 270^\circ - \frac{1}{2}\angle EAF$. Then $\angle EBF = 90^\circ + \angle EAF$, which implies B is the incenter of $\triangle EAF$ (because the incenter is the unique point P on the bisector of $\angle EAF$ such that $\angle EPF = 90^\circ - \frac{1}{2}\angle EAF$). Then $\angle AEB = \angle BEF = \angle EBM$ since $EF \parallel MN$. So $EBAM$ is an isosceles trapezoid. Hence $EA = MB$. Similarly $FA = NB$. Therefore, $MN = MB + NB = AE + AF$.

66. Point C lies on the minor arc AB of the circle centered at O. Suppose the tangent line at C cuts the perpendiculars to chord AB through A at E and through B at F. Let D be the intersection of chord AB and radius OC. Prove that $CE \cdot CF = AD \cdot BD$ and $CD^2 = AE \cdot BF$.

Solution. (Due to Wong Chun Wai) Note that $\angle EAD$, $\angle ECD$, $\angle FCD$, $\angle FBD$ are right angles. So A, D, C, E are concyclic and B, D, C, F are concyclic. Then $\angle ADE = \angle ACE = \angle ABC = \angle DFC$, say the measure of these angles is α. Also, $\angle BDF = \angle BCF = \angle BAC = \angle DEC$, say the measure of these angle is β. Then

$$CE \cdot CF = (DE\cos\beta)(DF\cos\alpha) = (DE\cos\alpha)(DF\cos\beta) = AD \cdot BD,$$

$$CD^2 = (DE\sin\beta)(DF\sin\alpha) = (DE\sin\alpha)(DF\sin\beta) = AE \cdot BF.$$

67. Quadrilaterals $ABCP$ and $A'B'C'P'$ are inscribed in two concentric circles. If triangles ABC and $A'B'C'$ are equilateral, prove that

$$P'A^2 + P'B^2 + P'C^2 = PA'^2 + PB'^2 + PC'^2.$$

Solution. Let O be the center of both circles and E be the midpoint of $A'B'$. From $\triangle PA'B'$ with median PE, by cosine law, we get $PA'^2 + PB'^2 = 2(PE^2 + EB'^2)$. From $\triangle PC'E$ with cevian PO (note $C'O = 2OE$), by cosine law again, we get $PC'^2 + 2PE^2 = 3(PO^2 + 2OE)^2$. Putting these together, we get

$$\begin{aligned} PA'^2 + PB'^2 + PC'^2 &= 2(EB'^2 + OE^2) + 3PO^2 + 4OE^2 \\ &= 2B'O^2 + 3PO^2 + C'O^2 \\ &= 3(PO^2 + P'O^2). \end{aligned}$$

Similarly, $P'A^2 + P'B^2 + PC'^2 = 3(PO^2 + P'O^2)$.

Alternatively, the problem can be solved using complex numbers. Without loss of generality, let the center be at the origin, A' be at $re^{i\theta} = r(\cos\theta + i\sin\theta)$ and P be at $Re^{i\alpha}$. Let $\omega = e^{2\pi i/3}$. We have

$$\begin{aligned} & PA'^2 + PB'^2 + PC'^2 \\ =& |Re^{i\alpha} - re^{i\theta}|^2 + |Re^{i\alpha} - re^{i\theta}\omega|^2 + |Re^{i\alpha} - re^{i\theta}\omega^2|^2 \\ =& 3R^2 - 2\mathrm{Re}\left(Rre^{i(\alpha-\theta)}(1+\omega+\omega^2)\right) + 3r^2 \\ =& 3R^2 + 3r^2. \end{aligned}$$

Similarly, $P'A^2 + P'B^2 + P'C^2 = 3R^2 + 3r^2$.

68. Let the inscribed circle of triangle ABC touchs side BC at D, side CA at E and side AB at F. Let G be the foot of perpendicular from D to EF. Show that $\dfrac{FG}{EG} = \dfrac{BF}{CE}$.

Solution. (Due to Wong Chun Wai) Let I be the incenter of $\triangle ABC$. Then $\angle BDI = 90^\circ = \angle EGD$. Also, $\angle DEG = \frac{1}{2}\angle DIF = \angle DIB$. So $\triangle BDI, \triangle EGD$ are similar. Then $BD/ID = DG/EG$. Likewise, $\triangle CDI, \triangle FGD$ are similar and $CD/ID = DG/FG$. Therefore,

$$\frac{FG}{EG} = \frac{DG/EG}{DG/FG} = \frac{BD/ID}{CD/ID} = \frac{BD}{CD} = \frac{BF}{CE}.$$

69. (1998 IMO shortlisted problem) Let $ABCDEF$ be a convex hexagon such that

$$\angle B + \angle D + \angle F = 360^\circ \quad \text{and} \quad \frac{AB}{BC} \cdot \frac{CD}{DE} \cdot \frac{EF}{FA} = 1.$$

Prove that

$$\frac{BC}{CA} \cdot \frac{AE}{EF} \cdot \frac{FD}{DB} = 1.$$

Solution. Let P be such that $\angle FEA = \angle DEP$ and $\angle EFA = \angle EDP$, where P is on the opposite side of lines DE and CD as A. Then $\triangle FEA, \triangle DEP$ are similar. So

$$\frac{FA}{EF} = \frac{DP}{PE} \quad \text{and} \quad (*) \ \frac{EF}{ED} = \frac{EA}{EP}.$$

Since $\angle B + \angle D + \angle F = 360°$, we get $\angle ABC = \angle PDC$. Also,

$$\frac{AB}{BC} = \frac{DE \cdot FA}{CD \cdot EF} = \frac{DP}{CD}.$$

Then $\triangle ABC, \triangle PDC$ are similar. Consequently, we get $\angle BCA = \angle DCP$ and (**) $CB/CD = CA/CP$. Since $\angle FED = \angle AEP$, by (*), $\triangle FED, \triangle AEP$ are similar. Also, since $\angle BCD = \angle ACP$, by (**), $\triangle BCD, \triangle ACP$ are similar. So $AE/EF = PA/FD$ and $BC/CA = DB/PA$. Multiplying these and moving all factors to the left side, we get the desired equation.

Using complex numbers, we can get an algebraic solution. Let a, b, c, d, e, f denote the complex numbers corresponding to A, B, C, D, E, F, respectively. (The origin may be taken anywhere on the plane.) Since $ABCDEF$ is convex, $\angle B, \angle D$ and $\angle F$ are the arguments of the complex numbers $(a-b)/(c-b), (c-d)/(e-d)$ and $(e-f)/(a-f)$, respectively. Then the condition $\angle B + \angle D + \angle F = 360°$ implies that the product of these three complex numbers is a positive real number. It is equal to the product of their absolute values $AB/BC, CD/DE$ and EF/FA. Since $(AB/BC)(CD/DE)(EF/FA) = 1$, we have

$$\frac{a-b}{c-b} \cdot \frac{c-d}{e-d} \cdot \frac{e-f}{a-f} = 1.$$

So

$$\begin{aligned} 0 &= (a-b)(c-d)(e-f) - (c-b)(e-d)(a-f) \\ &= (b-c)(a-e)(f-d) - (a-c)(f-e)(b-d). \end{aligned}$$

Then

$$\frac{BC}{CA} \cdot \frac{AE}{EF} \cdot \frac{FD}{DB} = \left| \frac{b-c}{a-c} \cdot \frac{a-e}{f-e} \cdot \frac{f-d}{b-d} \right| = 1.$$

Similar Triangles

70. (1984 British Math Olympiad) P, Q, and R are arbitrary points on the sides BC, CA, and AB respectively of triangle ABC. Prove that the three circumcentres of triangles AQR, BRP, and CPQ form a triangle similar to triangle ABC.

Solution. Let the circumcenters of triangles AQR, BRP and CPQ be A', B' and C', respectively. A good drawing suggests the circles pass through a common point! To prove this, let circumcircles of triangles AQR and BRP intersect at R and X. Then $\angle QXR = 180° - \angle CAB = \angle ABC + \angle BCA$ and $\angle RXP = 180° - \angle ABC = \angle CAB + \angle BCA$. So $\angle PXQ = 360° - \angle QXR - \angle RXP = 180° - \angle BCA$, which implies X is on the circumcircle of triangle CPQ. Now

$$\begin{aligned} \angle C'A'B' &= \angle C'A'X + \angle XA'B' \\ &= \frac{1}{2}\angle QA'X + \frac{1}{2}\angle RA'X \\ &= \frac{1}{2}\angle QA'R = \angle CAB. \end{aligned}$$

Similarly, $\angle A'B'C' = \angle ABC$ and $\angle B'C'A' = \angle BCA$. So, triangles $A'B'C'$ and ABC are similar.

71. Hexagon $ABCDEF$ is inscribed in a circle so that $AB = CD = EF$. Let P, Q, R be the points of intersection of AC and BD, CE and DF, EA and FB respectively. Prove that triangles PQR and BDF are similar.

Solution. (Due to Ng Ka Wing) Let O be the center of the circle and let L, M, N be the projections of O on BD, DF, FB, respectively. Then L, M, N are midpoints of BD, DF, FB, respectively. Let S be the projection of O on AE. Since $AB = EF$, we get $FB = AE$ and hence

$ON = OS$. Let $\angle AOB = \angle COD = \angle EOF = 2\alpha$. Then $\angle RON = \frac{1}{2}\angle SON = \frac{1}{2}\angle ARB = \frac{1}{4}(\angle AOB + \angle EOF) = \alpha$. Hence, $ON/OR = \cos\alpha$. Similarly, $\angle POL = \angle QOM = \alpha$ and $OL/OP = OM/OQ = \cos\alpha$.

Next rotate $\triangle PQR$ around O at angle α so that the image Q' of Q lies on the line OM, the image R' of R lies on the line ON and the image P' of P lies on line OL. Then $ON/OR' = OL/OP' = OM/OQ' = \cos\alpha$. So $\triangle P'Q'R', \triangle LMN$ are similar. Since L, M, N are midpoints of BD, DF, FB, respectively, we have $\triangle LMN, \triangle BDF$ are similar. Therefore, $\triangle PQR, \triangle BDF$ are similar.

72. (1998 IMO shortlisted problem) Let $ABCD$ be a cyclic quadrilateral. Let E and F be variable points on the sides AB and CD, respectively, such that $AE : EB = CF : FD$. Let P be the point on the segment EF such that $PE : PF = AB : CD$. Prove that the ratio between the areas of triangles APD and BPC does not depend on the choice of E and F.

Solution. Let $[UVW]$ denote the area of $\triangle UVW$ and let $d(X, YZ)$ denote the distance from X to line YZ. We have $AE : EB = CF : FD = a : b$, where $a + b = 1$. Since $PE : PF = AB : CD$, we have

$$d(P, AD) = \frac{CD}{AB+CD} d(E, AD) + \frac{AB}{AB+CD} d(F, AD),$$

$$\begin{aligned}[APD] &= \frac{CD}{AB+CD}[AED] + \frac{AB}{AB+CD}[AFD] \\ &= \frac{a \cdot CD}{AB+CD}[ABD] + \frac{b \cdot AB}{AB+CD}[ACD],\end{aligned}$$

$$d(P, BC) = \frac{CD}{AB+CD} d(E, BC) + \frac{AB}{AB+CD} d(F, BC),$$

$$\begin{aligned}[BPC] &= \frac{CD}{AB+CD}[BEC] + \frac{AB}{AB+CD}[BFC] \\ &= \frac{b \cdot CD}{AB+CD}[BAC] + \frac{a \cdot AB}{AB+CD}[BDC].\end{aligned}$$

Since A, B, C, D are concyclic, $\sin\angle BAD = \sin\angle BCD$ and $\sin\angle ABC = \sin\angle ADC$, So,

$$\begin{aligned}\frac{[APD]}{[BPC]} &= \frac{a \cdot CD \cdot [ABD] + b \cdot AB \cdot [ACD]}{b \cdot CD \cdot [BAC] + a \cdot AB \cdot [BDC]} \\ &= \frac{a \cdot CD \cdot AB \cdot AD \cdot \sin\angle BAD + b \cdot AB \cdot CD \cdot AD \cdot \sin\angle ADC}{b \cdot CD \cdot AB \cdot BC \cdot \sin\angle ABC + a \cdot AB \cdot CD \cdot BC \cdot \sin\angle BCD} \\ &= \frac{AD}{BC} \cdot \frac{a \cdot \sin\angle BAD + b \cdot \sin\angle ADC}{b \cdot \sin\angle ABC + a \cdot \sin\angle BCD} = \frac{AD}{BC}.\end{aligned}$$

Tangent Lines

73. Two circles intersect at points A and B. An arbitrary line through B intersects the first circle again at C and the second circle again at D. The tangents to the first circle at C and to the second circle at D intersect at M. The parallel to CM which passes through the point of intersection of AM and CD intersects AC at K. Prove that BK is tangent to the second circle.

Solution. Let L be the intersection of AM and CD. Since

$$\begin{aligned}\angle CMD + \angle CAD &= \angle CMD + \angle CAB + \angle DAB \\ &= \angle CMD + \angle BCM + \angle BDM = 180^\circ,\end{aligned}$$

so A, C, M, D are concyclic. Since $LK \parallel MC$,

$$\begin{aligned}\angle LKC &= 180^\circ - \angle KCM = 180^\circ - \angle KCL - \angle LCM \\ &= 180^\circ - \angle ACB - \angle CAB = \angle CBA = \angle LBA.\end{aligned}$$

So A, B, L, K are concyclic. Then

$$\angle KBA = \angle KLA = \angle CMA = \angle CDA = \angle BDA.$$

Therefore, BK is tangent to the circle passing through A, B, D.

74. (1999 IMO) Two circles Γ_1 and Γ_2 are contained inside the circle Γ, and are tangent to Γ at the distinct points M and N, respectively.

Γ_1 passes through the center of Γ_2. The line passing through the two points of intersection of Γ_1 and Γ_2 meets Γ at A and B, respectively. The lines MA and MB meets Γ_1 at C and D, respectively. Prove that CD is tangent to Γ_2.

Solution. (Due to Wong Chun Wai) Let X, Y be the centers of Γ_1, Γ_2, respectively. Extend $\overrightarrow{YX}$ to meet Γ_2 at Q. Join AN to meet Γ_2 at E. Since AB is the radical axis of Γ_1, Γ_2, so $AC \times AM = AE \times AN$. This implies C, M, N, E are concyclic. Let U be the intersection of line CE with the tangent to Γ_1 at M. Then $\angle UCM = \angle ENM = \angle ANM = \angle UMC$. So CE is tangent to Γ_1. Similarly, CE is tangent to Γ_2. Now $YE = YQ$ and

$$\begin{aligned}\angle CYE = 90^\circ - \angle ECY &= 90^\circ - \frac{1}{2}\angle CXY \\ &= 90^\circ - (90^\circ - \angle CYQ) = \angle CYQ.\end{aligned}$$

These imply $\triangle CYE, \triangle CYQ$ are congruent. Hence $\angle CQY = \angle CEY = 90^\circ$. Similarly $\angle DQY = 90^\circ$. Therefore, CD is tangent to Γ_2.

75. (Proposed by India for 1992 IMO) Circles G_1 and G_2 touch each other externally at a point W and are inscribed in a circle G. A, B, C are points on G such that A, G_1 and G_2 are on the same side of chord BC, which is also tangent to G_1 and G_2. Suppose AW is also tangent to G_1 and G_2. Prove that W is the incenter of triangle ABC.

Solution. Let P and Q be the points of tangency of G_1 with BC and arc BAC, respectively. Let D be the midpoint of the complementary arc BC of G (not containing A) and L be a point on G_1 so that DL is tangent to G_1 and intersects segment PC. Considering the homothety with center Q that maps G_1 onto G, we see that Q, P, D are collinear because the tangent at P (namely BC) and the tangent at D are parallel. Since $\angle BQD, \angle CBD$ subtend equal arcs, $\triangle BQD, \triangle PBD$ are similar. Hence $DB/DP = DQ/DB$. By the intersecting chord theorem, $DB^2 = DP \cdot DQ = DL^2$. So $DL = DB = DC$. Then D has the same power $DB^2 = DC^2$ with respect to G_1 and G_2. Hence D is on the radical axis AW of G_1 and G_2. So $L = W$ and DW is tangent to G_1 and G_2.

Since D is the midpoint of arc BC, so AW bisects $\angle BAC$. Also,

$$\angle ABW = \angle BWD - \angle BAD = \angle WBD - \angle CBD = \angle CBW$$

and BW bisects $\angle ABC$. Therefore W is the incenter of $\triangle ABC$.

Comments. The first part of the proof can also be done by inversion with respect to the circle centered at D and of radius $DB = DC$. It maps arc BC onto the chord BC. Both G_1 and G_2 are invariant because the power of D with respect to them is $DB^2 = DC^2$. Hence W is fixed and so DW is tangent to both G_1 and G_2.

Locus

76. Perpendiculars from a point P on the circumcircle of $\triangle ABC$ are drawn to lines AB, BC with feet at D, E, respectively. Find the locus of the circumcenter of $\triangle PDE$ as P moves around the circle.

Solution. Since $\angle PDB + \angle PEB = 180^\circ$, P, D, B, E are concyclic. Hence, the circumcircle of $\triangle PDE$ passes through B always. Then PB is a diameter and the circumcenter of $\triangle PDE$ is at the midpoint M of PB. Let O be the circumcenter of $\triangle ABC$, then $OM \perp PB$. It follows that the locus of M is the circle with OB as a diameter.

77. Suppose A is a point inside a given circle and is different from the center. Consider all chords (excluding the diameter) passing through A. What is the locus of the intersection of the tangent lines at the endpoints of these chords?

Solution. (Due to Wong Him Ting) Let O be the center and and r be the radius. Let A' be the point on OA extended beyond A such that $OA \times OA' = r^2$. Suppose BC is one such chord passing through A and the tangents at B and C intersect at D'. By symmetry, D' is on the line OD, where D is the midpoint of BC. Since $\angle OBD' = 90^\circ$, $OD \times OD' = OB^2 (= OA \times OA'.)$ So $\triangle OAD$ is similar to $\triangle OD'A'$. Since $\angle ODA = 90^\circ$, D' is on the line L perpendicular to OA at A'.

Conversely, for D' on L, let the chord through A perpendicular to OD' intersect the circle at B and C. Let D be the intersection of the chord with OD'. Now $\triangle OAD, \triangle OD'A'$ are similar right triangles. So $OD \times OD' = OA \times OA' = OB^2 = OC^2$, which implies $\angle OBD' = \angle OCD' = 90°$. Therefore, D' is on the locus. This shows the locus is the line L.

78. Given $\triangle ABC$. Let line EF bisects $\angle BAC$ and $AE \cdot AF = AB \cdot AC$. Find the locus of the intersection P of lines BE and CF.

Solution. For such a point P, since $AB/AE = AF/AC$ and $\angle BAE = \angle FAC$, so $\triangle BAE, \triangle FAC$ are similar. Then $\angle AEP = \angle PCA$. So A, E, C, P are concyclic. Hence $\angle BPC = \angle CAE = \angle BAC/2$. Therefore, P is on the circle $\mathcal{C}$ whose points X satisfy $\angle BXC = \angle BAC/2$ and whose center is on the same side of line BC as A.

Conversely, for P on $\mathcal{C}$, Let BP, CP intersect the angle bisector of $\angle BAC$ at E, F, respectively. Since $\angle BPC = \angle BCA/2$, so $\angle EPF = \angle EAC$. Hence A, E, C, P are concyclic. So $\angle BEA = \angle FCA$. Also $\angle BAE = \angle FAC$. So $\triangle BAE, \triangle FAC$ are similar. Then $AB \cdot AC = AE \cdot AF$. Therefore, the locus of P is the circle $\mathcal{C}$.

79. (1996 Putnam Exam) Let C_1 and C_2 be circles whose centers are 10 units apart, and whose radii are 1 and 3. Find the locus of all points M for which there exists points X on C_1 and Y on C_2 such that M is the midpoint of the line segment XY.

Solution. (Due to Poon Wai Hoi) Let O_1, O_2 be the centers of C_1, C_2, respectively. If we fix Y on C_2, then as X moves around C_1, M will trace a circle Γ_Y with radius $\frac{1}{2}$ centered at the midpoint m_Y of O_1Y. As Y moves around C_2, m_Y will trace a circle of radius $\frac{3}{2}$ centered at the midpont P of O_1O_2. So the locus is the solid annulus centered at P with inner radius $\frac{3}{2} - \frac{1}{2} = 1$ and outer radius $\frac{3}{2} + \frac{1}{2} = 2$.

<u>*Collinear or Concyclic Points*</u>

80. (1982 IMO) Diagonals AC and CE of the regular hexagon $ABCDEF$ are divided by the inner points M and N, respectively, so that

$$\frac{AM}{AC} = \frac{CN}{CE} = r.$$

Determine r if B, M and N are collinear.

Solution. (Due to Lee Tak Wing) Let $AC = x$, then $BC = x/\sqrt{3}$, $CN = xr$, $CM = x(1-r)$. Let $[XYZ]$ denote the area of $\triangle XYZ$. Since $\angle NCM = 60°$, $\angle BCM = 30°$ and $[BCM] + [CMN] = [BCN]$, so

$$\frac{x^2(1-r)\sin 30°}{2\sqrt{3}} + \frac{x^2r(1-r)\sin 60°}{2} = \frac{x^2r}{2\sqrt{3}}.$$

Cancelling x^2 and solving for r, we get $r = \dfrac{1}{\sqrt{3}}$.

81. (1965 Putnam Exam) If A, B, C, D are four distinct points such that every circle through A and B intersects or coincides with every circle through C and D, prove that the four points are either collinear or concyclic.

Solution. Suppose A, B, C, D are neither concyclic nor collinear. Then the perpendicular bisector p of AB cannot coincide with the perpendicular bisector q of CD. If lines p and q intersect, their common point is the center of two concentric circles, one through A and B, the other through C and D, a contradiction. If lines p and q are parallel, then lines AB and CD are also parallel. Consider points P and Q on p and q, respectively, midway between the parallel lines AB and CD. Clearly, the circles through A, B, P and C, D, Q have no common point, again a contradiction.

82. (1957 Putnam Exam) Given an infinite number of points in a plane, prove that if all the distances between every pair are integers, then the points are collinear.

Solution. Suppose there are three noncollinear points A, B, C such that $AB = r$ and $AC = s$. Observe that if P is one of the other points, then bt the triangle inequality, $|PA - PB| = 0, 1, 2, \ldots, r$. Hence P would be on the line H_0 joining A, B or on one of the hyperbolas $H_i = \{X : |XA - XB| = i\}$ for $i = 1, 2, \ldots, r-1$ or on the perpendicular bisector H_r of AB. Similarly, $|PA - PC| = 0, 1, 2, \ldots, s$. So P is on one of the sets $K_j = \{X : |XA - XC| = j\}$ for $j = 0, 1, \ldots, s$. Since lines AB and AC are distinct, every intersection $H_i \cap K_j$ is only a finite set. So there can only be finitely many points that are integral distances from A, B, C, a contradiction. Therefore, the given points must be collinear.

83. (1995 IMO shortlisted problem) The incircle of triangle ABC touches BC, CA and AB at D, E and F respectively. X is a point inside triangle ABC such that the incircle of triangle XBC touches BC at D also, and touches CX and XB at Y and Z respectively. Prove that $EFZY$ is a cyclic quadrilateral.

    Solution. If $EF \parallel BC$, then $AB = AC$ and AD is an axis of symmetry of $EFZY$. Hence $EFZY$ is a cyclic quadrilateral. If lines EF and BC intersect at P, then by Menelaus' theorem, $(AF \cdot BP \cdot CE)/(FB \cdot PC \cdot EA) = 1$. Since $BZ = BD = BF, CY = CD = CE$ and $AF/EA = 1 = XZ/YX$, we get $(XZ \cdot BP \cdot CY)/(ZB \cdot PC \cdot YX) = 1$. By the converse of the Menelaus' theorem, Z, Y, P are collinear. By the intersecting chord theorem, $PE \cdot PF = PD^2 = PY \cdot PZ$. Hence $EFZY$ is a cyclic quadrilateral by the converse of the intersecting chord theorem.

84. (1998 IMO) In the convex quadrilateral $ABCD$, the diagonals AC and BD are perpendicular and the opposite sides AB and DC are not parallel. Suppose the point P, where the perpendicular bisectors of AB and DC meet, is inside $ABCD$. Prove that $ABCD$ is a cyclic quadrilateral if and only if the triangles ABP and CDP have equal areas.

    Solution. (Due to Leung Wing Chung) Set the origin at P. Suppose A and C are on the line $y = p$ and B and D are on the line $x = q$. Let $AP = BP = r$, $CP = DP = s$. Then the coordinates of A, B, C, D are

$$(-\sqrt{r^2 - p^2}, p), (q, \sqrt{r^2 - q^2}), (\sqrt{s^2 - p^2}, p), (q, -\sqrt{s^2 - q^2}),$$

    respectively. Now $\triangle ABP$, $\triangle CDP$ have equal areas if and only if

$$\frac{1}{2}\begin{vmatrix} -\sqrt{r^2 - p^2} & p \\ q & \sqrt{r^2 - q^2} \end{vmatrix} = \frac{1}{2}\begin{vmatrix} \sqrt{s^2 - p^2} & p \\ q & -\sqrt{s^2 - q^2} \end{vmatrix},$$

i.e. $(-\sqrt{r^2 - p^2}\sqrt{r^2 - q^2} - pq)/2 = (-\sqrt{s^2 - p^2}\sqrt{s^2 - q^2} - pq)/2$. Since $f(x) = (-\sqrt{x^2 - p^2}\sqrt{x^2 - q^2} - pq)/2$ is strictly decreasing when $x \geq |p|$ and $|q|$, the determinants are equal if and only if $r = s$, which is equivalent to $ABCD$ cyclic.

85. (1970 Putnam Exam) Show that if a convex quadrilateral with side-lengths a, b, c, d and area $\sqrt{abcd}$ has an inscribed circle, then it is a cyclic quadrilateral.

    Solution. Since the quadrilateral has an inscribed circle, we have $a + c = b + d$. Let k be the length of a diagonal and angles α and β selected so that

$$k^2 = a^2 + b^2 - 2ab\cos\alpha = c^2 + d^2 - 2cd\cos\beta.$$

    If we subtract $(a - b)^2 = (c - d)^2$ and divide by 2, we get the equation (*) $ab(1 - \cos\alpha) = cd(1 - \cos\beta)$. From the area $(ab\sin\alpha + cd\sin\beta)/2 = \sqrt{abcd}$, we get

$$4abcd = a^2b^2(1 - \cos^2\alpha) + c^2d^2(1 - \cos^2\beta) + 2abcd\sin\alpha\sin\beta.$$

    Using (*), we can cancel $abcd$ to obtain the equation

$$\begin{aligned} 4 &= (1 + \cos\alpha)(1 - \cos\beta) + (1 + \cos\beta)(1 - \cos\alpha) + 2\sin\alpha\sin\beta \\ &= 2 - 2\cos(\alpha + \beta), \end{aligned}$$

    which implies $\alpha + \beta = 180°$. Therefore, the quadrilateral is cyclic.

Concurrent Lines

86. In $\triangle ABC$, suppose $AB > AC$. Let P and Q be the feet of the perpendiculars from B and C to the angle bisector of $\angle BAC$, respectively. Let D be on line BC such that $DA \perp AP$. Prove that lines BQ, PC and AD are concurrent.

    Solution. Let M be the intersection of PC and AD. Let B' be the mirror image of B with respect to line AP. Since $BB' \perp AP$ and

$AD \perp AP$, so $BB' \parallel AD$. Then $\triangle BCB', \triangle DAC$ are similar. Since P is the midpoint of BB', so PC intersects AD at its midpoint M. Now

$$\frac{AQ}{PQ} = \frac{MC}{PC} = \frac{AM}{B'P} = \frac{AM}{BP}.$$

$\triangle BPQ, \triangle MAQ$ are similar. This implies $\angle BQP = \angle MQA$. So line BQ passes through M, too.

87. (1990 Chinese National Math Competition) Diagonals AC and BD of a cyclic quadrilateral $ABCD$ meets at P. Let the circumcenters of $ABCD, ABP, BCP, CDP$ and DAP be O, O_1, O_2, O_3 and O_4, respectively. Prove that OP, O_1O_3, O_2O_4 are concurrent.

Solution. Let line PO_2 intersect the circumcircle of $\triangle BCP$ and segment AD at points Q and R, respectively. Now $\angle PDR = \angle BCA = \angle PQB$ and $\angle DPR = \angle QPB$. So $\angle DRP = \angle QBP = 90°$ and $PO_2 \perp AD$. Next circumcircles of $ABCD$ and DAP share the common chord AD, so $OO_4 \perp AD$. Hence PO_2 and OO_4 are parallel. Similarly, PO_4 and OO_2 are parallel. So PO_2OO_4 is a parallelogram and diagonal O_2O_4 passes through the midpoint G of OP. Similarly, PO_1OO_3 is a parallelogram and diagonal O_1O_3 passes through G. Therefore, OP, O_1O_3, O_2O_4 concur at G.

88. (1995 IMO) Let A, B, C and D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at the points X and Y. The line XY meets BC at the point Z. Let P be a point on the line XY different from Z. The line CP intersects the circle with diameter AC at the points C and M, and the line BP intersects the circle with diameter BD at the points B and N. Prove that the lines AM, DN and XY are concurrent.

Solution 1. (Due to Yu Chun Ling) Let AR be parallel to BP and DR' be parallel to CP, where R and R' are points on line XY. Since $BZ \cdot ZD = XZ \cdot ZY = CZ \cdot ZA$, we get $BZ/AZ = CZ/DZ$. Since $\triangle CZP$ is similar to $\triangle DZR'$ and $\triangle BZP$ is similar to $\triangle AZR$, so

$$\frac{ZP}{ZR} = \frac{BZ}{AZ} = \frac{CZ}{DZ} = \frac{ZP}{ZR'}.$$

Hence R and R' must coincide. Therefore, $\triangle BPC$ is similar to $\triangle ARD$.

Since $XY \perp AD$, $AM \perp CM$, $CM \parallel DR$, $DN \perp BN$ and $BN \parallel AR$, the lines AM, DN, XY are the extensions of the altitudes of $\triangle ARD$, hence they must be concurrent.

Solution 2. (Due to Mok Tze Tao) Set the origin at Z and the x-axis on line AD. Let the coordinates of the circumcenters of triangles AMC and BND be $(x_1, 0)$ and $(x_2, 0)$, and the circumradii be r_1 and r_2, respectively. Then the coordinates of A and C are $(x_1 - r_1, 0)$ and $(x_1 + r_1, 0)$, respectively. Let the coordinates of P be $(0, y_0)$. Since $AM \perp CP$ and the slope of CP is $-\dfrac{y_0}{x_1 + r_1}$, the equation of AM works out to be $(x_1 + r_1)x - y_0 y = x_1^2 - r_1^2$. Let Q be the intersection of AM with XY, then Q has coordinates $(0, \dfrac{r_1^2 - x_1^2}{y_0})$. Similarly, let Q' be the intersection of DN with XY, then Q' has coordinates $(0, \dfrac{r_2^2 - x_2^2}{y_0})$. Since $r_1^2 - x_1^2 = ZX^2 = r_2^2 - x_2^2$, so $Q = Q'$.

Solution 3. Let AM intersect XY at Q and DN intersect XY at Q'. Observe that the right triangles AZQ, AMC, PZC are similar, so $AZ/QZ = PZ/CZ$. Then $QZ = AZ \cdot CZ/PZ = XZ \cdot YZ/PZ$. Similarly, $Q'Z = XZ \cdot YZ/PZ$. Therefore $Q = Q'$.

89. AD, BE, CF are the altitudes of $\triangle ABC$. If P, Q, R are the midpoints of DE, EF, FD, respectively, then show that the perpendicular from P, Q, R to AB, BC, CA, respectively, are concurrent.

Solution. Let Y be the foot of the perpendicular from Q to BC and H be the orthocenter of $\triangle ABC$. Note that $\angle ACF = 90° - \angle CAB = \angle ABE$. Since $\angle CEH = 90° = \angle CDH$, C, D, H, E are concyclic. So $\angle ACF = \angle EDH$. Now $AH \parallel QY$, $ED \parallel QR$, so $\angle EDH = \angle RQY$. Hence, $\angle RQY = \angle EDH = \angle ACF$. Similarly, $\angle ABE = \angle FDH = \angle PQY$. Next, since $\angle ACF = \angle ABE$, QY bisects $\angle PQR$. From these, it follows the perpendiculars from P, Q, R to AB, BC, CA concur at the incenter of $\triangle PQR$.

90. (1988 Chinese Math Olympiad Training Test) $ABCDEF$ is a hexagon

inscribed in a circle. Show that the diagonals AD, BE, CF are concurrent if and only if $AB \cdot CD \cdot EF = BC \cdot DE \cdot FA$.

Solution. (Due to Yu Ka Chun) Suppose AD, BE, CF concurs at X. From similar triangles ABX and EDX, we get $AB/DE = BX/DX$. Similarly, $CD/FA = DX/FX$ and $EF/BC = FX/BX$. Multiplying thse, we get $(AB \cdot CD \cdot EF)/(DE \cdot FA \cdot BC) = 1$, so $AB \cdot CD \cdot EF = BC \cdot DE \cdot FA$.

For the converse, we use the so-called *method of false position.* Suppose (*) $AB \cdot CD \cdot EF = BC \cdot DE \cdot FA$ and AD intersect BE at X. Now let CX meet the circle again at F'. By the first part, we get $AB \cdot CD \cdot EF' = BC \cdot DE \cdot F'A$. Dividing this by (*), we have $EF'/EF = F'A/FA$. If F' is on open arc AF, then $F'A < FA$ and $EF < EF'$ yielding $F'A/FA < 1 < EF'/EF$, a contradiction. If F' is on the open arc EF, then $FA < F'A$ and $EF' < EF$ yielding $EF'/EF < 1 < F'A/FA$, a contradiction. So $F' = F$.

Alternatively, we can use Ceva's theorem and its converse. Let AC and BE meet at G, CE and AD meet at H, EA and CF meet at I. Let h, k be the distances from A, C to BE, respectively. Then

$$\frac{AG}{CG} = \frac{h}{k} = \frac{AB \sin \angle ABG}{BC \sin \angle CBG}.$$

Similarly,

$$\frac{CH}{EH} = \frac{CD \sin \angle CDH}{DE \sin \angle EDH} \quad \text{and} \quad \frac{EI}{AI} = \frac{EF \sin \angle EFI}{FA \sin \angle AFI}.$$

Now $\angle ABG = \angle EDH$, $\angle CBG = \angle EFI$, $\angle CDH = \angle AFI$. By Ceva's theorem and its converse, AD, BE, CF are concurrent if and only if

$$1 = \frac{AG \cdot CH \cdot EI}{CG \cdot EH \cdot AI} = \frac{AB \cdot CD \cdot EF}{BC \cdot DE \cdot FA}.$$

91. A circle intersects a triangle ABC at six points $A_1, A_2, B_1, B_2, C_1, C_2$, where the order of appearance along the triangle is $A, C_1, C_2, B, A_1, A_2, C, B_1, B_2, A$. Suppose B_1C_1, B_2C_2 meets at X, C_1A_1, C_2A_2 meets at Y and A_1B_1, A_2B_2 meets at Z. Show that AX, BY, CZ are concurrent.

Solution. Let D be the intersection of AX and B_1C_2. Since AX, B_1C_1, B_2C_2 are concurrent, by (the trigonometric form of) Ceva's theorem,

$$1 = \frac{DC_2 \cdot B_1B_2 \cdot AC_1}{DB_1 \cdot AB_2 \cdot C_1C_2} = \frac{\sin C_2AD \cdot \sin B_2B_1C_1 \cdot \sin B_1C_2B_2}{\sin DAB_1 \cdot \sin C_1B_1C_2 \cdot \sin B_2C_2C_1}.$$

Then $\dfrac{\sin BAX}{\sin XAC} = \dfrac{\sin C_2AD}{\sin DAB_1} = \dfrac{\sin C_1B_1C_2 \cdot \sin B_2C_2C_1}{\sin B_2B_1C_1 \cdot \sin B_1C_2B_2}$. Similarly,

$$\frac{\sin CBY}{\sin YBA} = \frac{\sin A_1C_1A_2 \cdot \sin C_2A_2A_1}{\sin C_2C_1A_1 \cdot \sin C_1A_2C_2},$$

$$\frac{\sin ACZ}{\sin ZCB} = \frac{\sin B_1A_1B_2 \cdot \sin A_2B_2A_1}{\sin A_2A_1B_1 \cdot \sin A_1B_2A_2}.$$

Using $\angle C_1B_1C_2 = \angle C_1A_2C_2$ and similar angle equality, we see that the product of the three equations involving X, Y, Z above is equal to 1. By the converse of the trigonometric form of Ceva's theorem, we see that AX, BY, CZ are concurrent.

92. (1995 IMO shortlisted problem) A circle passing through vertices B and C of triangle ABC intersects sides AB and AC at C' and B', respectively. Prove that BB', CC' and HH' are concurrent, where H and H' are the orthocenters of triangles ABC and $AB'C'$, respectively.

Solution. (Due to Lam Po Leung) Let $d(X, L)$ denote the distance from a point X to a line L. For the problem, we will use the following lemma.

Lemma. Let lines L_1, L_2 intersect at P (forming four angles with vertex P). Suppose H, H' lie on an opposite pair of these angles. If $d(H, L_1)/d(H', L_1) = d(H, L_2)/d(H', L_2)$, then H, P, H' are collinear.

Proof. Let HH' intersect L_1, L_2 at X, Y, respectively. Then

$$\frac{HH'}{H'X} = \frac{HX}{H'X} + 1 = \frac{d(H, L_1)}{d(H', L_1)} + 1 = \frac{d(H, L_2)}{d(H', L_2)} + 1 = \frac{HY}{H'Y} + 1 = \frac{HH'}{H'Y}.$$

So $X = Y$ is on both L_1 and L_2, hence it is P. Therefore, H, P, H' are collinear.

For the problem, let BB', CC' intersect at P. Since $\angle ABH = 90^\circ - \angle A = \angle AC'H'$, so $BH \parallel C'H'$. Similarly, $CH \parallel B'H'$. Let BH, CC' intersect at L and CH, BB' intersect at K. Now

$$\begin{aligned}\angle PBH &= \angle ABH - \angle C'BP = (90^\circ - \angle A) - \angle B'CP \\ &= \angle ACH - \angle B'CP = \angle PCH.\end{aligned}$$

So, K, B, C, L are concyclic. Then $\triangle LHK, \triangle BHC$ are similar. Also, $\triangle BHC, \triangle B'H'C'$ are similar because

$$\angle CBH = 90^\circ - \angle ACB = 90^\circ - \angle AC'B' = \angle C'B'H'$$

and similarly $\angle BCH = \angle B'C'H'$. Therefore $\triangle LHK, \triangle B'H'C'$ are similar. So $KH/B'H' = LH/C'H'$. Since $BH \parallel C'H'$ and $CH \parallel B'H'$, so $d(H, BB')/d(H', BB') = d(H, CC')/d(H', CC')$. By the lemma, HH' also passes through P.

Perpendicular Lines

93. (1998 APMO) Let ABC be a triangle and D the foot of the altitude from A. Let E and F be on a line passing through D such that AE is perpendicular to BE, AF is perpendicular to CF, and E and F are different from D. Let M and N be the midpoints of the line segments BC and EF, respectively. Prove that AN is perpendicular to NM.

Solution. (Due to Cheung Pok Man) There are many different pictures, so it is better to use coordinate geometry to cover all cases. Set A at the origin and let $y = b \neq 0$ be the equation of the line through D, E, F. (Note the case $b = 0$ implies $D = E = F$, which is not allowed.) Let the coordinates of D, E, F be $(d, b), (e, b), (f, b)$, respectively. Since $BE \perp AE$ and slope of AE is b/e, so the equation of line AE is $ex + by - (b^2 + e^2) = 0$. Similarly, the equation of line CF is $fx + by - (b^2 + f^2) = 0$ and the equation of line BC is $dx + by - (b^2 + d^2) = 0$.

From these, we found the coordinates of B, C are $(d+e, b-\frac{de}{b}), (d+f, b-\frac{df}{b})$, respectively. Then the coordinates of M, N are $(d+\frac{e+f}{2}, b-\frac{de+df}{2b}), (\frac{e+f}{2}, b)$, respectively. So the slope of AN is $2b/(e+f)$ and the slope of MN is $-(\frac{de+df}{2b})/d = -\frac{e+f}{2b}$. The product of these slopes is -1. Therefore, $AN \perp MN$.

94. (2000 APMO) Let ABC be a triangle. Let M and N be the points in which the median and the angle bisector, respectively, at A meet the side BC. Let Q and P be the points in which the perpendicular at N to NA meets MA and BA, respectively, and O the point in which the perpendicular at P to BA meets AN produced. Prove that QO is perpendicular to BC.

Solution 1. (Due to Wong Chun Wai) Set the origin at N and the x-axis on line NO. Let the equation of line AB be $y = ax + b$, then the equation of lines AC and PO are $y = -ax - b$ and $y = -\dfrac{1}{a}x + b$, respectively. Let the equation of BC be $y = cx$. Then B has coordinates $(\dfrac{b}{c-a}, \dfrac{bc}{c-a})$, C has coordinates $(-\dfrac{b}{c+a}, -\dfrac{bc}{c+a})$, M has coordinates $(\dfrac{ab}{c^2-a^2}, \dfrac{abc}{c^2-a^2})$, A has coordinates $(-\dfrac{b}{a}, 0)$, O has coordinates $(ab, 0)$ and Q has coordinates $(0, \dfrac{ab}{c})$. Then BC has slope c and QO has slope $-\dfrac{1}{c}$. Therefore, $QO \perp BC$.

Solution 2. (Due to Poon Wai Hoi) The case $AB = AC$ is clear. Without loss of generality, we may assume $AB > AC$. Let AN intersect the circumcircle of $\triangle ABC$ at D. Then

$$\angle DBC = \angle DAC = \frac{1}{2}\angle BAC = \angle DAB = \angle DCB.$$

So $DB = DC$ and $MD \perp BC$.

With A as the center of homothety, shrink D to O, B to B' and C to C'. Then $\angle OB'C' = \frac{1}{2}\angle BAC = \angle OC'B'$ and $BC \parallel B'C'$. Let $B'C'$ cut PN at K. Then $\angle OB'K = \angle DAB = \angle OPK$. So P, B', O, K are concyclic. Hence $\angle B'KO = \angle B'PO = 90^\circ$ and so $B'K = C'K$. Since

$BC \parallel B'C'$, this implies A, K, M are collinear. Therefore, $K = Q$. Since $\angle B'KO = 90°$ and $BC \parallel B'C'$, we get $QO \perp BC$.

95. Let BB' and CC' be altitudes of triangle ABC. Assume that $AB \neq AC$. Let M be the midpoint of BC, H the orthocenter of ABC and D the intersection of $B'C'$ and BC. Prove that $DH \perp AM$.

Solution. Let A' be the foot of the altitude from A to BC. Since $A', B', C'M$ lie on the nine-point circle of $\triangle ABC$, so by the intersecting chord theorem, $DB' \cdot DC' = DA' \cdot DM$. Since $\angle AC'H = 90° = \angle AB'H$, points A, C', H, B' lie on a circle Γ_1 with the midpoint X of AH as center. Since $\angle HA'M = 90°$, so the circle Γ_2 through H, A', M has the midpoint Y of HM as center. Since $DB' \cdot DC' = DA' \cdot DM$, the powers of D with respect to Γ_1 and Γ_2 are the same. So D (and H) are on the radical axis of Γ_1, Γ_2. Then $DH \perp XY$. By the midpoint theorem, $XY \parallel AM$. Therefore, $DH \perp AM$.

96. (1996 Chinese Team Selection Test) The semicircle with side BC of $\triangle ABC$ as diameter intersects sides AB, AC at points D, E, respectively. Let F, G be the feet of the perpendiculars from D, E to side BC respectively. Let M be the intersection of DG and EF. Prove that $AM \perp BC$.

Solution. Let H be the foot of the perpendicular from A to BC. Now $\angle BDC = 90° = \angle BEC$. So $DF = BD \sin B = BC \cos B \sin B$ and similarly $EG = BC \cos C \sin C$. Now

$$\frac{GM}{MD} = \frac{EG}{FD} = \frac{\cos C \sin C}{\cos B \sin B} = \frac{\cos C}{\cos B}\frac{AB}{AC}.$$

Since $BH = AB \cos B,\ HG = AE \cos C$, we get

$$\frac{BH}{HG} = \frac{AB \cos B}{AE \cos C} = \frac{AC \cos B}{AD \cos C} \quad \text{and} \quad \frac{BH}{HG}\frac{GM}{MD}\frac{DA}{AB} = 1.$$

By the converse of Menelaus' theorem on $\triangle BDG$, points A, M, H are collinear. Therefore, $AM \perp BC$.

97. (1985 IMO) A circle with center O passes through the vertices A and C of triangle ABC and intersects the segments AB and AC again at distinct points K and N, respectively. The circumcircles of triangles ABC and KBN intersect at exactly two distinct points B and M. Prove that $OM \perp MB$.

Solution. Let CM intersect the circle with center O at a point L. Since $\angle BMC = 180° - \angle BAC = 180° - \angle KAC = \angle KLC$, so BM is parallel to KL. Now

$$\begin{aligned}\angle LKM &= \angle LKN + \angle NKM = \angle LCN + \angle NBM \\ &= 180° - \angle BMC = \angle BAC = \angle KLM.\end{aligned}$$

Then $KM = LM$. Since $KO = LO$, so $OM \perp KL$. Hence $OM \perp BM$.

98. (1997 Chinese Senoir High Math Competition) A circle with center O is internally tangent to two circles inside it at points S and T. Suppose the two circles inside intersect at M and N with N closer to ST. Show that $OM \perp MN$ if and only if S, N, T are collinear.

Solution. (Due to Leung Wai Ying) Consider the tangent lines at S and at T. (Suppose they are parallel, then S, O, T will be collinear so that M and N will be equidistant from ST, contradicting N is closer to ST.) Let the tangent lines meet at K, then $\angle OSK = 90° = \angle OTK$ implies O, S, K, T lie on a circle with diameter OK. Also, $KS^2 = KT^2$ implies K is on the radical axis MN of the two inside circles. So M, N, K are collinear.

If S, N, T are collinear, then

$$\angle SMT = \angle SMN + \angle TMN = \angle NSK + \angle KTN = 180° - \angle SKT.$$

So M, S, K, T, O are concyclic. Then $\angle OMN = \angle OMK = \angle OSK = 90°$.

Conversely, if $OM \perp MN$, then $\angle OMK = 90° = \angle OSK$ implies M, S, K, T, O are concyclic. Then

$$\begin{aligned}\angle SKT &= 180° - \angle SMT \\ &= 180° - \angle SMN - \angle TMN \\ &= 180° - \angle NSK - \angle KTN.\end{aligned}$$

Thus, $\angle TNS = 360° - \angle NSK - \angle SKT - \angle KTN = 180°$. Therefore, S, N, T are collinear.

99. AD, BE, CF are the altitudes of $\triangle ABC$. Lines EF, FD, DE meet lines BC, CA, AB in points L, M, N, respectively. Show that L, M, N are collinear and the line through them is perpendicular to the line joining the orthocenter H and circumcenter O of $\triangle ABC$.

Solution. Since $\angle ADB = 90° = \angle AEB$, A, B, D, E are concyclic. By the intersecting chord theorem, $NA \cdot NB = ND \cdot NE$. So the power of N with respect to the circumcircles of $\triangle ABC, \triangle DEF$ are the same. Hence N is on the radical axis of these circles. Similarly, L, M are also on this radical axis. So L, M, N are collinear.

Since the circumcircle of $\triangle DEF$ is the nine-point circle of $\triangle ABC$, the center N of the nine-point circle is the midpoint of H and O. Since the radical axis is perpendicular to the line of centers O and N, so the line through L, M, N is perpendicular to the line HO.

Geometric Inequalities, Maximum/Minimum

100. (1973 IMO) Let $P_1, P_2, \ldots, P_{2n+1}$ be distinct points on some half of the unit circle centered at the origin O. Show that

$$|\overrightarrow{OP_1} + \overrightarrow{OP_2} + \cdots + \overrightarrow{OP_{2n+1}}| \geq 1.$$

Solution. When $n = 0$, then $|\overrightarrow{OP_1}| = 1$. Suppose the case $n = k$ is true. For the case $n = k+1$, we may assume $P_1, P_2, \ldots, P_{2k+3}$ are arranged clockwise. Let $\overrightarrow{OR} = \overrightarrow{OP_2} + \cdots + \overrightarrow{OP_{2k+2}}$ and $\overrightarrow{OS} = \overrightarrow{OP_1} + \overrightarrow{OP_{2k+3}}$. By the case $n = k$, $|\overrightarrow{OR}| \geq 1$. Also, $\overrightarrow{OR}$ lies inside $\angle P_1OP_{2k+3}$. Since $|\overrightarrow{OP_1}| = 1 = |\overrightarrow{OP_{2k+3}}|$, OS bisects $\angle P_1OP_{2k+3}$. Hence $\angle ROS \leq 90°$. Then $|\overrightarrow{OP_1} + \cdots + \overrightarrow{OP_{2k+3}}| = |\overrightarrow{OR} + \overrightarrow{OS}| \geq |\overrightarrow{OR}| \geq 1$.

101. Let the angle bisectors of $\angle A, \angle B, \angle C$ of triangle ABC intersect its circumcircle at P, Q, R, respectively. Prove that

$$AP + BQ + CR > BC + CA + AB.$$

Solution. (Due to Lau Lap Ming) Since $\angle ABQ = \angle CBQ$, we have $AQ = CQ$. By cosine law,

$$AQ^2 = AB^2 + BQ^2 - 2AB \cdot BQ \cos \angle ABQ$$
$$CQ^2 = CB^2 + BQ^2 - 2CB \cdot BQ \cos \angle CBQ.$$

If $AB \neq CB$, then subtracting these and simplifying, we get $AB + CB = 2BQ \cos \angle ABQ < 2BQ$. If $AB = CB$, then BQ is a diameter and we again get $AB + CB = 2AB < 2BQ$. Similarly, $BC + AC < 2CR$ and $CA + BA < 2AP$. Adding these inequalities and dividing by 2, we get the desired inequality.

102. (1997 APMO) Let ABC be a triangle inscribed in a circle and let $l_a = m_a/M_a, l_b = m_b/M_b, l_c = m_c/M_c$, where m_a, m_b, m_c are the lengths of the angle bisectors (internal to the triangle) and M_a, M_b, M_c are the lengths of the angle bisectors extended until they meet the circle. Prove that

$$\frac{l_a}{\sin^2 A} + \frac{l_b}{\sin^2 B} + \frac{l_c}{\sin^2 C} \geq 3,$$

and that equality holds iff ABC is equilateral.

Solution. (Due to Fung Ho Yin) Let A' be the point the angle bisector of $\angle A$ extended to meet the circle. Applying sine law to $\triangle ABA'$, we get $AB/\sin C = M_a/\sin(B + \frac{A}{2})$. Applying sine law to $\triangle ABD$, we get $AB/\sin(C + \frac{A}{2}) = m_a/\sin B$. So

$$l_a = \frac{m_a}{M_a} = \frac{\sin B \sin C}{\sin(B + \frac{A}{2}) \sin(C + \frac{A}{2})} \geq \sin B \sin C.$$

By the AM-GM inequality,

$$\frac{l_a}{\sin^2 A} + \frac{l_b}{\sin^2 B} + \frac{l_c}{\sin^2 C} \geq \frac{\sin B \sin C}{\sin^2 A} + \frac{\sin C \sin A}{\sin^2 B} + \frac{\sin A \sin B}{\sin^2 C} \geq 3$$

with equality if and only if $\sin A = \sin B = \sin C$ and $C + \frac{A}{2} = B + \frac{A}{2} = \cdots = 90°$, which is equivalent to $\angle A = \angle B = \angle C$.

103. (Mathematics Magazine, Problem 1506) Let I and O be the incenter and circumcenter of $\triangle ABC$, respectively. Assume $\triangle ABC$ is not equilateral (so $I \neq O$). Prove that

$$\angle AIO \leq 90^\circ \quad \text{if and only if} \quad 2BC \leq AB + CA.$$

Solution. (Due to Wong Chun Wai) Let D be the intersection of ray AI and the circumcircle of $\triangle ABC$. It is well-known that $DC = DB = DI$. ($DC = DB$ because $\angle CAD = \angle BAD$ and $DB = DI$ because $\angle BID = \angle BAD + \angle ABI = \angle CAD + \angle CBI = \angle DBC + \angle CBI = \angle DBI$.) Since $ABDC$ is a cyclic quadrilateral, by Ptolemy's theorem, $AD \cdot BC = AB \cdot DC + AC \cdot DB = (AB + AC) \cdot DI$. Then $DI = AD \cdot BC/(AB + AC)$. Since $\triangle AOD$ is isosceles, $\angle AIO = 90^\circ$ if and only if $DI \leq AD/2$, which is equivalent to $2BC \leq AB + AC$.

Comments. In the solution above, we see $\angle AIO = 90^\circ$ if and only if $2BC = AB + AC$. Also, the converse of the well-known fact is true, i.e. the point I on AD such that $DC = DB = DI$ is the incenter of $\triangle ABC$. This is because $\angle BID = \angle DBI$ if and only if $\angle CBI = \angle ABI$, since $\angle DBC = \angle BAD$ always.)

104. Squares $ABDE$ and $ACFG$ are drawn outside $\triangle ABC$. Let P, Q be points on EG such that BP and CQ are perpendicular to BC. Prove that $BP + CQ \geq BC + EG$. When does equality hold?

Solution. Let M, N, O be midpoints of BC, PQ, EG, respectively. Let H be the point so that $HEAG$ is a parallelogram. Translating by $\overrightarrow{GA}$, then rotating by 90° about A, $\triangle GHA$ will coincide with $\triangle ABC$ and O will move to M. So $HA = BC$, $HA \perp BC$, $OE = OG = MA$, $EG \perp MA$. Let L be on MN such that $AL \parallel EG$. Since $NL \parallel PB, PB \perp BC, BC \perp HA$, so $LNOA$ is a parallelogram. Then $AO = LN$. Since $MA \perp EG$, so $MA \perp AL$, which implies $ML \geq MA$. Therefore

$$\begin{aligned} BP + CQ &= 2MN = 2(LN + ML) \\ &\geq 2(AO + MA) = 2(BM + OE) = BC + EG. \end{aligned}$$

Equality holds if and only if L coincides with A, i.e. $AB = AC$.

105. Point P is inside $\triangle ABC$. Determine points D on side AB and E on side AC such that $BD = CE$ and $PD + PE$ is minimum.

Solution. The minimum is attained when $ADPE$ is a cyclic quadrilateral. To see this, consider the point G such that G lies on the opposite side of line AC as B, $\angle ABP = \angle ACG$ and $CG = BP$. Let E be the intersection of lines AC and PG. Let D be the intersection of AB with the circumcircle of APE. Since $ADPE$ is a cyclic quadrilateral, $\angle BDP = \angle AEP = \angle CEG$. Using the definition of G, we have $\triangle BDP, \triangle CEG$ are congruent. So $BD = CE$ and $PD + PE = GE + PE = GP$.

For other D', E' on sides AB, AC, respectively such that $BD' = CE'$, by the definition of G, we have $\triangle BPD', \triangle CGE'$ are congruent. Then $PD' = GE'$ and $PD' + PE' = GE' + PE' > GP$.

<u>*Solid or Space Geometry*</u>

106. (Proposed by Italy for 1967 IMO) Which regular polygons can be obtained (and how) by cutting a cube with a plane?

Solution. (Due to Fan Wai Tong, Kee Wing Tao and Tam Siu Lung) Observe that if two sides of a polygon is on a face of the cube, then the whole polygon lies on the face. Since a cube has 6 faces, only regular polygon with 3, 4, 5 or 6 sides are possible. Let the vertices of the bottom face of the cube be A, B, C, D and the vertices on the top face be A', B', C', D' with A' on top of A, B' on top of B and so on. Then the plane through A, B', D' cuts an equilateral triangle. The perpendicular bisecting plane to edge AA' cuts a square. The plane through the midpoints of edges $AB, BC, CC', C'D', D'A', A'A$ cuts a regular hexagon. Finally, a regular pentagon is impossible, otherwise the five sides will be on five faces of the cube implying two of the sides are on parallel planes, but no two sides of a regular pentagon are parallel.

107. (1995 Israeli Math Olympiad) Four points are given in space, in general position (i.e., they are not coplanar and any three are not collinear).

A plane π is called an *equalizing* plane if all four points have the same distance from π. Find the number of equalizing planes.

Solution. The four points cannot all lie on one side of an equalizing plane, otherwise they would lie in a plane parallel to the equalizing plane. Hence either three lie on one side and one on the other or two lie on each side. In the former case, there is exactly one equalizing plane, which is parallel to the plane P containing the three points and passing through the midpoint of the segment joining the fourth point x and the foot of the perpendicular from x to P. In the latter case, again there is exactly one equalizing plane. The two pair of points determine two skew lines in space. Consider the two planes, each containing one of the line and is parallel to the other line. The equalizing plane is the plane midway between these two plane. Since there are $4+3=7$ ways of dividing the four points into these two cases, there are exactly 7 equalizing planes.

Solutions to Number Theory Problems

<u>*Digits*</u>

108. (1956 Putnam Exam) Prove that every positive integer has a multiple whose decimal representation involves all ten digits.

Solution. Let n be a positive integer and $p = 1234567890 \times 10^k$, where k is so large that $10^k > n$. Then the n consecutive integers $p+1, p+2, \cdots, p+n$ have decimal representations beginning with $1234567890\cdots$ and one of them is a multiple of n.

109. Does there exist a positive integer a such that the sum of the digits (in base 10) of a is 1999 and the sum of the digits (in base 10) of a^2 is 1999^2?

Solution. Yes. In fact, there is such a number whose digits consist of 0's and 1's. Let $k = 1999$. Consider $a = 10^{2^1} + 10^{2^2} + \cdots + 10^{2^k}$. Then the sum of the digits of a is k. Now

$$a^2 = 10^{2^2} + 10^{2^3} + \cdots + 10^{2^{k+1}} + 2 \sum_{1 \le i < j \le k} 10^{2^i + 2^j}.$$

Observe that the exponent are all different by the uniqueness of base 2 representation. Therefore, the sum of the digits of a^2 in base 10 is $k + 2C_2^k = k^2$.

110. (Proposed by USSR for 1991 IMO) Let a_n be the last nonzero digit in the decimal representation of the number $n!$. Does the sequence $a_1, a_2, \ldots, a_n, \ldots$ become periodic after a finite number of terms?

Solution. Suppose after N terms, the sequence becomes periodic with period T. Then $a_{i+jT} = a_i$ for $i \ge N, j = 1, 2, 3 \ldots$. By the pigeonhole principle, there are two numbers among $10^{N+1}, 10^{N+2}, 10^{N+3}, \ldots$ that have the same remainder when divided by T, say $10^m \equiv 10^k \pmod{T}$ with $N < m < k$. Then $10^k - 10^m = jT$ for some integer j.

Observe that $10^k! = 10^k(10^k - 1)!$ implies $a_{10^k} = a_{10^k-1}$. Let $n = 10^k - 1 + jT$. Since $10^k - 1 \ge N$, $a_{n+1} = a_{10^k+jT} = a_{10^k-1+jT} = a_n$. Since $(n+1)! = (2 \times 10^k - 10^m)(n!) = 199\cdots 90\cdots 0(n!)$, so $9a_n = 9a_{n-1} \equiv a_n \pmod{10}$. This implies $a_n = 5$. However, in the prime factorization of $n!$, the exponent of 2 is greater than the exponent of 5, which implies a_n is even, a contradiction.

Modulo Arithmetic

111. (1956 Putnam Exam) Prove that the number of odd binomial coefficients in any row of the Pascal triangle is a power of 2.

Solution. By induction, $(1+x)^{2^m} \equiv 1 + x^{2^m} \pmod 2$. If we write n in base 2, say $n = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_k}$, where the a_i's are distinct nonnegative integers, then

$$(1+x)^n = (1+x)^{2^{a_1}} \cdots (1+x)^{2^{a_k}} \equiv (1+x^{2^{a_1}}) \cdots (1+x^{2^{a_k}}) \pmod 2.$$

In expanding the expression in front of (mod 2), we get the sum of x^{n_S}, where for each subset S of $\{1,2,\ldots,k\}$, $n_S = \sum_{i\in S} 2^{a_i}$. Since there are 2^k subsets of $\{1,2,\ldots,k\}$, there are exactly 2^k terms, each with coefficient 1. This implies there are exactly 2^k odd binomial coefficients in the n-th row of the Pascal triangle.

112. Let $a_1, a_2, a_3, \ldots, a_{11}$ and $b_1, b_2, b_3, \ldots, b_{11}$ be two permutations of the natural numbers $1, 2, 3, \ldots, 11$. Show that if each of the numbers a_1b_1, $a_2b_2, a_3b_3, \ldots, a_{11}b_{11}$ is divided by 11, then at least two of them will have the same remainder.

Solution. Suppose a_1b_1,$a_2b_2,\ldots$,$a_{11}b_{11}$ have distinct remainders when divided by 11. By symmetry, we may assume $a_1b_1 \equiv 0 \pmod{11}$. Let $x = (a_2b_2)\cdots(a_{11}b_{11})$. On one hand, $x \equiv 10! \equiv 10 \pmod{11}$. On the other hand, since $a_ib_i \not\equiv 0 \pmod{11}$, for $i = 2,\ldots,11$, we get $a_1 = 11 = b_1$. So $x = (a_2\cdots a_{11})(b_2\cdots b_{11}) = (10!)^2 \equiv 10^2 \equiv 1 \pmod{11}$, a contradiction.

113. (1995 Czech-Slovak Match) Let $a_1, a_2, \ldots$ be a sequence satisfying $a_1 = 2, a_2 = 5$ and

$$a_{n+2} = (2 - n^2)a_{n+1} + (2 + n^2)a_n$$

for all $n \ge 1$. Do there exist indices p, q and r such that $a_pa_q = a_r$?

Solution. (Due to Lau Lap Ming) The first few terms are $2, 5, 11, 8, 65, -766, \ldots$. Since the differences of consecutive terms are multiples of 3, we suspect $a_n \equiv 2 \pmod 3$ for all n. Clearly, $a_1, a_2 \equiv 2 \pmod 3$. If $a_n, a_{n+1} \equiv 2 \pmod 3$, then

$$a_{n+2} \equiv (2-n^2)2 + (2+n^2)2 = 8 \equiv 2 \pmod 3.$$

So by induction, all $a_n \equiv 2 \pmod 3$. Then $a_pa_q \ne a_r$ for all p, q, r as $4 \not\equiv 2 \pmod 3$.

Prime Factorization

114. (American Mathematical Monthly, Problem E2684) Let A_n be the set of positive integers which are less than n and are relatively prime to n. For which $n > 1$, do the integers in A_n form an arithmetic progression?

Solution. Suppose A_n is an arithmetic progression. If n is odd and $n \ge 3$, then $1, 2 \in A_n$ implies $A_n = \{1, 2, \ldots, n-1\}$, which implies n is prime. If n is even and not divisible by 3, then $1, 3 \in A_n$, $2 \notin A_n$ imply $A_n = \{1, 3, 5, \ldots, n-1\}$, which implies n is a power of 2. Finally, if n is even and divisible by 3, then let p be the smallest prime not dividing n. Either $p \equiv 1 \pmod 6$ or $p \equiv 5 \pmod 6$. In the former case, since $1, p$ are the first two elements of A_n and $n - 1 \in A_n$, so $1 + k(p-1) = n - 1$ for some k. This implies $n \equiv 2 \pmod 6$, a contradiction. So $p \equiv 5 \pmod 6$. Then $2p - 1$ is divisible by 3 and so $2p - 1 \notin A_n$. Consequently, $A_n = \{1, p\}$, which implies $n = 6$ by considering the prime factorization of n. Therefore, A_n is an arithmetic progression if and only if n is a prime, a power of 2 or $n = 6$.

115. (1971 IMO) Prove that the set of integers of the form $2^k - 3$ ($k = 2, 3, \ldots$) contains an infinite subset in which every two members are relatively prime.

Solution. We shall give a recipe for actually constructing an infinite set of integers of the form $a_i = 2^{k_i} - 3$, $i = 1, 2, \ldots$, each relatively prime to all the others. Let $a_1 = 2^2 - 3 = 1$. Suppose we have n pairwise relatively prime numbers $a_1 = 2^{k_1} - 3$, $a_2 = 2^{k_2} - 3$, $\ldots$, $a_n = 2^{k_n} - 3$. We form the product $s = a_1 a_2 \cdots a_n$, which is odd. Now consider the $s + 1$ numbers $2^0, 2^1, 2^2, \ldots, 2^s$. At least two of these will be congruent (mod s), say $2^\alpha \equiv 2^\beta \pmod{s}$, or equivalently $2^\beta(2^{\alpha-\beta} - 1) = ms$ for some integer m. The odd number s does not divide 2^β, so it must divide $2^{\alpha-\beta} - 1$; hence $2^{\alpha-\beta} - 1 = ls$ for some integer l. Since $2^{\alpha-\beta} - 1$ is divisible by s and s is odd, $2^{\alpha-\beta} - 3$ is relatively prime to s. This implies $2^{\alpha-\beta} - 3 \neq 2^{k_i} - 3$ for $i = 1, 2, \ldots, n$. So we may define $a_{n+1} = 2^{\alpha-\beta} - 3$. This inductive construction can be repeated to form an infinite sequence.

Comments. By Euler's theorem, we may take the exponent $\alpha - \beta$ to be $\phi(s)$, the Euler ϕ-function of s, which equals the number of positive integers less than s that are relatively prime to s, then $2^{\phi(s)} \equiv 1 \pmod{s}$.

116. (1988 Chinese Math Olympiad Training Test) Determine the smallest value of the natural number $n > 3$ with the property that whenever the set $S_n = \{3, 4, \ldots, n\}$ is partitioned into the union of two subsets, at least one of the subsets contains three numbers a, b and c (not necessarily distinct) such that $ab = c$.

Solution. (Due to Lam Pei Fung) We first show that $3^5 = 243$ has the property, then we will show it is the least solution.

Suppose S_{243} is partitioned into two subsets X_1, X_2. Without loss of generality, let 3 be in X_1. If $3^2 = 9$ is in X_1, then we are done. Otherwise, 9 is in X_2. If $9^2 = 81$ is in X_2, then we are done. Otherwise, 81 is in X_1. If $81/3 = 27$ is in X_1, then we are done. Otherwise, 27 is in X_2. Finally, either $3 \times 81 = 243$ is in X_1 or $9 \times 27 = 243$ is in X_2. In either case we are done.

To show 243 is the smallest, we will show that S_{242} can be partitioned into two subsets, each of which does not contain products of its elements. Define C to be "prime" in S_{242} if C is not the product of elements of S_{242}. The "primes" in S_{242} consist of $4, 8, p, 2p$ where $p < 242$ is a usual prime number. Since the smallest prime in S_{242} is

3, no number in S_{242} is the product of more than four "primes". Put all the "primes" and numbers that can be written as products of four "primes" in one subset X_1, and let $X_2 = S_{242} \setminus X_1$.

No products in X_2 are in X_2 because numbers in X_2 have at least two "prime" factors, so their products can be written with at least four "prime" factors. Next looking at the product of $4, 8, p, 2p$ (p odd prime < 242), we see that a product of two "primes" cannot be factored into a product of four "primes". So no products in X_1 are in X_1.

Base n Representations

117. (1983 IMO) Can you choose 1983 pairwise distinct nonnegative integers less than 10^5 such that no three are in arithmetic progression?

Solution. We consider the *greedy algorithm* for constructing such a sequence: start with 0, 1 and at each step add the *smallest* integer which is not in arithmetic progression with any two preceding terms. We get $0, 1, 3, 4, 9, 10, 12, 13, 27, 28, \ldots$. In base 3, this sequence is

$$0, 1, 10, 11, 100, 101, 110, 111, 1000, 1001, \ldots.$$

(Note this sequence is the nonnegative integers in base 2.) Since 1982 in base 2 is 11110111110, so switching this from base 3 to base 10, we get the 1983th term of the sequence is $87843 < 10^5$. To see this sequence works, suppose x, y, z with $x < y < z$ are three terms of the sequence in arithmetic progression. Consider the rightmost digit in base 3 where x differs from y, then that digit for z is a 2, a contradiction.

118. (American Mathematical Monthly, Problem 2486) Let p be an odd prime number and r be a positive integer *not* divisible by p. For any positive integer k, show that there exists a positive integer m such that the rightmost k digits of m^r, when expressed in the base p, are all 1's.

Solution. We prove by induction on k. For $k = 1$, take $m = 1$. Next, suppose m^r, in base p, ends in k 1's, i.e.

$$m^r = 1 + p + \cdots + p^{k-1} + (ap^k + \text{higher terms}).$$

Clearly, $\gcd(m,p)=1$. Then

$$\begin{aligned}(m+cp^k)^r &= m^r + rm^{r-1}cp^k + \cdots + c^r p^{kr} i \\ &= 1 + p + \cdots + p^{k-1} + (a + rm^{r-1}c)p^k + \text{higher terms.}\end{aligned}$$

Since $\gcd(mr,p)=1$, the congruence $a + rm^{r-1}c \equiv 1 \pmod{p}$ is solvable for c. If c_0 is a solution, then $(m+c_0p^k)^r$ will end in $(k+1)$ 1's as required.

119. (Proposed by Romania for 1985 IMO) Show that the sequence $\{a_n\}$ defined by $a_n = [n\sqrt{2}]$ for $n = 1,2,3,\ldots$ (where the brackets denote the greatest integer function) contains an infinite number of integral powers of 2.

Solution. Write $\sqrt{2}$ in base 2 as $b_0.b_1b_2b_3\ldots$, where each $b_i = 0$ or 1. Since $\sqrt{2}$ is irrational, there are infinitely many $b_k = 1$. If $b_k = 1$, then in base 2, $2^{k-1}\sqrt{2} = b_0 \cdots b_{k-1}.b_k \cdots$ Let $m = [2^{k-1}\sqrt{2}]$, then

$$2^{k-1}\sqrt{2} - 1 < [2^{k-1}\sqrt{2}] = m < 2^{k-1}\sqrt{2} - \frac{1}{2}.$$

Multiplying by $\sqrt{2}$ and adding $\sqrt{2}$, we get $2^k < (m+1)\sqrt{2} < 2^k + \frac{\sqrt{2}}{2}$. Then $[(m+1)\sqrt{2}] = 2^k$.

<u>*Representations*</u>

120. Find all (even) natural numbers n which can be written as a sum of two odd composite numbers.

Solution. Let $n \ge 40$ and d be its units digit. If $d=0$, then $n = 15 + (n-15)$ will do. If $d=2$, then $n = 27 + (n-27)$ will do. If $d=4$, then $n = 9 + (n-9)$ will do. If $d=6$, then $n = 21 + (n-21)$ will do. If $d=8$, then $n = 33 + (n-33)$ will do. For $n < 40$, direct checking shows only $18 = 9+9,\ 24 = 9+15,\ 30 = 9+21,\ 34 = 9+25,\ 36 = 9+27$ can be so expressed.

121. Find all positive integers which cannot be written as the sum of two or more consecutive positive integers.

Solution. (Due to Cheung Pok Man) For odd integer $n = 2k+1 \ge 3$, $n = k + (k+1)$. For even integer $n \ge 2$, suppose $n = m + (m+1) + \cdots + (m+r) = (2m+r)(r+1)/2$ with $m, r \ge 1$. Then $2m+r, r+1 \ge 2$ and one of $2m+r, r+1$ is odd. So n must have an odd divisor greater than 1. In particular, $n = 2^j,\ j = 0,1,2,\ldots$, cannot be written as the sum of consecutive positive integers. For the other even integers, $n = 2^j(2k+1)$ with $j,k \ge 1$. If $2^j > k$, then $n = (2^j - k) + (2^j - k + 1) + \cdots + (2^j + k)$. If $2^j \le k$, then $n = (k - 2^j + 1) + (k - 2^j + 2) + \cdots + (k + 2^j)$.

122. (Proposed by Australia for 1990 IMO) Observe that $9 = 4+5 = 2+3+4$. Is there an integer N which can be written as a sum of 1990 consecutive positive integers and which can be written as a sum of (more than one) consecutive integers in exactly 1990 ways?

Solution. For such N, we have $N = \sum_{i=0}^{1989}(m+i) = 995(2m+1989)$. So N is odd and is divisible by $995 = 5 \times 199$. Also, there are exactly 1990 positive integer pairs (n,k) such that $N = \sum_{i=0}^{k}(n+i) = \frac{(k+1)(n+2k)}{2}$. Hence $2N$ can be factored as $(k+1)(2n+k)$ in exactly 1990 ways. (Note if $2N = ab$ with $2 \le a < b$, then $n = (1+b-a)/2, k = a-1$.) This means $2N$ has exactly $2 \times 1991 = 2 \times 11 \times 181$ positive divisors. Now write $2N$ in prime factorization as $2 \times 5^{e_1} \times 199^{e_2} \times \cdots$. Then we get $2 \times 11 \times 181 = 2(e_1+1)(e_2+1)\cdots$. So $\{e_1, e_2\} = \{10, 180\}$. Therefore, $N = 5^{10} \times 199^{180}$ or $5^{180} \times 199^{10}$. As all the steps can be reversed, these are the only answers.

123. Show that if $p > 3$ is prime, then p^n cannot be the sum of two positive cubes for any $n \ge 1$. What about $p = 2$ or 3?

Solution. Suppose n is the smallest positive integer such that p^n is the sum of two positive cubes, say $p^n = a^3 + b^3 = (a+b)(a^2 - ab + b^2)$. Then $a + b = p^k$ and $a^2 - ab + b^2 = p^{n-k}$. Since $a + b \ge 2$, so $k > 0$. Since $a^2 - ab + b^2 \ge ab > 1$, so $n > k$. Now $3ab = (a+b)^2 - (a^2 - ab + b^2) = p^{2k} - p^{n-k}$ and $0 < k < n$, so $p \mid 3ab$. Since $p > 3$, so $p \mid a$ or $p \mid b$. Since $a + b = p^k$, so $p \mid a$ and $p \mid b$, say $a = pA$ and $b = pB$. Then $A^3 + B^3 = p^{n-3}$, contradicting the smallest property of n.

For $p=2$, suppose $a^3+b^3=2^n$. If $a+b>2$, then $2\mid a,\ 2\mid b$ and $(\frac{a}{2})^3+(\frac{b}{2})^3=2^{n-3}$. So $a=b=2^k$ and $n=3k+1$.

For $p=3$, suppose $a^3+b^3=3^n$. If $a+b=3^k$ and $a^2-ab+b^2=3^{n-k}\geq 3^2$, then $9\mid 3ab$ implies $3\mid a,\ 3\mid b$ and $(\frac{a}{3})^3+(\frac{b}{3})^3=3^{n-3}$. Otherwise we have $3=a^2-ab+b^2\geq ab$ and then $a+b=3$. So in this case, a,b are $2\cdot 3^k, 3^k$ and $n=3k+2$.

124. (Due to Paul Erdös and M. Surányi) Prove that every integer k can be represented in infinitely many ways in the form $k=\pm 1^2\pm 2^2\pm\cdots\pm m^2$ for some positive integer m and some choice of signs $+$ or $-$.

Solution. We first show every integer can be so represented in at least one way. If k can be represented, then changing all the signs, we see $-k$ also can be represented. So it suffices to do the nonnegative cases. The key observation is the identity

$$(m+1)^2-(m+2)^2-(m+3)^2+(m+4)^2=4.$$

Now $0=4-4=(-1^2-2^2+3^2)-4^2+5^2+6^2-7^2,\ 1=1^2,\ 2=-1^2-2^2-3^2+4^2,\ 3=-1^2+2^2$. By the identity, if k can be represented, then $k+4$ can be represented. So by induction, every nonnegative integer (and hence every integer) can be represented. To see there are infinitely many such representations, we use the identity again. Observe $0=4-4=(m+1)^2-(m+2)^2-(m+3)^2+(m+4)^2-(m+5)^2+(m+6)^2+(m+7)^2-(m+8)^2$. So for every representation, we can add 8 more terms to get another representation.

125. (1996 IMO shortlisted problem) A finite sequence of integers $a_0,a_1,\ldots,a_n$ is called *quadratic* if for each $i\in\{1,2,\ldots,n\}$, $|a_i-a_{i-1}|=i^2$.

(a) Prove that for any two integers b and c, there exists a natural number n and a quadratic sequence with $a_0=b$ and $a_n=c$.

(b) Find the least natural number n for which there exists a quadratic sequence with $a_0=0$ and $a_n=1996$.

Solution. Part (a) follows from the last problem by letting $k=c-b$. For part (b), consider a_k in such a quadratic sequence. We have

$a_k\leq 1^2+2^2+\cdots+k^2=k(k+1)(2k+1)/6$. So $a_{17}\leq 1785$. Also $a_k\equiv 1^2+2^2+\cdots+k^2 \pmod 2$. Since $1^2+2^2+\cdots+18^2$ is odd, $n\geq 19$. To construct such a quadratic sequence with $n=19$, first note $1^2+2^2+\cdots+19^2=2470$. Now we write $(2470-1996)/2=237=14^2+5^2+4^2$. Then

$$1996=1^2+2^2+3^2-4^2-5^2+6^2+\cdots+13^2-14^2+15^2+\cdots+19^2.$$

126. Prove that every integer greater than 17 can be represented as a sum of three integers >1 which are pairwise relatively prime, and show that 17 does not have this property.

Solution. (Due to Chan Kin Hang and Ng Ka Wing) Let $k\geq 3$. From $18=2+3+13$, we see $2+3+(6k-5)$ works for $6k$. From $20=3+4+13$, we see $3+4+(6k-4)$ works for $6k+2$. From $22=2+3+17$, we see $2+3+(6k-1)$ works for $6k+4$.

For $6k+1$, we split into cases $12k'+1$ and $12k'+7$. We have $12k'+1=9+(6k'-1)+(6k'-7)$ and $12k'+7=3+(6k'-1)+(6k'+5)$.

For $6k+3$, we split into cases $12k'+3$ and $12k'+9$. We have $12k'+3=3+(6k'-1)+(6k'+1)$ and $12k'+9=9+(6k'-1)+(6k'+1)$.

For $6k+5$, we split into cases $12k'+5$ and $12k'+11$. We have $12k'+5=9+(6k'-5)+(6k'+1)$ and $12k'+11=3+(6k'+1)+(6k'+7)$.

Finally, 17 does not have the property. Otherwise, $17=a+b+c$, where a,b,c are relatively prime and $a<b<c$. Then a,b,c are odd. If $a=3$, then $3+5+7<a+b+c<3+5+11$ shows this is impossible. If $a\geq 5$, then $b\geq 7, c\geq 9$ and $a+b+c\geq 21>17$, again impossible.

Chinese Remainder Theorem

127. (1988 Chinese Team Selection Test) Define $x_n=3x_{n-1}+2$ for all positive integers n. Prove that an integer value can be chosen for x_0 so that x_{100} is divisible by 1998.

Solution. Let $y_n = \frac{x_n}{3^n}$, then $y_n = y_{n-1} + \frac{2}{3^n}$, which implies

$$y_n = y_0 + \frac{2}{3} + \frac{2}{3^2} + \cdots + \frac{2}{3^n}.$$

This gives $x_n = (x_0+1)3^n - 1$. We want $x_{100} = (x_0+1)3^{100} - 1$ to be divisible by $1998 = 4 \times 7 \times 71$, which means

$$x_0 \equiv 0 \pmod 4), \quad x_0 \equiv 1 \pmod 7), \quad x_0 \equiv 45 \pmod{71}.$$

Since $4, 7, 71$ are pairwise relatively prime, by the Chinese remainder theorem, such x_0 exists.

128. (Proposed by North Korea for 1992 IMO) Does there exist a set M with the following properties:

(a) The set M consists of 1992 natural numbers.

(b) Every element in M and the sum of any number of elements in M have the form m^k, where m, k are positive integers and $k \geq 2$?

Solution. (Due to Cheung Pok Man) Let $n = 1+2+\cdots+1992$. Choose n distinct prime numbers $p_1, p_2, \ldots, p_n$. Let $d = 2^{e_2}3^{e_3}4^{e_4}\cdots n^{e_n}$, where e_i is a solution of the n equations $x \equiv -1 \pmod{p_i}$ and $x \equiv 0 \pmod{p_j}$ for every $1 \leq j \leq n$, $j \neq i$. (Since the p_i's are pairwise relatively prime, such a solution exists by the Chinese remainder theorem.) Since $e_2, e_3, \ldots, e_n \equiv 0 \pmod{p_1}$, d is a p_1-th power. Since $e_2+1, e_3, \ldots, e_n \equiv 0 \pmod{p_2}$, $2d$ is a p_2-th power and so on. It follows $d, 2d, \ldots, 1992d$ are all perfect powers and any sum of them is a multiple of d, less than or equal to nd, hence is also a perfect power.

<u>*Divisibility*</u>

129. Find all positive integers a, b such that $b > 2$ and $2^a + 1$ is divisible by $2^b - 1$.

Solution. Since $b > 2$, so $2^b - 1 < 2^a + 1$, hence $b < a$. Let $a = qb + r, 0 \leq r < b$, then by division, we get

$$\frac{2^a+1}{2^b-1} = 2^{a-b} + 2^{a-2b} + \cdots + 2^{a-qb} + \frac{2^r+1}{2^b-1}.$$

Since $0 < \frac{2^r+1}{2^b-1} < 1$, there are no solutions.

130. Show that there are infinitely many composite n such that $3^{n-1} - 2^{n-1}$ is divisible by n.

Solution. We use the fact $x - y \mid x^k - y^k$ for positive integer k. Consider $n = 3^{2^t} - 2^{2^t}$ for $t = 2, 3, \ldots$. By induction, we can show $2^t \mid 3^{2^t} - 1 = n - 1 + 2^{2^t}$. (Alternatively, by Euler's theorem, $3^{2^t} = (3^{\phi(2^t)})^2 \equiv 1 \pmod{2^t}$.) Then $n - 1 = 2^t k$. So $n = 3^{2^t} - 2^{2^t} \mid (3^{2^t})^k - (2^{2^t})^k = 3^{n-1} - 2^{n-1}$.

131. Prove that there are infinitely many positive integers n such that $2^n + 1$ is divisible by n. Find all such n's that are prime numbers.

Solution. Looking at the cases $n = 1$ to 10 suggest for $n = 3^k, k = 0, 1, 2, \ldots$, we should have $n \mid 2^n + 1$. The case $k = 0$ is clear. Suppose case k is true. Now $2^{3^{k+1}} + 1 = (2^{3^k} + 1)(2^{3^k 2} - 2^{3^k} + 1)$. By case k, $2^{3^k} \equiv 1 \pmod 3$, so $2^{3^k 2} - 2^{3k} + 1 \equiv (-1)^2 - (-1) + 1 \equiv 0 \pmod 3$. So $2^{3^{k+1}} + 1$ is divisible by 3^{k+1}, completing the induction.

If a prime n divides 2^n+1, then by Fermat's little theorem, $n \mid 2^n - 2$, too. Then $n \mid (2^n+1) - (2^n - 2) = 3$, so $n = 3$.

132. (1998 Romanian Math Olympiad) Find all positive integers (x, n) such that $x^n + 2^n + 1$ is a divisor of $x^{n+1} + 2^{n+1} + 1$.

Solution. (Due to Cheng Kei Tsi and Leung Wai Ying) For $x = 1$, $2(1^n+2^n+1) > 1^{n+1}+2^{n+1}+1 > 1^n+2^n+1$. For $x = 2$, $2(2^n+2^n+1) > 2^{n+1} + 2^{n+1} + 1 > 2^n + 2^n + 1$. For $x = 3$, $3(3^n + 2^n + 1) > 3^{n+1} + 2^{n+1} + 1 > 2(3^n + 2^n + 1)$. So there are no solutions with $x = 1, 2, 3$.

For $x \geq 4$, if $n \geq 2$, then we get $x(x^n + 2^n + 1) > x^{n+1} + 2^{n+1} + 1$. Now

$$\begin{aligned} & x^{n+1} + 2^{n+1} + 1 \\ =& (x-1)(x^n + 2^n + 1) \\ & \quad + x^n - (2^n+1)x + 3 \cdot 2^n + 2 \\ >& (x-1)(x^n + 2^n + 1) \end{aligned}$$

because for $n = 2$, $x^n - (2^n + 1)x + 2^{n+1} = x^2 - 5x + 8 > 0$ and for $n \geq 3$, $x^n - (2^n + 1)x \geq x(4^{n-1} - 2^n - 1) > 0$. Hence only $n = 1$ and $x \geq 4$ are possible. Now $x^n + 2^n + 1 = x + 3$ is a divisor of $x^{n+1} + 2^{n+1} + 1 = x^2 + 5 = (x - 3)(x + 3) + 14$ if and only if $x + 3$ is a divisor of 14. Since $x + 3 \geq 7$, $x = 4$ or 11. So the solutions are $(x, y) = (4, 1)$ and $(11, 1)$.

133. (1995 Bulgarian Math Competition) Find all pairs of positive integers (x, y) for which $\dfrac{x^2 + y^2}{x - y}$ is an integer and divides 1995.

Solution. Suppose (x, y) is such a pair. We may assume $x > y$, otherwise consider (y, x). Then $x^2 + y^2 = k(x - y)$, where $k \mid 1995 = 3 \times 5 \times 7 \times 19$. If $p = 3$ or 7 or 19 divides k, then by the fact that prime $p \equiv 3 \pmod 4$ dividing $x^2 + y^2$ implies p divides x or y, we may cancel p^2 to get an equation $x_0^2 + y_0^2 = k_0(x_0 - y_0)$ with k_0 not divisible by 3, 7, 19. Since $x_0^2 + y_0^2 > x_0^2 > x_0 > x_0 - y_0$, we must have $x_0^2 + y_0^2 = 5(x_0 - y_0)$. Completing squares, we get $(2x_0 - 5)^2 + (2y_0 + 5)^2 = 50$, which gives $(x_0, y_0) = (3, 1)$ or $(2, 1)$. It follows $(x, y) = (3c, c)$, $(2c, c)$, $(c, 3c)$, $(c, 2c)$, where c is a positive divisor of $3 \times 7 \times 19$.

134. (1995 Russian Math Olympiad) Is there a sequence of natural numbers in which every natural number occurs just once and moreover, for any $k = 1, 2, 3, \ldots$ the sum of the first k terms is divisible by k?

Solution. Let $a_1 = 1$. Suppose $a_1, \ldots, a_k$ has been chosen to have the property. Let n be the smallest natural number not yet appeared. By the Chinese remainder theorem, there is an integer m such that $m \equiv -a_1 - \cdots - a_k \pmod{k+1}$ and $m \equiv -a_1 - \cdots - a_k - n \pmod{k+2}$. We can increase m by a large multiple of $(k + 1)(k + 2)$ to ensure it is positive and not equal to anyone of $a_1, \ldots, a_k$. Let $a_{k+1} = m$ and $a_{k+2} = n$. The sequence constructed this way have the property.

135. (1998 Putnam Exam) Let $A_1 = 0$ and $A_2 = 1$. For $n > 2$, the number A_n is defined by concatenating the decimal expansions of A_{n-1} and A_{n-2} from left to right. For example, $A_3 = A_2A_1 = 10, A_4 = A_3A_2 =$

$101, A_5 = A_4A_3 = 10110$, and so forth. Determine all n such that A_n is divisible by 11.

Solution. The *Fibonacci numbers* F_n is defined by $F_1 = 1, F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n > 2$. Note A_n has F_n digits. So we have the recursion $A_n = 10^{F_{n-2}}A_{n-1}+A_{n-2} \equiv (-1)^{F_{n-2}}A_{n-1}+A_{n-2} \pmod{11}$. By induction, the sequence $F_n \pmod 2$ is $1, 1, 0, 1, 1, 0, \ldots$. The first eight terms of $A_n \pmod{11}$ are $0, 1, -1, 2, 1, 1, 0, 1$. (Note the numbers start to repeat after the sixth term.) In fact, the recursion implies $A_{n+6} \equiv A_n \pmod{11}$ by induction. So A_n is divisible by 11 if and only if $n = 6k + 1$ for some positive integer k.

136. (1995 Bulgarian Math Competition) If $k > 1$, show that k does not divide $2^{k-1} + 1$. Use this to find all prime numbers p and q such that $2^p + 2^q$ is divisible by pq.

Solution. Suppose $k \mid 2^{k-1} + 1$ for some $k > 1$. Then k is odd. Write $k = p_1^{e_1} \cdots p_r^{e_r}$, where p_i's are distinct primes. Let $p_i - 1 = 2^{m_i}q_i$ withe q_i odd. Let $m_j = \min\{m_1, \ldots, m_r\}$. Since $m_i \equiv 1 \pmod{m_j}$, we get $p_i^{e_i} \equiv 1 \pmod{m_j}$ and so $k = 2^{m_j}q + 1$ for some positive integer q. Since $p_j \mid k$ and $k \mid 2^{k-1} + 1$, so $2^{2^{m_j}q} = 2^{k-1} \equiv -1 \pmod{p_j}$. Then $2^{(p_j-1)q} = (2^{2^{m_j}q})^{q_j} \equiv -1 \pmod{p_j}$ because q_j is odd. However, $2^{p_j-1} \equiv 1 \pmod{p_j}$ by Fermat's little theorem since $\gcd(2, p_j) = 1$. So $2^{(p_j-1)q} \equiv 1 \pmod{p_j}$, a contradiction.

Suppose p, q are prime and $2^p + 2^q$ is divisible by pq. Then $2^p \equiv -2^q \pmod p$. If p, q are odd, then $2^p \equiv 2 \pmod p$ by Fermat's little theorem and $2^q \equiv -2^p \equiv -2 \pmod p$. So $2^{pq} \equiv (-2)^p \equiv -2 \pmod p$. Similarly, $2^{pq} \equiv -2 \pmod q$. Then $2^{pq-1} \equiv -1 \pmod{pq}$, contradicting the first part of the problem. If $p = 2$, then $q = 2$ or $q > 2$. If $q > 2$, then $2^2 \equiv -2^q \equiv -2 \pmod q$ by Fermat's little theorem, which implies $q = 3$. Therefore, the solutions are $(p, q) = (2, 2), (2, 3), (3, 2)$.

137. Show that for any positive integer n, there is a number whose decimal representation contains n digits, each of which is 1 or 2, and which is divisible by 2^n.

Solution. We will prove that the 2^n numbers with n digits of 1's or 2's have different remainders when divided by 2^n. Hence one of them is divisible by 2^n. For $n = 1$, this is clear. Suppose this is true for $n = k$. Now if a, b are $(k+1)$-digit numbers, where each digit equals 1 or 2, and $a \equiv b \pmod{2^{k+1}}$, then the units digits of a, b are the same. If $a = 10a' + i, b = 10b' + i$, where i is the units digit, then 2^{k+1} divides $a - b = 10(a' - b')$ is equivalent to 2^k divides $a' - b'$. Since a', b' are k-digit numbers (with digits equal 1 or 2), we have $a' = b'$. So $a = b$, completing the induction.

138. For a positive integer n, let $f(n)$ be the largest integer k such that 2^k divides n and $g(n)$ be the sum of the digits in the binary representation of n. Prove that for any positive integer n,

(a) $f(n!) = n - g(n)$;

(b) 4 divides $\binom{2n}{n} = \dfrac{(2n)!}{n!n!}$ if and only if n is not a power of 2.

Solution. (Due to Ng Ka Man and Poon Wing Chi) (a) Write n in base 2 as $(a_r a_{r-1} \cdots a_0)_2$. Then

$$a_i = (a_r \cdots a_{i+1} a_i)_2 - (a_r \cdots a_{i+1} 0)_2 = \left[\frac{n}{2^i}\right] - 2\left[\frac{n}{2^{i+1}}\right].$$

So

$$g(n) = \sum_{i=0}^{r} a_i = \sum_{i=0}^{r} \left(\left[\frac{n}{2^i}\right] - 2\left[\frac{n}{2^{i+1}}\right] \right) = n - \sum_{i=0}^{r} \left[\frac{n}{2^i}\right] = n - f(n!).$$

(b) Let $M_n = (2n)!/(n!)^2$. Since $g(2n) = g(n)$, using (a), we get

$$f(M_n) = f((2n)!) - 2f(n!) = 2g(n) - g(2n) = g(n).$$

So the largest k such that 2^k divides M_n is $k = g(n)$. Now 4 divides M_n if and only if $g(n) \geq 2$, which is equivalent to n not being a power of 2.

139. (Proposed by Australia for 1992 IMO) Prove that for any positive integer m, there exist an infinite number of pairs of integers (x, y) such that

(a) x and y are relatively prime;

(b) y divides $x^2 + m$;

(c) x divides $y^2 + m$.

Solution. Note $(x, y) = (1, 1)$ is such a pair. Now, if (x, y) is such a pair with $x \leq y$, then consider (y, z), where $y^2 + m = xz$. Then every common divisor of z and y is a divisor of m, and hence of x. So $\gcd(z, y) = 1$. Now

$$x^2(z^2 + m) = (y^2 + m) + x^2 m = y^4 + 2my^2 + m(x^2 + m)$$

is divisible by y. Since $\gcd(x, y) = 1$, $y \mid z^2 + m$, so (y, z) is another such pair with $y \leq y^2/x < z$. This can be repeated infinitely many times.

140. Find all integers $n > 1$ such that $1^n + 2^n + \cdots + (n-1)^n$ is divisible by n.

Solution. For odd $n = 2j + 1 > 1$, since $(n-k)^n \equiv k^n \pmod{n}$ for $1 \leq k \leq j$, so $1^n + 2^n + \cdots + (n-1)^n$ is divisible by n. For even n, write $n = 2^s t$, where t is odd. Then $2^s \mid 1^n + 2^n + \cdots + (n-1)^n$. Now if k is even and less than n, then $2^s \mid k^n$. If k is odd and less than n, then by Euler's theorem, $k^{2^{s-1}} \equiv 1 \pmod{2^s}$, so $k^n \equiv 1 \pmod{2^s}$. Then $0 \equiv 1^n + 2^n + \cdots + (n-1)^n \equiv \frac{n}{2} \pmod{2^s}$, which implies $2^{s+1} \mid n$, a contradiction. So only odd $n > 1$ has the property.

141. (1972 Putnam Exam) Show that if n is an integer greater than 1, then n does not divide $2^n - 1$.

Solution. Suppose $n \mid 2^n - 1$ for some $n > 1$. Since $2^n - 1$ is odd, so n is odd. Let p be the smallest prime divisor of n. Then $p \mid 2^n - 1$, so $2^n \equiv 1 \pmod{p}$. By Fermat's little theorem, $2^{p-1} \equiv 1 \pmod{p}$. Let k be the smallest positive integer such that $2^k \equiv 1 \pmod{p}$. Then $k \mid n$ (because otherwise $n = kq + r$ with $0 < r < k$ and $1 \equiv 2^n = (2^k)^q 2^r \equiv 2^r \pmod{p}$, contradicting k being smallest). Similarly $k \mid p - 1$. So $k \mid \gcd(n, p-1)$. Now $d = \gcd(n, p-1)$ must be 1 since $d \mid n$, $d \leq p-1$ and p is the smallest prime divisor of n. So $k = 1$ and $2 = 2^k \equiv 1 \pmod{p}$, a contradiction.

142. (Proposed by Romania for 1985 IMO) For $k \geq 2$, let $n_1, n_2, \ldots, n_k$ be positive integers such that

$$n_2 \mid (2^{n_1} - 1),\ n_3 \mid (2^{n_2} - 1),\ \ldots,\ n_k \mid (2^{n_{k-1}} - 1),\ n_1 \mid (2^{n_k} - 1).$$

Prove that $n_1 = n_2 = \cdots = n_k = 1$.

Solution. Observe that if $n_i = 1$ for some i, then n_{i+1} will equal 1 and the chain effect causes all of them to be 1. So assume no n_i is 1. Let p_k be the smallest prime number dividing n_k. Then $p_k \mid 2^{n_{k-1}} - 1$. So $2^{n_{k-1}} \equiv 1 \pmod{p_k}$. Let m_k be the smallest positive integer m such that $2^m \equiv 1 \pmod{p_k}$. Then $m_k \mid n_{k-1}$ and $m_k \mid p_k - 1$ by Fermat's little theorem. In particular $1 < m_k \leq p_k - 1 < p_k$ and so the smallest prime divisor p_{k-1} of n_{k-1} is less than p_k. Then we get the contradiction that $p_k > p_{k-1} > \cdots > p_1 > p_k$.

143. (1998 APMO) Determine the largest of all integer n with the property that n is divisible by all positive integers that are less than $\sqrt[3]{n}$.

Solution. (Due to Lau Lap Ming) The largest n is 420. Since $420 = 3 \cdot 4 \cdot 5 \cdot 7$ and $7 < \sqrt[3]{420} < 8$, 420 has the property. Next, if n has the property and $n > 420$, then $3, 4, 5, 7$ divide n. Hence $n \geq 840 > 729 = 9^3$. Then $5, 7, 8, 9$ divide n, so $n \geq 5 \cdot 7 \cdot 8 \cdot 9 = 2420 > 2197 = 13^3$. Then $5, 7, 8, 9, 11, 13$ divide n, so $n \geq 5 \cdot 7 \cdot 8 \cdot 9 \cdot 11 \cdot 13 > 8000 > 19^3$. Let k be the integer such that $19^k < \sqrt[3]{n} \leq 19^{k+1}$. Then $5^k, 7^k, 9^k, 11^k, 13^k, 16^k, 17^k, 19^k$ divide n, and we get the following contradiction

$$\begin{aligned} n &\geq 5^k 7^k 9^k 11^k 13^k 16^k 17^k 19^k \\ &= 19^k (4 \cdot 5)^k (3 \cdot 7)^k (2 \cdot 11)^k (2 \cdot 13)^k (3 \cdot 17)^k > 19^{3k+3} \geq n. \end{aligned}$$

144. (1997 Ukrainian Math Olympiad) Find the smallest integer n such that among any n integers (with possible repetitions), there exist 18 integers whose sum is divisible by 18.

Solution. Taking seventeen 0's and seventeen 1's, we see that the smallest such integer n cannot be 34 or less. We will show 35 is the answer. Consider the statement "among any $2k-1$ integers, there exist k of them whose sum is divisible by k." We will first show that if the statement is true for $k = k_1$ and k_2, then it is true for $k = k_1 k_2$.

Suppose it is true for $k = k_1$ and k_2. Since the case $k = k_1$ is true, for $2k_1k_2 - 1$ integers, we can take out $2k_1 - 1$ of them and pick k_1 of them with sum divisible by k_1 to form a group. Then return the other $k_1 - 1$ integers to the remaining integers and repeat the taking and picking. Totally we we will get $2k_2 - 1$ groups. Since the case $k = k_2$ is true, from the $2k_2 - 1$ sums $s_1, \ldots, s_{2k_2-1}$ of the groups, considering the numbers $d_i = s_i / \gcd(k_1, k_2)$, we can get k_2 of them whose sum is divisible by k_2. The union of the k_2 groups with sum s_i's consists of k_1k_2 numbers whose sum is then divisible by k_1k_2.

To finish the problem, since $18 = 2 \cdot 3^2$, we have to show the statement is true for $k = 2$ and 3. Among $2 \cdot 2 - 1 = 3$ numbers, there are two odd or two even numbers, their sum is even. Among $2 \cdot 3 - 1 = 5$ integers, consider (mod 3) of the integers. If 0,1,2 each appears, then the sum of those three will be 0 (mod 3), otherwise there are two choices for 5 integers and three of them will be congruent (mod 3), their sum is 0 (mod 3).

Comments. The statement is true for every positive integer k. All we have to consider is the case $k = p$ is prime. Suppose $2p - 1$ integers are given. There are

$$m = \binom{2p-1}{p} = \frac{(2p-1)(2p-2)\cdots(p+1)}{(p-1)!}$$

ways in picking p of them. *If no p of them have a sum divisible by p*, then consider

$$S = \sum (a_1 + \cdots + a_p)^{p-1},$$

where the sum is over all m pickings $a_1, \ldots, a_p$. By Fermat's little theorem,

$$S \equiv 1 + \cdots + 1 = m \not\equiv 0 \pmod{p}.$$

On the other hand, in expansion, the terms $a_1^{e_1} \cdots a_p^{e_p}$ have exponent sum $e_1 + \cdots + e_p = p - 1$. Hence the numbers of nonzero exponents e_i

in the terms will be positive integers $j \le p-1$. Since $p-j$ of the e_i is 0, the coefficient of the term in the full expansion of S is

$$\binom{2p-1-j}{p-j}\binom{p-1}{e_1,\ldots,e_p} = \frac{(2p-1-j)\cdots p \cdots (p-j+1)}{(p-j)!}\binom{p-1}{e_1,\ldots,e_p},$$

which is divisible by p. So all coefficients are divisible by p, hence $S \equiv 0 \pmod{p}$, a contradiction.

Perfect Squares, Perfect Cubes

145. Let a, b, c be positive integers such that $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$. If the greatest common divisor of a, b, c is 1, then prove that $a+b$ must be a perfect square.

Solution. By algebra, $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$ is equivalent to $\frac{a-c}{c} = \frac{c}{b-c}$. Suppose $\frac{a-c}{c} = \frac{c}{b-c} = \frac{p}{q}$, where p, q are positive integers and $\gcd(p,q) = 1$. Then $\frac{a}{p+q} = \frac{c}{q}$ and $\frac{b}{p+q} = \frac{c}{p}$ by simple algebra. So

$$\frac{a}{p(p+q)} = \frac{b}{q(p+q)} = \frac{c}{pq}.$$

Now $\gcd(p,q) = 1$ implies $\gcd(p(p+q), q(p+q), pq) = 1$. Since $\gcd(a,b,c) = 1$, we have $a = p(p+q)$, $b = q(p+q)$ and $c = pq$. Therefore $a + b = (p+q)^2$.

146. (1969 Eötvös-Kürschák Math Competition) Let n be a positive integer. Show that if $2 + 2\sqrt{28n^2+1}$ is an integer, then it is a square.

Solution. If $2 + 2\sqrt{28n^2+1} = m$, an integer, then $4(28n^2+1) = (m-2)^2$. This implies m is even, say $m = 2k$. So $28n^2 = k^2 - 2k$. This implies k is even, say $k = 2j$. Then $7n^2 = j(j-1)$. Since $\gcd(j, j-1) = 1$, either $j = 7x^2, j-1 = y^2$ or $j = x^2, j-1 = 7y^2$. In the former case, we get $-1 \equiv y^2 \pmod{7}$, which is impossible. In the latter case, $m = 2k = 4j = 4x^2$ is a square.

147. (1998 Putnam Exam) Prove that, for any integers a, b, c, there exists a positive integer n such that $\sqrt{n^3 + an^2 + bn + c}$ is not an integer.

Solution. Let $P(x) = x^3 + ax^2 + bx + c$ and $n = |b| + 1$. Observe that $P(n) \equiv P(n+2) \pmod{2}$. Suppose both $P(n)$ and $P(n+2)$ are perfect squares. Since perfect squares are congruent to 0 or 1 (mod 4), so $P(n) \equiv P(n+2) \pmod{4}$. However, $P(n+2) - P(n) = 2n^2 + 2b$ is not divisible by 4, a contradiction. So either $P(n)$ or $P(n+2)$ is not a perfect square. Therefore, either $\sqrt{P(n)}$ or $\sqrt{P(n+2)}$ is not an integer.

148. (1995 IMO shortlisted problem) Let k be a positive integer. Prove that there are infinitely many perfect squares of the form $n2^k - 7$, where n is a positive integer.

Solution. It suffices to show there is a sequence of positive integers a_k such that $a_k^2 \equiv -7 \pmod{2^k}$ and the a_k's have no maximum. Let $a_1 = a_2 = a + 3 = 1$. For $k \ge 3$, suppose $a_k^2 \equiv -7 \pmod{2^k}$. Then either $a_k^2 \equiv -7 \pmod{2^{k+1}}$ or $a_k^2 \equiv 2^k - 7 \pmod{2^{k+1}}$. In the former case, let $a_{k+1} = a_k$. In the latter case, let $a_{k+1} = a_k + 2^{k-1}$. Then since $k \ge 3$ and a_k is odd,

$$a_{k+1}^2 = a_k^2 + 2^k a_k + 2^{2k-2} \equiv a_k^2 + 2^k a_k \equiv a_k^2 + 2^k \equiv -7 \pmod{2^{k+1}}.$$

Since $a_k^2 \ge 2^k - 7$ for all k, the sequence has no maximum.

149. Let a, b, c be integers such that $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = 3$. Prove that abc is the cube of an integer.

Solution. Without loss of generality, we may assume $\gcd(a,b,c) = 1$. (Otherwise, if $d = \gcd(a,b,c)$, then for $a' = a/d, b' = b/d, c' = c/d$, the equation still holds for a', b', c' and $a'b'c'$ is a cube if and only if abc is a cube.) Multiplying by abc, we get a new equation $a^2c + b^2a + c^2b = 3abc$.

If $abc = \pm 1$, then we are done. Otherwise, let p be a prime divisor of abc. Since $\gcd(a,b,c) = 1$, the new equation implies that p divides exactly two of a, b, c. By symmetry, we may assume p divides

a, b, but not c. Suppose the largest powers of p dividing a, b are m, n, respectively.

If $n < 2m$, then $n+1 \le 2m$ and $p^{n+1} \mid a^2c,\ b^2c,\ 3abc$. Hence $p^{n+1} \mid c^2b$, forcing $p \mid c$, a contradiction. If $n > 2m$, then $n \ge 2m+1$ and $p^{2m+1} \mid c^2b,\ b^2a,\ 3abc$. Hence $p^{2m+1} \mid a^2c$, forcing $p \mid c$, a contradiction. Therefore, $n = 2m$ and $abc = \prod_{p|abc} p^{3m}$ is a cube.

Diophantine Equations

150. Find all sets of positive integers x, y and z such that $x \le y \le z$ and $x^y + y^z = z^x$.

Solution. (Due to Cheung Pok Man) Since $3^{1/3} > 4^{1/4} > 5^{1/5} > \cdots$, we have $y^z \ge z^y$ if $y \ge 3$. Hence the equation has no solution if $y \ge 3$. Since $1 \le x \le y$, the only possible values for (x, y) are $(1,1), (1,2)$ and $(2,2)$. These lead to the equations $1+1 = z, 1+2^z = z$ and $4+2^z = z^2$. The third equation has no solution since $2^z \ge z^2$ for $z \ge 4$ and $(2,2,3)$ is not a solution to $x^y + y^z = z^x$. The second equation has no solution either since $2^z > z$. The first equation leads to the unique solution $(1,1,2)$.

151. (Due to W. Sierpinski in 1955) Find all positive integral solutions of $3^x + 4^y = 5^z$.

Solution. We will show there is exactly one set of solution, namely $x = y = z = 2$. To simplify the equation, we consider modulo 3. We have $1 = 0 + 1^y \equiv 3^x + 4^y = 5^z \equiv (-1)^z \pmod 3$. It follows that z must be even, say $z = 2w$. Then $3^x = 5^z - 4^y = (5^w + 2^y)(5^w - 2^y)$. Now $5^w + 2^y$ and $5^w - 2^y$ are not both divisible by 3, since their sum is not divisible by 3. So, $5^w + 2^y = 3^x$ and $5^w - 2^y = 1$. Then, $(-1)^w + (-1)^y \equiv 0 \pmod 3$ and $(-1)^w - (-1)^y \equiv 1 \pmod 3$. From these, we get w is odd and y is even. If $y > 2$, then $5 \equiv 5^w + 2^y = 3^x \equiv 1$ or $3 \pmod 8$, a contradiction. So $y = 2$. Then $5^w - 2^y = 1$ implies $w = 1$ and $z = 2$. Finally, we get $x = 2$.

152. (Due to Euler, also 1985 Moscow Math Olympiad) If $n \ge 3$, then prove that 2^n can be represented in the form $2^n = 7x^2 + y^2$ with x, y odd positive integers.

Solution. After working out solutions for the first few cases, a pattern begins to emerge. If (x, y) is a solution to case n, then the pattern suggests the following: If $(x+y)/2$ is odd, then $((x+y)/2, |7x-y|/2)$ should be a solution for the case $n+1$. If $(x+y)/2$ is even, then $(|x-y|/2, (7x+y)/2)$ should be a solution for the case $n+1$. Before we confirm this, we observe that since $(x+y)/2 + |x-y|/2 = \max(x, y)$ is odd, exactly one of $(x+y)/2,\ |x-y|/2$ is odd. Similarly, exactly one of $(7x+y)/2,\ |7x-y|/2$ is odd. Also, if (x, y) is a solution and one of x, y is odd, then the other is also odd.

Now we confirm the pattern by induction. For the case $n = 3$, $(x, y) = (1,1)$ with $(1+1)/2 = 1$ leads to a solution $(1,3)$ for case $n = 4$. Suppose in case n, we have a solution (x, y). If $(x+y)/2$ is odd, then $7\left(\frac{x+y}{2}\right)^2 + \left(\frac{|7x-y|}{2}\right)^2 = 14x^2 + 2y^2 = 2^{n+1}$. If $(x+y)/2$ is even, then $7\left(\frac{|x-y|}{2}\right)^2 + \left(\frac{7x+y}{2}\right)^2 = 14x^2 + 2y^2 = 2^{n+1}$. Therefore, the pattern is true for all cases by induction.

153. (1995 IMO shortlisted problem) Find all positive integers x and y such that $x + y^2 + z^3 = xyz$, where z is the greatest common divisor of x and y.

Solution. Suppose (x, y) is a pair of solution. Let $x = az,\ y = bz$, where a, b are positive integers (and $\gcd(a, b) = 1$). The equation implies $a + b^2z + z^2 = abz^2$. Hence $a = cz$ for some integer c and we have $c + b^2 + z = cbz^2$, which gives $c = \dfrac{b^2+z}{bz^2-1}$. If $z = 1$, then $c = \dfrac{b^2+1}{b-1} = b + 1 + \dfrac{2}{b-1}$. It follows that $b = 2$ or 3, so $(x, y) = (5,2)$ or $(5,3)$. If $z = 2$, then $16c = \dfrac{16b^2+32}{4b-1} = 4b + 1 + \dfrac{33}{4b-1}$. It follows that $b = 1$ or 3, so $(x, y) = (4,2)$ or $(4,6)$.

In general, $cz^2 = \dfrac{b^2z^2+z^3}{bz^2-1} = b + \dfrac{b+z^3}{bz^2-1}$. Now integer $cz^2 - b = \dfrac{b+z^3}{bz^2-1} \geq 1$ implies $b \leq \dfrac{z^2-z+1}{z-1}$. If $z \geq 3$, then $\dfrac{z^2-z+1}{z-1} < z+1$, so $b \leq z$. It follows that $c = \dfrac{b^2+z}{bz^2-1} \leq \dfrac{z^2+z}{z^2-1} < 2$, so $c = 1$. Now b is an integer solution of $w^2 - z^2w + z + 1 = 0$. So the discriminant $z^4 - 4z - 4$ is a square. However, it is between $(z^2-1)^2$ and $(z^2)^2$, a contradiction. Therefore, the only solutions are $(x,y) = (4,2),\ (4,6),\ (5,2)$ and $(5,3)$.

154. Find all positive integral solutions to the equation $xy + yz + zx = xyz + 2$.

Solution. By symmetry, we may assume $x \leq y \leq z$. Dividing both sides by xyz, we get $\dfrac{1}{z} + \dfrac{1}{y} + \dfrac{1}{x} = 1 + \dfrac{2}{xyz}$. So

$$1 < 1 + \frac{2}{xyz} = \frac{1}{z} + \frac{1}{y} + \frac{1}{x} \leq \frac{3}{x}.$$

Then $x = 1$ or 2. If $x = 1$, then the equation implies $y = z = 1$. If $x = 2$, then $\dfrac{1}{z} + \dfrac{1}{y} = \dfrac{1}{2} + \dfrac{1}{yz}$. So $\dfrac{1}{2} < \dfrac{1}{2} + \dfrac{1}{yz} = \dfrac{1}{z} + \dfrac{1}{y} \leq \dfrac{2}{y}$. Then $y < 4$. Simple checkings yield $y = 3, z = 4$. Therefore, the required solutions are $(x,y,z) = (1,1,1), (2,3,4), (2,4,3), (3,2,4), (3,4,2), (4,2,3), (4,3,2)$.

155. Show that if the equation $x^2 + y^2 + 1 = xyz$ has positive integral solutions x, y, z, then $z = 3$.

Solution. (Due to Chan Kin Hang) Suppose the equation has positive integral solutions x, y, z with $z \neq 3$. Then $x \neq y$ (for otherwise $2x^2 + 1 = x^2z$ would give $x^2(z-2) = 1$ and so $x = 1, z = 3$). As the equation is symmetric in x, y, we may assume $x > y$. Among the positive integral solutions (x,y,z) with $x \geq y$ and $z \neq 3$, let (x_0, y_0, z_0) be a solution with x_0 least possible. Now $x^2 - y_0z_0x + (y_0^2+1) = 0$ has x_0 as a root. The other root is $x_1 = y_0z_0 - x_0 = (y_0^2+1)/x_0$. We have $0 < x_1 = (y_0^2+1)/x_0 \leq (y_0^2+1)/(y_0+1) \leq y_0$. Now (y_0, x_1, z_0) is also a positive integral solution with $y_0 \geq x_1$ and $z_0 \neq 3$. However $y_0 < x_0$ contradicts x_0 being least possible.

156. (1995 Czech-Slovak Match) Find all pairs of nonnegative integers x and y which solve the equation $p^x - y^p = 1$, where p is a given odd prime.

Solution. If (x,y) is a solution, then

$$p^x = y^p + 1 = (y+1)(y^{p-1} - \cdots + y^2 - y + 1)$$

and so $y + 1 = p^n$ for some n. If $n = 0$, then $(x,y) = (0,0)$ and p may be arbitrary. Otherwise,

$$\begin{aligned} p^x &= (p^n-1)^p + 1 \\ &= p^{np} - p\cdot p^{n(p-1)} + \binom{p}{2}p^{n(p-2)} + \cdots - \binom{p}{p-2}p^{2n} + p\cdot p^n. \end{aligned}$$

Since p is prime, all of the binomial coefficients are divisible by p. Hence all terms are divisible by p^{n+1}, and all but the last by p^{n+2}. Therefore the highest power of p dividing the right side is p^{n+1} and so $x = n+1$. We also have

$$0 = p^{np} - p\cdot p^{n(p-1)} + \binom{p}{2}p^{n(p-2)} + \cdots - \binom{p}{p-2}p^{2n}.$$

For $p = 3$, this gives $0 = 3^{3n} - 3\cdot 3^{2n}$, which implies $n = 1$ and $(x,y) = (2,2)$. For $p \geq 5$, $\binom{p}{p-2}$ is not divisible by p^2, so every term but the lat on the right is divisible by p^{2n+2}, while the last term is not. Since 0 is divisible by p^{2n+2}, this is a contradiction.

Therefore, the only solutions are $(x,y) = (0,0)$ for all odd prime p and $(x,y) = (2,2)$ for $p = 3$.

157. Find all integer solutions of the system of equations

$$x + y + z = 3 \quad \text{and} \quad x^3 + y^3 + z^3 = 3.$$

Solution. Suppose (x,y,z) is a solution. From the identity

$$(x+y+z)^3 - (x^3+y^3+z^3) = 3(x+y)(y+z)(z+x),$$

we get $8 = (3-z)(3-x)(3-y)$. Since $6 = (3-z) + (3-x) + (3-y)$. Checking the factorization of 8, we see that the solutions are $(1,1,1),\ (-5,4,4),\ (4,-5,4),\ (4,4,-5)$.

Solutions to Combinatorics Problems

Counting Methods

158. (1996 Italian Mathematical Olympiad) Given an alphabet with three letters a, b, c, find the number of words of n letters which contain an even number of a's.

Solution 1. (Due to Chao Khek Lun and Ng Ka Wing) For a nonnegative even integer $2k \le n$, the number of n letter words with $2k$ a's is $C^n_{2k} 2^{n-2k}$. The answer is the sum of these numbers, which can be simplified to $((2+1)^n + (2-1)^n)/2$ using binomial expansion.

Solution 2. (Due to Tam Siu Lung) Let S_n be the number of n letter words with even number of a's and T_n be the number of n letter words with odd number of a's. Then $S_n + T_n = 3^n$. Among the S_n words, there are T_{n-1} words ended in a and $2S_{n-1}$ words ended in b or c. So we get $S_n = T_{n-1} + 2S_{n-1}$. Similarly $T_n = S_{n-1} + 2T_{n-1}$. Subtracting these, we get $S_n - T_n = S_{n-1} - T_{n-1}$. So $S_n - T_n = S_1 - T_1 = 2 - 1 = 1$. Therefore, $S_n = (3^n + 1)/2$.

159. Find the number of n-words from the alphabet $A = \{0, 1, 2\}$, if any two neighbors can differ by at most 1.

Solution. Let x_n be the number of n-words satisfying the condition. So $x_1 = 3, x_2 = 7$. Let y_n be the number of n-words satisfying the condition and beginning with 0. (By interchanging 0 and 2, y_n is also the number of n-words satisfying the condition and beginning with 2.) Considering a 0, 1 or 2 in front of an n-word, we get $x_{n+1} = 3x_n - 2y_n$ and $y_{n+1} = x_n - y_n$. Solving for y_n in the first equation, then substituting into the second equation, we get $x_{n+2} - 2x_{n+1} - x_n = 0$. For convenience, set $x_0 = x_2 - 2x_1 = 1$. Since $r^2 - 2r - 1 = 0$ has roots $1 \pm \sqrt{2}$ and $x_0 = 1, x_1 = 3$, we get $x_n = \alpha(1+\sqrt{2})^n + \beta(1-\sqrt{2})^n$, where $\alpha = (1+\sqrt{2})/2, \beta = (1-\sqrt{2})/2$. Therefore, $x_n = ((1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1})/2$.

160. (1995 Romanian Math Olympiad) Let $A_1, A_2, \ldots, A_n$ be points on a circle. Find the number of possible colorings of these points with p colors, $p \ge 2$, such that any two neighboring points have distinct colors.

Solution. Let C_n be the answer for n points. We have $C_1 = p$, $C_2 = p(p-1)$ and $C_3 = p(p-1)(p-2)$. For $n+1$ points, if A_1 and A_n have different colors, then $A_1, \ldots, A_n$ can be colored in C_n ways, while A_{n+1} can be colored in $p-2$ ways. If A_1 and A_n have the same color, then $A_1, \ldots, A_n$ can be colored in C_{n-1} ways and A_{n+1} can be colored in $p-1$ ways. So $C_{n+1} = (p-2)C_n + (p-1)C_{n-1}$ for $n \ge 3$, which can be written as $C_{n+1} + C_n = (p-1)(C_n + C_{n-1})$. This implies $C_{n+1} + C_n = (p-1)^{n-2}(C_3 + C_2) = p(p-1)^n$. Then $C_n = (p-1)^n + (-1)^n(p-1)$ for $n > 3$ by induction.

Pigeonhole Principle

161. (1987 Austrian-Polish Math Competition) Does the set $\{1, 2, \ldots, 3000\}$ contain a subset A consisting of 2000 numbers such that $x \in A$ implies $2x \notin A$?

Solution. Let A_0 be the subset of $S = \{1, 2, \ldots, 3000\}$ containing all numbers of the form $4^n k$, where n is a nonnegative integer and k is an odd positive integer. Then no two elements of A_0 have ratio 2. A simple count shows A_0 has 1999 elements. Now for each $x \in A_0$, form a set $S_x = \{x, 2x\} \cap S$. Note the union of all S_x's contains S. So, by the pigeonhole principle, any subset of S having more than 1999 elements must contain a pair in some S_x, hence of ratio 2. So no subset of 2000 numbers in S has the property.

162. (1989 Polish Math Olympiad) Suppose a triangle can be placed inside a square of unit area in such a way that the center of the square is not inside the triangle. Show that one side of the triangle has length less than 1.

Solution. (Due to To Kar Keung) Through the center c of the square, draw a line L_1 parallel to the closest side of the triangle and a second line L_2 perpendicular to L_1 at c. The lines L_1 and L_2 divide the square into four congruent quadrilaterals. Since c is not inside the triangle, the triangle can lie in at most two (adjacent) quadrilaterals. By the pigeonhole principle, two of the vertices of the triangle must belong to the same quadrilateral. Now the furthest distance between two points in the quadrilateral is the distance between two opposite vertices, which is at most 1. So the side of the triangle with two vertices lying in the same quadrilateral must have length less than 1.

163. The cells of a 7×7 square are colored with two colors. Prove that there exist at least 21 rectangles with vertices of the same color and with sides parallel to the sides of the square.

Solution. (Due to Wong Chun Wai) Let the colors be black and white. For a row, suppose there are k black cells and $7-k$ white cells. Then there are $C_2^k + C_2^{7-k} = k^2 - 7k + 21 \geq 9$ pairs of cells with the same color. So there are at least $7\times 9 = 63$ pairs of cells on the same row with the same color. Next there are $C_2^7 = 21$ pairs of columns. So there are $21\times 2 = 42$ combinations of color and pair of columns. For combination $i = 1$ to 42, if there are j_i pairs in the same combination, then there are at least $j_i - 1$ rectangles for that combination. Since the sum of the j_i's is at least 63, so there are at least $\sum_{i=1}^{42}(j_i - 1) \geq 63 - 42 = 21$ such rectangles.

164. For $n > 1$, let $2n$ chess pieces be placed at the centers of $2n$ squares of an $n \times n$ chessboard. Show that there are four pieces among them that formed the vertices of a parallelogram. If $2n$ is replaced by $2n-1$, is the statement still true in general?

Solution. (Due to Ho Wing Yip) Let m be the number of rows that have at least 2 pieces. (Then each of the remaining $n-m$ rows contains at most 1 piece.) For each of these m rows, locate the leftmost square that contains a piece. Record the distances (i.e. number of squares) between this piece and the other pieces on the same row. The distances can only be $1, 2, \ldots, n-1$ because there are n columns.

Since the number of pieces in these m rows altogether is at least $2n - (n-m) = n + m$, there are at least $(n+m) - m = n$ distances recorded altogether for these m rows. By the pigeonhole principle, at least two of these distances are the same. This implies there are at least two rows each containing 2 pieces that are of the same distance apart. These 4 pieces yield a parallelogram.

For the second question, placing $2n-1$ pieces on the squares of the first row and first column shows there are no parallelograms.

165. The set $\{1, 2, \ldots, 49\}$ is partitioned into three subsets. Show that at least one of the subsets contains three different numbers a, b, c such that $a + b = c$.

Solution. By the pigeonhole principle, one of the subsets, say X, must contain at least 49/3 elements, say $x_1 < x_2 < \ldots < x_{17}$. Form the differences $x_2 - x_1, x_3 - x_1, \ldots, x_{17} - x_1$ and remove x_1 (because a, b, c are to be different) if it appears on the list. If one of the remaining differences belongs to X, then we are done.

Otherwise, by the pigeonhole principle again, one of the subsets, say $Y(\neq X)$, must contain at least 15/2 elements from these differences $y_j = x_{i_j} - x_1$, say $y_1 < y_2 < \ldots < y_8$. Consider the differences $y_2 - y_1, y_3 - y_1, \ldots, y_8 - y_1$ and remove y_1 and x_{i_1} if they appear on the list. If one of these differences belong to Y, then we are done. If one of them, say $y_j - y_1 = x_{i_j} - x_{i_1} (\neq x_{i_1}, x_{i_j})$, belong to X, then let $x_{i_1}, x_{i_j}, x_{i_j} - x_{i_1}$ are different elements of X and $(x_{i_j} - x_{i_1}) + x_{i_1} = x_{i_j}$ and we are done.

Thus, we may assume 5 of these differences $z_k = y_{j_k} - y_1$, belong to the remainging subset Z and say $z_1 < z_2 < \ldots < z_5$. Form the difference $z_2 - z_1, z_3 - z_1, z_4 - z_1, z_5 - z_1$ and remove $z_1, y_{j_1}, x_{i_{j_1}}$ if they appear on the list. The remaining difference $z_k - z_1 = y_{j_k} - y_{j_1} = x_{i_{j_k}} - y_{i_{j_1}}$ must belong to one of X, Y or Z. As above, we get three distinct elements a, b, c in one of X, Y or Z such that $a + b = c$.

Inclusion-Exclusion Principle

166. Let $m \geq n > 0$. Find the number of surjective functions from $B_m = \{1, 2, \ldots, m\}$ to $B_n = \{1, 2, \ldots, n\}$.

Solution. For $i = 1, 2, \ldots, n$, let A_i be the set of functions $f : B_m \to$

B_n such that $i \neq f(1), \ldots, f(m)$. By the inclusion-exclusion principle,

$$\begin{aligned}
&|A_1 \cup \cdots \cup A_n| \\
&= \sum_{1\le i\le n} |A_i| - \sum_{1\le i<j\le n} |A_i \cap A_j| + \sum_{1\le i<j<k\le n} |A_i \cap A_2 \cap A_3| - \cdots \\
&= \binom{n}{1}(n-1)^m - \binom{n}{2}(n-2)^m + \binom{n}{3}(n-3)^m - \cdots .
\end{aligned}$$

The number of surjections from B_m to B_n is

$$n^m - |A_1 \cup \cdots \cup A_n| = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^m.$$

167. Let A be a set with 8 elements. Find the maximal number of 3-element subsets of A, such that the intersection of any two of them is not a 2-element set.

Solution. Let $|S|$ denote the number of elements in a set S. Let $B_1, \cdots, B_n \subseteq A$ be such that $|B_i| = 3$, $|B_i \cap B_j| \neq 2$ for $i, j = 1, \ldots, n$. If $a \in A$ belongs to $B_1, \ldots, B_k$, then $|B_i \cap B_j| = 1$ for $i, j = 1, \ldots, k$. Since $8 = |A| \geq |B_1 \cup \cdots \cup B_k| = 1 + 2k$, we get $k \leq 3$. From this, we see that every element of A is in at most 3 B_i's. Then $3n \leq 8 \times 3$, so $n \leq 8$. To show 8 is possible, just consider

$$B_1 = \{1,2,3\},\ B_2 = \{1,4,5\},\ B_3 = \{1,6,7\},\ B_4 = \{8,3,4\},$$

$$B_5 = \{8,2,6\},\ B_6 = \{8,5,7\},\ B_7 = \{3,5,6\},\ B_8 = \{2,4,7\}.$$

168. (a) (1999 China Hong Kong Math Olympiad) Students have taken a test paper in each of n ($n \geq 3$) subjects. It is known that for any subject exactly three students get the best score in the subject, and for any two subjects excatly one student gets the best score in every one of these two subjects. Determine the smallest n so that the above conditions imply that exactly one student gets the best score in every one of the n subjects.

(b) (1978 Austrian-Polish Math Competition) There are 1978 clubs. Each has 40 members. If every two clubs have exactly one common member, then prove that all 1978 clubs have a common member.

Solution. (a) (Due to Fan Wai Tong) For $i = 1, 2, \ldots, n$, let S_i be the set of students who get the best score in the i-th subject. Suppose nobody gets the best score in every one of the n subjects. Let x be one student who is best in most number of subjects, say m ($m < n$) subjects. Without loss of generality, suppose x is in $S_1, S_2, \ldots, S_m$. For $i = 1, 2, \ldots, m$, let $S_i' = S_i \setminus \{x\}$. Then the m sets S_i' are pairwise disjoint and so each shares a (distinct) common member with S_{m+1}. Since S_{m+1} has three members, so $m \leq 3$. This means each student is best in at most three subjects. By the inclusion-exclusion principle,

$$\begin{aligned}
&|S_1 \cup S_2 \cup \cdots \cup S_n| \\
&= \sum_{1\le i\le n} |S_i| - \sum_{1\le i<j\le n} |S_i \cap S_j| + \sum_{1\le i<j<k\le n} |S_i \cap S_j \cap S_k| \\
&\le\ 3n - \binom{n}{2} + |S_1 \cup S_2 \cup \cdots \cup S_n|,
\end{aligned}$$

which implies $n \leq 7$. Therefore, if $n \geq 8$, then there is at least one student who get the best score in every one of the n subjects. There is exactly one such students because only one student gets the best score in a pair of subjects.

Finally, we give an example of the case $n = 7$ with nobody best in all subjects:

$$S_1 = \{x_1, x_2, x_3\}, \quad S_2 = \{x_1, x_4, x_5\}, \quad S_3 = \{x_2, x_4, x_6\},$$

$$S_4 = \{x_3, x_5, x_6\}, \quad S_5 = \{x_1, x_6, x_7\}, \quad S_6 = \{x_2, x_5, x_7\},$$

$$S_7 = \{x_3, x_4, x_7\}.$$

(b) Let $n = 1978$ and $k = 40$. Let $C_1, C_2, \ldots, C_n$ be the n clubs. For each member of C_1, form a list of the indices of the other clubs that this member also belongs to. Since C_1 and any other club C_i have exactly one common member, the k lists of the k members of C_1 are disjoint and together contain all integers from 2 to n. By the pigeonhole principle, one of the lists, say x's list, will contain at least $m = \lceil \frac{n-1}{k} \rceil$ numbers. (The notation means m is the least integer greater than or equal to $\frac{n-1}{k}$.)

Next we will show this x is a member of all n clubs. Suppose x is not a member of some club C_i. Then each of the $m+1$ clubs that x belong to will share a different member with C_i (otherwise two of the $m+1$ clubs will share a member y in C_i and also x, a contradiction). Since C_i has k members, so $k \geq m+1 \geq \frac{n-1}{k}+1$, which implies $k^2-k+1 \geq n$. Since $k^2-k+1 = 1561 < n = 1978$, this is a contradiction. So x must be a member of all n clubs.

Comments. It is clear that the two problems are essentially the same. As the number of members in the sets gets large, the inclusion-exclusion principle in (a) will be less effective. The argument in part (b) is more convenient and shows that for n sets, each having k members and each pair having exactly one common member, if $n > k^2-k+1$, then all n sets have a common member.

Combinatorial Designs

169. (1995 Byelorussian Math Olympiad) In the begining, 65 beetles are placed at different squares of a 9×9 square board. In each move, every beetle creeps to a horizontal or vertical adjacent square. If no beetle makes either two horizontal moves or two vertical moves in succession, show that after some moves, there will be at least two beetles in the same square.

Solution. (Due to Cheung Pok Man and Yung Fai) Assign an ordered pair (a,b) to each square with $a,b = 1,2,\ldots,9$. Divide the 81 squares into 3 types. Type A consists of squares with both a and b odd, type B consists of squares with both a and b even and type C consists of the remaining squares. The numbers of squares of the types A, B and C are 25, 16 and 40, respectively.

Assume no collision occurs. After two successive moves, beetles in type A squares will be in type B squares. So the number of beetles in

type A squares are at most 16 at any time. Then there are at most 32 beetles in type A or type B squares at any time. Also, after one move, beetles in type C squares will go to type A or type B squares. So there are at most 32 beetles in type C squares at any time. Hence there are at most 64 beetles on the board, a contradiction.

170. (1995 Greek Math Olympiad) Lines $l_1, l_2, \ldots, l_k$ are on a plane such that no two are parallel and no three are concurrent. Show that we can label the C_2^k intersection points of these lines by the numbers $1, 2, \ldots, k-1$ so that in each of the lines $l_1, l_2, \ldots, l_k$ the numbers $1, 2, \ldots, k-1$ appear exactly once if and only if k is even.

Solution. (Due to Ng Ka Wing) If such labeling exists for an integer k, then the label 1 must occur once on each line and each point labeled 1 lies on exactly 2 lines. Hence there are $k/2$ 1's, i.e. k is even.

Conversely, if k is even, then the following labeling works: for $1 \leq i < j \leq k-1$, give the intersection of lines l_i and l_j the label $i+j-1$ when $i+j \leq k$, the label $i+j-k$ when $i+j > k$. For the intersection of lines l_k and l_i $(i = 1,2,\ldots,k-1)$, give the label $2i-1$ when $2i \leq k$, the label $2i-k$ when $2i > k$.

Alternatively, we can make use of the symmetry of an odd number sided regular polygon to construct the labeling as follows: for k even, consider the $k-1$ sided regular polygon with the vertices labeled $1, 2, \ldots, k-1$. For $1 \leq i < j \leq k-1$, the perpendicular bisector of the segment joining vertices i and j passes through a unique vertex, give the intersection of lines l_i and l_j the label of that vertex. For the intersection of lines l_k and l_i $(i = 1,2,\ldots,k-1)$, give the label i.

171. (1996 Tournaments of the Towns) In a lottery game, a person must select six distinct numbers from $1, 2, 3, \ldots, 36$ to put on a ticket. The lottery commitee will then draw six distinct numbers randomly from $1, 2, 3, \ldots, 36$. Any ticket with numbers *not* containing any of these six numbers is a winning ticket. Show that there is a scheme of buying 9 tickets guaranteeing at least a winning ticket, but 8 tickets is not enough to guarantee a winning ticket in general.

Solution. Consider the nine tickets with numbers

$$(1,2,3,4,5,6),\quad (1,2,3,7,8,9),\quad (4,5,6,7,8,9),$$

$$(10,11,12,13,14,15),\quad (10,11,12,16,17,18),\quad (13,14,15,16,17,18),$$

$$(19,20,21,22,23,24),\quad (25,26,27,28,29,30),\quad (31,32,33,34,35,36).$$

For the first three tickets, if they are not winning, then two of the six numbers drawn must be among $1,2,\ldots,9$. For the next three tickets, if they are not winning, then two of the six sumbers must be $10,11,\ldots,18$. For the last three tickets, if they are not winning, then three of the six numbers must be among $19,20,\ldots,36$. Since only six numbers are drawn, at least one of the nine tickets is a winning ticket.

For any eight tickets, if one number appears in three tickets, then this number and one number from each of the five remaining tickets may be the six numbers drawn, resulting in no winning tickets.

So of the 48 numbers on the eight tickets, we may assume (at least) 12 appeared exactly 2 times, say they are $1,2,\ldots,12$. Consider the two tickets with 1 on them. The remaining 10 numbers on them will miss (at least) one of the numbers $2,3,\ldots,12$, say 12. Now 12 appears in two other tickets. Then 1, 12 and one number from each of the four remaining tickets may be the six numbers drawn by the committee, resulting in no winning tickets.

172. (1995 Byelorussian Math Olympiad) By dividing each side of an equilateral triangle into 6 equal parts, the triangle can be divided into 36 smaller equilateral triangles. A beetle is placed on each vertex of these triangles at the same time. Then the beetles move along different edges with the same speed. When they get to a vertex, they must make a $60°$ or $120°$ turn. Prove that at some moment two beetles must meet at some vertex. Is the statement true if 6 is replaced by 5?

Solution. We put coordinates at the vertices so that (a,b), for $0 \le b \le a \le 6$, corresponds to the position of $\binom{a}{b}$ in the Pascal triangle. First mark the vertices

$$(0,0),\ (2,0),\ (2,2),\ (4,0),\ (4,2),\ (4,4),\ (6,0),\ (6,2),\ (6,4),\ (6,6).$$

After one move, if no beetles meet, then the 10 beetles at the marked vertices will move to 10 unmarked vertices and 10 other beetles will move to the marked vertices. After another move, these 20 beetles will be at unmarked vertices. Since there are only 18 unmarked vertices, two of them will meet.

If 6 is replaced by 5, then divide the vertices into groups as follows:

$$\{(0,0),(1,0),(1,1)\},\quad \{(2,0),(3,0),(3,1)\},$$

$$\{(2,1),(3,2),(3,3),(2,2)\},\quad \{(4,0),(5,0),(5,1)\},$$

$$\{(4,1),(5,2),(5,3),(4,2)\},\quad \{(4,3),(5,4),(5,5),(4,4)\}.$$

Let the beetles in each group move in the counterclockwise direction along the vertices in the group. Then the beetles will not meet at any moment.

<u>*Covering, Convex Hull*</u>

173. (1991 Australian Math Olympiad) There are n points given on a plane such that the area of the triangle formed by every 3 of them is at most 1. Show that the n points lie on or inside some triangle of area at most 4.

Solution. (Due to Lee Tak Wing) Let the n points be $P_1, P_2, \ldots, P_n$. Suppose $\triangle P_iP_jP_k$ have the maximum area among all triangles with vertices from these n points. No P_l can lie on the opposite side of the line through P_i parallel to P_jP_k as P_jP_k, otherwise $\triangle P_jP_kP_l$ has larger area than $\triangle P_iP_jP_k$. Similarly, no P_l can lie on the opposite side of the line through P_j parallel to P_iP_k as P_iP_k or on the opposite side of the line through P_k parallel to P_iP_j as P_iP_j. Therefore, each of the n points lie in the interior or on the boundary of the triangle having P_i, P_j, P_k as midpoints of its sides. Since the area of $\triangle P_iP_jP_k$ is at most 1, so the area of this triangle is at most 4.

174. (1969 Putnam Exam) Show that any continuous curve of unit length can be covered by a closed rectangles of area 1/4.

Solution. Place the curve so that its endpoints lies on the x-axis. Then take the smallest rectangle with sides parallel to the axes which covers the curve. Let its horizontal and vertical dimensions be a and b, respectively. Let P_0 and P_5 be its endpoints. Let P_1, P_2, P_3, P_4 be the points on the curve, in the order named, which lie one on each of the four sides of the rectangle. The polygonal line $P_0P_1P_2P_3P_4P_5$ has length at most one.

The horizontal projections of the segments of this polygonal line add up to at least a, since the line has points on the left and right sides of the rectangle. The vertical projections of the segments of this polygonal line add up to at least $2b$, since the endpoints are on the x-axis and the line also has points on the top and bottom side of the rectangle.

So the polygonal line has length at least $\sqrt{a^2+4b^2} \geq 1$. By the AM-GM inequality, $4ab \leq a^2+4b^2 \leq 1$ and so the area is at most $1/4$.

175. (1998 Putnam Exam) Let $\mathcal{F}$ be a finite collection of open discs in the plane whose union covers a set E. Show that there is a pairwise disjoint subcollection $D_1, \ldots, D_n$ in $\mathcal{F}$ such that the union of $3D_1, \ldots, 3D_n$ covers E, where $3D$ is the disc with the same center as D but having three times the radius.

Solution. We construct such D_i's by the greedy algorithm. Let D_1 be a disc of largest radius in $\mathcal{F}$. Suppose $D_1, \ldots, D_j$ has been picked. Then we pick a disc D_{j+1} disjoint from each of $D_1, \ldots, D_j$ and has the largest possible radius. Since $\mathcal{F}$ is a finite collection, the algorithm will stop at a final disc D_n. For x in E, suppose x is not in the union of $D_1, \ldots, D_n$. Then x is in some disc D of radius r in $\mathcal{F}$. Now D is not one of the D_j's implies it intersects some disc D_j of radius $r_j \geq r$. By the triangle inequality, the centers is at most $r + r_j$ units apart. Then D is contained in $3D_j$. In particular, x is in $3D_j$. Therefore, E is contained in the union of $3D_1, \ldots, 3D_n$.

176. (1995 IMO) Determine all integers $n > 3$ for which there exist n points $A_1, A_2, \ldots, A_n$ in the plane, and real numbers $r_1, r_2, \ldots, r_n$ satisfying the following two conditions:

(a) no three of the points $A_1, A_2, \ldots, A_n$ lie on a line;
(b) for each triple i, j, k $(1 \leq i < j < k \leq n)$ the triangle $A_iA_jA_k$ has area equal to $r_i + r_j + r_k$.

Solution. (Due to Ho Wing Yip) For $n = 4$, note $A_1 = (0,0), A_2 = (1,0), A_3 = (1,1), A_4 = (0,1), r_1 = r_2 = r_3 = r_4 = 1/6$ satisfy the conditions. Next we will show there are no solutions for $n \geq 5$. Suppose the contrary, consider the convex hull of A_1, A_2, A_3, A_4, A_5. (This is the smallest convex set containing the five points.) There are three cases.

<u>Triangular Case.</u> We may assume the points are named so A_1, A_2, A_3 are the vertices of the convex hull, with A_4, A_5 inside such that A_5 is outside $\triangle A_1A_2A_4$ and A_4 is outside $\triangle A_1A_3A_5$. Denote the area of $\triangle XYZ$ by $[XYZ]$. We get a contradiction as follows:

$$\begin{aligned}[A_1A_4A_5] + [A_1A_2A_3] &= (r_1+r_4+r_5) + (r_1+r_2+r_3) \\ &= (r_1+r_2+r_4) + (r_1+r_3+r_5) \\ &= [A_1A_2A_4] + [A_1A_3A_5] < [A_1A_2A_3].\end{aligned}$$

<u>Pentagonal Case.</u> We may assume $r_1 = \min\{r_1, r_2, r_3, r_4, r_5\}$. Draw line L through A_1 parallel to A_3A_4. Since $[A_1A_3A_4] = r_1 + r_3 + r_4 \leq r_2 + r_3 + r_4 = [A_2A_3A_4]$, A_2 is on line L or on the half plane of L opposite A_3, A_4 and similarly for A_5. Since A_1, A_2, A_5 cannot all be on L, we get $\angle A_2A_1A_5 > 180°$ contradicting convexity.

<u>Quadrilateral Case.</u> We may assume A_5 is inside the convex hull. First obseve that $r_1 + r_3 = R_2 + r_4$. This is because

$$(r_1+r_2+r_3) + (r_3+r_4+r_1) = (r_1+r_2+r_4) + (r_2+r_4+r_3)$$

is the area S of the convex hull. So $2S = 3(r_1+r_2+r_3+r_4)$. Also

$$\begin{aligned}S &= [A_1A_2A_5] + [A_2A_3A_5] + [A_3A_4A_5] + [A_4A_1A_5] \\ &= 2(r_1 + r_2r_3 + r_4) + r_5.\end{aligned}$$

From the last equation, we get $r_5 = -(r_1+r_2+r_3+r_4)/8 = -S/12 < 0$.

Next observe that A_1, A_5, A_3 not collinear implies one side of $\angle A_1A_5A_3$ is less than $180°$. Then one of the quadrilaterals $A_1A_5A_3A_4$

or $A_1A_5A_3A_2$ is convex. By the first observation of this case, $r_1+r_3 = r_5+r_i$, where $r_i = r_4$ or r_2. Since $r_1+r_3 = r_2+r_4$, we get $r_5 = r_2$ or r_4. Similarly, considering A_2, A_5, A_4 not collinear, we also get $r_5 = r_1$ or r_3. Therefore, three of the numbers r_1, r_2, r_3, r_4, r_5 are negative, but the area of the corresponding triangle is positive, a contradiction.

177. (1999 IMO) Determine all finite sets S of at least three points in the plane which satisfy the following condition: for any two distinct points A and B in S, the perpendicular bisector of the line segment AB is an axis of symmetry of S.

Solution. Clearly, no three points of such a set is collinear (otherwise considering the perpendicular bisector of the two furthest points of S on that line, we will get a contradiction). Let H be the convex hull of such a set, which is the smallest convex set containing S. Since S is finite, the boundary of H is a polygon with the vertices $P_1, P_2, \ldots, P_n$ belonging to S. Let $P_i = P_j$ if $i \equiv j \pmod{n}$. For $i = 1, 2, \ldots, n$, the condition on the set implies P_i is on the perpendicular bisector of $P_{i-1}P_{i+1}$. So $P_{i-1}P_i = P_iP_{i+1}$. Considering the perpendicular bisector of $P_{i-1}P_{i+2}$, we see that $\angle P_{i-1}P_iP_{i+1} = \angle P_iP_{i+1}P_{i+2}$. So the boundary of H is a regular polygon.

Next, there cannot be any point P of S inside the regular polygon. (To see this, assume such a P exists. Place it at the origin and the furthest point Q of S from P on the positive real axis. Since the origin P is in the interior of the convex polygon, not all the vertices can lie on or to the right of the y-axis. So there exists a vertex P_j to the left of the y-axis. Since the perpendicular bisector of PQ is an axis of symmetry, the mirror image of P_j will be a point in S further than Q from P, a contradiction.) So S is the set of vertices of some regular polygon. Conversely, such a set clearly has the required property.

Comments. The official solution is shorter and goes as follows: Suppose $S = \{X_1, \ldots, X_n\}$ is such a set. Consider the *barycenter* of S, which is the point G such that

$$\overrightarrow{OG} = \frac{\overrightarrow{OX_1} + \cdots + \overrightarrow{OX_n}}{n}.$$

(The case $n = 2$ yields the midpoint of segment X_1X_2 and the case $n = 3$ yields the centroid of triangle $X_1X_2X_3$.) Note the barycenter does not depend on the origin. To see this, suppose we get a point G' using another origin O', i.e. $\overrightarrow{O'G'}$ is the average of $\overrightarrow{O'X_i}$ for $i = 1, \ldots, n$. Subtracting the two averages, we get $\overrightarrow{OG} - \overrightarrow{O'G'} = \overrightarrow{OO'}$. Adding $\overrightarrow{O'G'}$ to both sides, we get $\overrightarrow{OG} = \overrightarrow{OG'}$, so $G = G'$.

By the condition on S, after reflection with respect to the perpendicular bisector of every segment X_iX_j, the points of S are permuted only. So G is unchanged, which implies G is on every such perpendicular bisector. Hence G is equidistant from all X_i's. Therefore, the X_i's are concyclic. For three consecutive points of S, say X_i, X_j, X_k, on the circle, considering the perpendicular bisector of segment X_iX_k, we have $X_iX_j = X_jX_k$. It follows that the points of S are the vertices of a regular polygon and the converse is clear.

178. (1995 Russian Math Olympiad) There are n seats at a merry-go-around. A boy takes n rides. Between each ride, he moves clockwise a certain number (less than n) of places to a new horse. Each time he moves a different number of places. Find all n for which the boy ends up riding each horse.

Solution. The case $n = 1$ works. If $n > 1$ is odd, the boy's travel $1+2+\cdots+(n-1) = n(n-1)/2$ places between the first and the last rides. Since $n(n-1)/2$ is divisible by n, his last ride will repeat the first horse. If n is even, this is possible by moving forward $1,\ n-2,\ 3,\ n-4,\ \ldots,\ n-1$ places corresponding to horses $1,\ 2,\ n,\ 3,\ n-1,\ \ldots,\ \frac{n}{2}+1$.

179. (1995 Israeli Math Olympiad) Two players play a game on an infinite board that consists of 1×1 squares. Player I chooses a square and marks it with an O. Then, player II chooses another square and marks it with X. They play until one of the players marks a row or a column of 5 consecutive squares, and this player wins the game. If no player can achieve this, the game is a tie. Show that player II can prevent player I from winning.

Solution. (Due to Chao Khek Lun) Divide the board into 2×2 blocks. Then bisect each 2×2 block into two 1×2 tiles so that for every pair of blocks sharing a common edge, the bisecting segment in one will be horizontal and the other vertical. Since every five consecutive squares on the board contains a tile, after player I chose a square, player II could prevent player I from winning by choosing the other square in the tile.

180. (1995 USAMO) A calculator is broken so that the only keys that still work are the sin, cos, tan, $\sin^{-1}, \cos^{-1}$, and $\tan^{-1}$ buttons. The display initially shows 0. Given any positive rational number q, show that pressing some finite sequence of buttons will yield q. Assume that the calculator does real number calculations with infinite precision. All functions are in terms of radians.

Solution. We will show that all numbers of the form $\sqrt{m/n}$, where m, n are positive integers, can be displayed by induction on $k = m+n$. (Since $r/s = \sqrt{r^2/s^2}$, these include all positive rationals.)

For $k = 2$, pressing cos will display 1. Suppose the statement is true for integer less than k. Observe that if x is displayed, then using the facts $\theta = \tan^{-1} x$ implies $\cos^{-1}(\sin\theta) = (\pi/2) - \theta$ and $\tan((\pi/2) - \theta) = 1/x$. So, we can display $1/x$. Therefore, to display $\sqrt{m/n}$ with $k = m + n$, we may assume $m < n$. By the induction step, $n < k$ implies $\sqrt{(n-m)/m}$ can be displayed. Then using $\phi = \tan^{-1}\sqrt{(n-m)/m}$ and $\cos\phi = \sqrt{m/n}$, we can display $\sqrt{m/n}$. This completes the induction.

181. (1977 Eötvös-Kürschák Math Competition) Each of three schools is attended by exactly n students. Each student has exactly $n + 1$ acquaintances in the other two schools. Prove that one can pick three students, one from each school, who know one another. It is assumed that acquaintance is mutual.

Solution. (Due to Chan Kin Hang) Consider a student who has the highest number, say k, of acquaintances in another school. Call this student x, his school X and the k acquaintances in school Y. Since $n+1 > n \geq k$, x must have at least one acquaintance, say z, in the third school Z. Now z has at most k acquaintances in school X and hence z has at least $(n+1)-k$ acquaintances in school Y. Adding the number of acquaintances of x and z in school Y, we get $k+(n+1)-k = n+1 > n$ and so x and z must have a common acquaintance y in school Y.

182. Is there a way to pack 250 $1 \times 1 \times 4$ bricks into a $10 \times 10 \times 10$ box?

Solution. Assign coordinate (x, y, z) to each of the cells, where $x, y, z = 0, 1, \ldots, 9$. Let the cell (x, y, z) be given color $x + y + z \pmod 4$. Note each $1 \times 1 \times 4$ brick contains all 4 colors exactly once. If the packing is possible, then there are exactly 250 cells of each color. However, a direct counting shows there are 251 cells of color 0, a contradiction. So such a packing is impossible.

183. Is it possible to write a positive integer into each square of the first quadrant such that each column and each row contains every positive integer exactly once?

Solution. Yes, it is possible. Define $A_1 = (1)$ and $A_{n+1} = \begin{pmatrix} B_n & A_n \\ A_n & B_n \end{pmatrix}$, where the entries of B_n are those of A_n plus 2^{n-1}. So

$$A_1 = (1), \quad A_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 3 & 4 & 1 & 2 \\ 2 & 1 & 4 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \quad \ldots.$$

Note that if every column and every row of A_n contain $1, 2, \ldots, 2^{n-1}$ exactly once, then every column and every row of B_n will contain $2^{n-1}+1, \ldots, 2^n$ exactly once. So, every column and every row of A_{n+1} will contain $1, 2, \ldots, 2^n$ exactly once. Now fill the first quadrant using the A_n's.

184. There are n identical cars on a circular track. Among all of them, they have just enough gas for one car to complete a lap. Show that there is a car which can complete a lap by collecting gas from the other cars on its way around the track in the clockwise direction.

Solution. (Due to Chan Kin Hang) The case $n = 1$ is clear. Suppose the case $n = k$ is true. For the case $n = k + 1$, first observe that there is a car A which can reach the next car B. (If no car can reach the next car, then the gas for all cars would not be enough for completing a lap.) Let us empty the gas of B into A and remove B. Then the k cars left satisfy the condition. So there is a car that can complete a lap. This same car will also be able to complete the lap collecting gas from other cars when B is included because when this car gets to car A, the gas collected from car A will be enough to get it to car B.

185. (1996 Russian Math Olympiad) At the vertices of a cube are written eight pairwise distinct natural numbers, and on each of its edges is written the greatest common divisor of the numbers at the endpoints of the edge. Can the sum of the numbers written at the vertices be the same as the sum of the numbers written at the edges?

Solution. Observe that if $a > b$, then $\gcd(a, b) \le b$ and $\gcd(a, b) \le a/2$. So $3\gcd(a, b) \le a + b$. If the sum of the vertex numbers equals the sum of the edge numbers, then we will have $\gcd(a, b) = (a + b)/3$ for every pair of adjacent vertex numbers, which implies $a = 2b$ or $b = 2a$ at the two ends of every edge. At every vertex, there are 3 adjacent vertices. The $a = 2b$ or $b = 2a$ condition implies two of these adjacent vertex numbers must be the same, a contradiction.

186. Can the positive integers be partitioned into infinitely many subsets such that each subset is obtained from any other subset by adding the same integer to each element of the other subset?

Solution. Yes. Let A be the set of positive integers whose odd digit positions (from the right) are zeros. Let B be the set of positive integers whose even digit positions (from the right) are zeros. Then A and B are infinite set and the set of positive integers is the union of $a + B = \{a + b : b \in B\}$ as a ranges over the elements of A. (For example, $12345 = 2040 + 10305 \in 2040 + B$.)

187. (1995 Russian Math Olympiad) Is it possible to fill in the cells of a 9×9 table with positive integers ranging from 1 to 81 in such a way that the sum of the elements of every 3×3 square is the same?

Solution. Place $0, 1, 2, 3, 4, 5, 6, 7, 8$ on the first, fourth and seventh rows. Place $3, 4, 5, 6, 7, 8, 0, 1, 2$ on the second, fifth and eigth rows. Place $6, 7, 8, 0, 1, 2, 3, 4, 5$ on the third, sixth and ninth rows. Then every 3×3 square has sum 36. Consider this table and its $90°$ rotation. For each cell, fill it with the number $9a + b + 1$, where a is the number in the cell originally and b is the number in the cell after the table is rotated by $90°$. By inspection, 1 to 81 appears exactly once each and every 3×3 square has sum $9 \times 36 + 36 + 9 = 369$.

188. (1991 German Mathematical Olympiad) Show that for every positive integer $n \ge 2$, there exists a permutation $p_1, p_2, \ldots, p_n$ of $1, 2, \ldots, n$ such that p_{k+1} divides $p_1 + p_2 + \cdots + p_k$ for $k = 1, 2, \ldots, n - 1$.

Solution. (The cases $n = 2, 3, 4, 5$ suggest the following permutations.) For even $n = 2m$, consider the permutation

$$m+1,\ 1,\ m+2,\ 2,\ \ldots,\ m+m,\ m.$$

For odd $n = 2m+1$, consider the permutation

$$m+1,\ 1,\ m+2,\ 2,\ \ldots,\ m+m,\ m,\ 2m+1.$$

If $k = 2j-1, (1 \le j \le m)$ then $(m+1)+1+\cdots+(m+j) = j(m+j)$. If $k = 2j, (1 \le j \le m)$ then $(m+1)+1+\cdots+(m+j)+j = j(m+j+1)$.

189. Each lattice point of the plane is labeled by a positive integer. Each of these numbers is the arithmetic mean of its four neighbors (above, below, left, right). Show that all the numbers are equal.

Solution. Consider the smallest number m labelled at a lattice point. If the four neighboring numbers are a, b, c, d, then $(a+b+c+d)/4 = m$ and $a, b, c, d \ge m$ imply $a = b = c = d = m$. Since any two lattice points can be connected by horizontal and vertical segments, if one end is labelled m, then along this path all numbers will equal to m. Therefore, every number equals m.

190. (1984 Tournament of the Towns) In a party, n boys and n girls are paired. It is observed that in each pair, the difference in height is less than 10 cm. Show that the difference in height of the k-th tallest boy and the k-th tallest girl is also less than 10 cm for $k = 1, 2, \ldots, n$.

Solution. (Due to Andy Liu, University of Alberta, Canada) Let $b_1 \ge b_2 \ge \cdots \ge b_n$ be the heights of the boys and $g_1 \ge g_2 \ge \cdots \ge g_n$ be those of the girls. Suppose for some k, $|b_k - g_k| \ge 10$. In the case $g_k - b_k \ge 10$, we have $g_i - b_j \ge g_k - b_k \ge 10$ for $1 \le i \le k$ and $k \le j \le n$. Consider the girls of height g_i, where $1 \le i \le k$, and the boys of height b_j, where $k \le j \le n$. By the pigeonhole principle, two of these $n+1$ people must be paired originally. However, for that pair, $g_i - b_j \ge 10$ contradicts the hypothesis. (The case $b_k - g_k \ge 10$ is handled similarly.) So $|b_k - g_k| < 10$ for all k.

191. (1991 Leningrad Math Olympiad) One may perform the following two operations on a positive integer:

(a) multiply it by any positive integer and
(b) delete zeros in its decimal representation.

Prove that for every positive integer X, one can perform a sequence of these operations that will transform X to a one-digit number.

Solution. By the pigeonhole principle, at least two of the $X+1$ numbers

$$1,\ 11,\ 111,\ \ldots,\ \underbrace{111\cdots1}_{X+1 \text{ digits}}$$

have the same remainder when divided by X. So taking the difference of two of these numbers, we get a number of the form $11\cdots100\cdots0$, which is a multiple of X. Perform operation (a) on X to get such a multiple. Then perform operation (b) to delete the zeros (if any). If the new number has more than one digits, we do the following steps: (1) multiply by 82 to get a number $911\cdots102$, (2)delete the zero and multiply by 9 to get a number $8200\cdots08$, (3) delete the zeros to get 828, (4) now $828 \cdot 25 = 20700$, $27 \cdots 4 = 108$, $18 \cdot 5 = 90$ and delete zero, we get the single digit 9.

192. (1996 IMO shortlisted problem) Four integers are marked on a circle. On each step we simultaneously replace each number by the difference between this number and next number on the circle in a given direction (that is, the numbers a, b, c, d are replaced by $a-b, b-c, c-d, d-a$). Is it possible after 1996 such steps to have numbers a, b, c, d such that the numbers $|bc-ad|, |ac-bd|, |ab-cd|$ are primes?

Solution. (Due to Ng Ka Man and Ng Ka Wing) If the initial numbers are $a = w, b = x, c = y, d = z$, then after 4 steps, the numbers will be

$$a = 2(w-2x+3y-2z), \quad b = 2(x-2y+3z-2w),$$

$$c = 2(y-2z+3w-2x), \quad d = 2(z-2w+3y-2z).$$

From that point on, a, b, c, d will always be even, so $|bc-ad|, |ac-bd|, |ab-cd|$ will always be divisible by 4.

193. (1989 Nanchang City Math Competition) There are 1989 coins on a table. Some are placed with the head sides up and some the tail sides up. A group of 1989 persons will perform the following operations: the first person is allowed turn over any one coin, the second person is allowed turn over any two coins, ..., the k-th person is allowed turn over any k coins, ..., the 1989th person is allowed to turn over every coin. Prove that

(1) no matter which sides of the coins are up initially, the 1989 persons can come up with a procedure turning all coins the same sides up at the end of the operations,

(2) in the above procedure, whether the head or the tail sides turned up at the end will depend on the initial placement of the coins.

Solution. (Due to Chan Kin Hang) (1) The number 1989 may not be special. So let us replace it by a variable n. The cases $n = 1$ and 3 are true, but the case $n = 2$ is false (when both coins are heads up initially). So we suspect the statement is true for odd n and do induction on k, where $n = 2k-1$. The cases $k = 1, 2$ are true. Suppose the case k is true. For the case $k+1$, we have $n = 2k+1$ coins.

First, suppose all coins are the same side up initially. For $i = 1, 2, \ldots, k$, let the i-th person flip any i coins and let the $(2k+1-i)$-th person flips the remaining $2k+1-i$ coins. Then each coin is flipped $k+1$ times and at the end all coins will be the same side up.

Next, suppose not all coins are the same sides up initially. Then there is one coin head up and another tail up. Mark these two coins. Let the first $2k-1$ persons flip the other $2k-1$ coins the same side up by the case k. Then there are exactly $2k$ coins the same side up and one coin opposite side up. The $2k$-th person flips the $2k$ coins the same side up and the $2k+1$-st person flips all coins and this subcase is solved.

So the $k+1$ case is true in either way and the induction step is complete, in particular, case $n = 1989$ is true.

(2) If the procedure does not depend on the initial placement, then in some initial placements of the coins, the coins may be flipped with

all heads up and may also be flipped with all tails up. Reversing the flippings on the heads up case, we can then go from all coins heads up to all tails up in $2(1+2+\cdots+1989)$ flippings. However, for each coin to go from head up to tail up, each must be flipped an odd number of times and the 1989 coins must total to an odd number of flippings, a contradiction.

194. (Proposed by India for 1992 IMO) Show that there exists a convex polygon of 1992 sides satisfying the following conditions:

(a) its sides are $1, 2, 3, \ldots, 1992$ in some order;
(b) the polygon is circumscribable about a circle.

Solution. For $n = 1, 2, \ldots, 1992$, define

$$x_n = \begin{cases} n-1 & \text{if } n \equiv 1, 3 \pmod 4 \\ 3/2 & \text{if } n \equiv 2 \pmod 4 \\ 1/2 & \text{if } n \equiv 0 \pmod 4 \end{cases}$$

and $a_n = x_n + x_{n+1}$ with $x_{1993} = x_1$. The sequence a_n is

$$1,\ 2,\ 4,\ 3,\ 5,\ 6,\ 8,\ 7,\ \ldots,\ 1989,\ 1990,\ 1992,\ 1991.$$

Consider a circle centered at O with large radius r and wind a polygonal line $A_1A_2\cdots A_{1992}A_{1993}$ with length $A_iA_{i+1} = a_i$ around the circle so that the segments A_iA_{i+1} are tangent to the circle at some point P_i with $A_iP_i = x_i$ and $P_iA_{i+1} = x_{i+1}$. Then $OA_1 = \sqrt{x_1^2 + r^2} = OA_{1993}$. Define

$$\begin{aligned} f(r) &= 2\tan^{-1}\frac{x_1}{r} + 2\tan^{-1}\frac{x_2}{r} + \cdots + 2\tan^{-1}\frac{x_{1992}}{r} \\ &= (\angle A_1OP_1 + \angle P_{1992}OA_{1993}) + \angle P_1OP_2 + \angle P_2OP_3 \\ &\quad + \cdots + \angle P_{1991}OP_{1992}. \end{aligned}$$

Now f is continuous, $\lim_{r\to 0+} f(r) = 1992\pi$ and $\lim_{r\to+\infty} f(r) = 0$. By the intermediate value theorem, there exists r such that $f(r) = 2\pi$. For such r, A_{1993} will coincide with A_1, resulting in the desired polygon.

Comments. The key fact that makes the polygon exists is that there is a permutation $a_1, a_2, \ldots, a_{1992}$ of $1, 2, \ldots, 1992$ such that the system of equations

$$x_1 + x_2 = a_1,\ x_2 + x_3 = a_2,\ \ldots,\ x_{1992} + x_1 = a_{1992}$$

have positive real solutions.

195. There are 13 white, 15 black, 17 red chips on a table. In one step, you may choose 2 chips of different colors and replace each one by a chip of the third color. Can all chips become the same color after some steps?

Solution. Write (a, b, c) for a white, b black, c red chips. So from (a, b, c), in one step, we can get $(a+2, b-1, c-1)$ or $(a-1, b+2, c-1)$ or $(a-1, b-1, c+2)$. Observe that in all 3 cases, the difference

$$(a+2)-(b-1),\ (a-1)-(b+2),\ (a-1)-(b-1) \equiv a-b \pmod 3.$$

So $a - b \pmod 3$ is an invariant. If all chips become the same color, then at the end, we have $(45, 0, 0)$ or $(0, 45, 0)$ or $(0, 0, 45)$. So $a - b \equiv 0 \pmod 3$ at the end. However, $a - b = 13 - 15 \not\equiv 0 \pmod 3$ in the beginning. So the answer is no.

196. The following operations are permitted with the quadratic polynomial $ax^2 + bx + c$:

(a) switch a and c,
(b) replace x by $x + t$, where t is a real number.

By repeating these operations, can you transform $x^2 - x - 2$ into $x^2 - x - 1$?

Solution. Consider the discriminant $\Delta = b^2 - 4ac$. After operation (a), $\Delta = b^2 - 4ca = b^2 - 4ac$. After operation (b), $a(x+t)^2 + b(x+t) + c = ax^2 + (2at+b)x + (at^2 + bt + c)$ and $\Delta = (2at+b)^2 - 4a(at^2 + bt + c) = b^2 - 4ac$. So Δ is an invariant. For $x^2 - x - 2$, $\Delta = 9$. For $x^2 - x - 1$, $\Delta = 5$. So the answer is no.

197. Five numbers $1, 2, 3, 4, 5$ are written on a blackboard. A student may erase any two of the numbers a and b on the board and write the

numbers $a+b$ and ab replacing them. If this operation is performed repeatedly, can the numbers $21, 27, 64, 180, 540$ ever appear on the board?

Solution. Observe that the number of multiples of 3 among the five numbers on the blackboard cannot decrease after each operation. (If a, b are multiples of 3, then $a + b, ab$ will also be multiples of 3. If one of them is a multiple of 3, then ab will also be a multiple of 3.) The number of multiples of 3 can increase in only one way, namely when one of a or b is $1 \pmod 3$ and the other is $2 \pmod 3$, then $a+b \equiv 0 \pmod 3$ and $ab \equiv 2 \pmod 3$. Now note there is one multiple of 3 in $\{1, 2, 3, 4, 5\}$ and four multiples of 3 in $\{21, 27, 64, 180, 540\}$. So when the number of multiples of 3 increases to four, the fifth number must be $2 \pmod 3$. Since $64 \equiv 1 \pmod 3$, so $21, 27, 64, 180, 540$ can never appear on the board.

198. Nine 1×1 cells of a 10×10 square are infected. In one unit time, the cells with at least 2 infected neighbors (having a common side) become infected. Can the infection spread to the whole square? What if nine is replaced by ten?

Solution. (Due to Cheung Pok Man) Color the infected cells black and record the perimeter of the black region at every unit time. If a cell has four, three, two infected neighbors, then it will become infected and the perimeter will decrease by 4, 2, 0, respectively, when that cell is colored black. If a cell has one or no infected neighbors, then it will not be infected. Observe that the perimeter of the black region cannot increase. Since in the beginning, the perimeter of the black region is at most $9 \times 4 = 36$, and a 10×10 black region has perimeter 40, the infection cannot spread to the whole square.

If nine is replaced by ten, then it is possible as the ten diagonal cells when infected can spread to the whole square.

199. (1997 Colombian Math Olympiad) We play the following game with an equilateral triangle of $n(n+1)/2$ dollar coins (with n coins on each side). Initially, all of the coins are turned heads up. On each turn, we may turn over three coins which are mutually adjacent; the goal is to

make all of the coins turned tails up. For which values of n can this be done?

Solution. This can be done only for all $n \equiv 0, 2 \pmod 3$. Below by a triangle, we will mean three coins which are mutually adjacent. For $n = 2$, clearly it can be done and for $n = 3$, flip each of the four triangles. For $n \equiv 0, 2 \pmod 3$ and $n > 3$, flip every triangle. Then the coins at the corners are flipped once. The coins on the sides (not corners) are flipped three times each. So all these coins will have tails up. The interior coins are flipped six times each and have heads up. Since the interior coins have side length $n - 3$, by the induction step, all of them can be flipped so to have tails up.

Next suppose $n \equiv 1 \pmod 3$. Color the *heads* of each coin red, white and blue so that adjacent coins have different colors and any three coins in a row have different colors. Then the coins in the corner have the same color, say red. A simple count shows that there are one more red coins than white or blue coins. So the (odd or even) parities of the red and white coins are different in the beginning. As we flip the triangles, at each turn, either (a) both red and white coins increase by 1 or (b) both decrease by 1 or (c) one increases by 1 and the other decreases by 1. So the parities of the red and white coins stay different. In the case all coins are tails up, the number of red and white coins would be zero and the parities would be the same. So this cannot happen.

200. (1990 Chinese Team Selection Test) Every integer is colored with one of 100 colors and all 100 colors are used. For intervals $[a, b], [c, d]$ having integers endpoints and same lengths, if a, c have the same color and b, d have the same color, then the intervals are colored the same way, which means $a + x$ and $c + x$ have the same color for $x = 0, 1, \ldots, b - a$. Prove that -1990 and 1990 have different colors.

Solution. We will show that x, y have the same color if and only if $x \equiv y \pmod{100}$, which implies -1990 and 1990 have different colors.

Let the colors be $1, 2, \ldots, 100$ and let $f(x)$ be the color (number) of x. Since all 100 colors were used, there is an integer m_i such that $f(m_i) = i$ for $i = 1, 2, \ldots, 100$. Let $M = \min(m_1, m_2, \ldots, m_{100}) - 100^2$.

Consider a fixed integer $a \leq M$ and an arbitrary positive integer n. Since there are 100^2 ways of coloring a pair of integers, at least two of the pairs $a + i, a + i + n$ $(i = 0, 1, 2, \ldots, 100^2)$ are colored the same way, which means $f(a+i_1) = f(a+i_2)$ and $f(a+i_1+n) = f(a+i_2+n)$ for some integers i_1, i_2 such that $0 \leq i_1 < i_2 \leq 100^2$. Let $d = i_2 - i_1$. Since there are finitely many combinations of ordered pairs (i_1, d) and n is arbitrary, there are infinitely many n's, say $n_1, n_2, \ldots$, having the same i_1's and d's.

Since these n_k's may be arbitrarily large, the union of the intervals $[a+i_1, a+i_1+n_k]$ will contain every integer $x \geq a + 100^2$. So for every such x, there is an interval $[a + i_1, a + i_1 + n_k]$ containing x. Since $f(a + i_1) = f(a + i_1 + d)$ and $f(a + i_1 + n_k) = f(a + i_1 + d + n_k)$, so intervals $[a + i_1, a + i_1 + n_k], [a + i_1 + d, a + i_1 + d + n_k]$ are colored the same way. In particular, $f(x) = f(x + d)$. So $f(x)$ has period d when $x \geq a + 100^2$. Since $a \leq M$, 100 colors are used for the integers $x \geq a + 100^2$ and so $d \geq 100$. Consider the least possible such period d.

Next, by the pigeonhole principle, two of $f(a+100^2), f(a+100^2+1), \ldots, f(a+100^2+100)$ are the same, say $f(b) = f(c)$ with $a+100^2 \leq b < c \leq a + 100^2 + 100$. For every $x \geq a + 100^2 + 100$, choose a large integer m so that x is in $[b, b + md]$. Since $f(b + md) = f(b) = f(c) = f(c+md)$, intervals $[b, b+md], [c, c+md]$ are colored the same way. In particular, $f(x) = f(x + c - b)$. So $f(x)$ has period $c - b \leq 100$ when $x \geq a + 100^2 + 100$. So the least period of $f(x)$ for $x \geq a + 100^2 + 100$ must be 100. Finally, since a can be as close to $-\infty$ as we like, f must have period 100 on the set of integers. Since all 100 colors are used, no two of 100 consecutive intgers can have the same color.