MATH301 Real Analysis (2008 Fall)
Tutorial Note #5

Limit Superior and Limit Inferior

(*Note: In the following, we will consider extended real number system $\mathbf{R} \cup \{-\infty, \infty\}$

In MATH202, we study the limit of some sequences, we also see some theorems related to limit. (Say root test, ratio test etc). In general, we may meet some sequences which does not converge (i.e. $\lim_{n\to\infty} a_n$ does not exist or $= \pm\infty$). Here is one classical example:

$$\{a_1, a_2, a_3, a_4, \ldots\} = \{1, -1, 1, -1, 1, -1, 1, -1, \ldots \ldots \ldots\}$$

We can see the limit does not exist as the sequence oscillates. To study the "limit" of this sequence, consider some **subsequences** of original sequence

$$\{a_1, a_3, a_5, \ldots \ldots\} = \{1, 1, 1, \ldots \ldots\}$$

$$\{a_2, a_4, a_6, \ldots \ldots\} = \{-1, -1, -1, \ldots \ldots\}$$

We see the frontier subsequence converges to 1 and the later one converges to -1.

After considering all subsequences of $\{a_n\}$, we have **a set of collection of limit of subsequence L**.

$$L = \{-1, 1\}$$

We define the **limit superior** and **limit inferior** of a_n to be

$$\limsup_{n\to\infty} a_n = \sup L \quad and \quad \liminf_{n\to\infty} a_n = \inf L$$

Definition: (Limit superior and Limit inferior)

Given a sequence $\{a_1, a_2, a_3, \ldots.\}$, we define the limit superior and limit inferior by

$$\limsup_{n\to\infty} a_n = \sup L = \sup\{z : z \text{ is limit of some subsequences } a_{n_i}\}$$

$$\liminf_{n\to\infty} a_n = \inf L = \inf\{z : z \text{ is limit of some subsequences } a_{n_i}\}$$

Remark:

1. $\limsup_{n\to\infty} a_n = \sup L \in L$ and $\liminf_{n\to\infty} a_n = \inf L \in L$, i.e. there is subsequence $\{a_{n_k}\}$ and $\{b_{n_k}\}$ such that $\lim_{k\to\infty} a_{n_k} = \limsup_{n\to\infty} a_n$ and $lim_{k\to\infty} b_{n_k} = \liminf_{n\to\infty} a_n$

2. If $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = a$
   $\Leftrightarrow \sup L = \inf L = a \Leftrightarrow L = \{a\}$
   $\Leftrightarrow \lim_{n\to\infty} a_n = a$

Example 1

Find the $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ of

$$a_n = n^{\sin\left(\frac{n\pi}{2}\right)}$$

for $n = 1,2,3,\dots$

IDEA:

By writing out the sequence, we have

$$\{a_1, a_2, a_3, a_4, \dots.\} = \left\{1^{\sin\frac{\pi}{2}}, 2^{\sin\pi}, 3^{\sin\frac{3\pi}{2}}, 4^{\sin 2\pi}, 5^{\sin\frac{5\pi}{2}}, 6^{\sin 3\pi}, \dots\right\}$$

$$= \{1, 2^0, 3^{-1}, 4^0, 5, 6^0, \dots\} = \left\{1, 1, \frac{1}{3}, 1, 5, 1, \frac{1}{7}, 1, 9, \dots.\right\}$$

By observation, there are 3 possible limits of subsequence:

0 for $\{a_3, a_7, a_{11}, \dots\} = \left\{\frac{1}{3}, \frac{1}{7}, \frac{1}{11}, \dots\right\}$;

1 for $\{a_2, a_4, a_6, a_8, \dots\} = \{1,1,1,1,\dots\}$;

and $+\infty$ for $\{a_1, a_5, a_9, a_{13}, \dots\} = \{1,5,9,13,\dots.\}$

One may guess the $\limsup_{n\to\infty} a_n = +\infty$ and $liminf_{n\to\infty} a_n = 0$, but we still need to prove it formally

☺**Solution**

(Step 1: Guess the range of L by considering $\{a_n\}$)

Since $0 < a_n < +\infty$ for $n = 1,2,3 \dots$, hence for any converging subsequence $\{a_{n_k}\}$, the limit a should lies in $[0, \infty]$,

Thus $0 \le \inf L \le \sup L \le +\infty$ (or $L \subseteq [0, \infty]$)

(Step 2: Verify $infL = 0$ and $supL = +\infty$ by picking 2 specific subsequences)

By picking $\{a_{n_k}\} = \{a_3, a_7, a_{11}, \dots\}$, we have $\lim_{k\to\infty} a_{n_k} = 0$

By picking $\{a'_{n_k}\} = \{a_1, a_5, a_9, \dots\}$, we have $\lim_{k\to\infty} a'_{n_k} = +\infty$

Example 2

Find the $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ of

$$a_n = \begin{cases} \dfrac{n}{n+1} & if\ \ n\ is\ odd \\ \dfrac{1}{n+1} & if\ n\ is\ even \end{cases}$$

IDEA:

By writing the first few terms, we have

$$\{a_1, a_2, a_3, a_4, a_5, a_6, \dots\} = \left\{\frac{1}{2}, \frac{1}{3}, \frac{3}{4}, \frac{1}{5}, \frac{5}{6}, \frac{1}{7}, \frac{7}{8}, \frac{1}{9}, \dots.\right\}$$

By simple observation, we see there is 2 possible limits:

1 for $\{a_1, a_3, a_5, a_7, \ldots\} = \left\{\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \ldots\right\}$;

0 for $\{a_2, a_4, a_6, a_8, \ldots\} = \left\{\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \ldots\right\}$.

We guess $limsup_{n\to\infty} a_n = 1$ and $liminf_{n\to\infty} a_n = 0$

☺**Solution**

(Step 1: Guess the range of L by considering $\{a_n\}$)

Since $0 < \frac{n}{n+1} < 1$ and $0 < \frac{1}{n+1} < 1$,

Hence $0 < a_n < 1$ for $n = 1,2,3 \ldots$, for any converging subsequence $\{a_{n_k}\}$, the limit a should lies in $[0,1]$,

Thus $0 \leq \inf L \leq \sup L \leq 1$ (or $L \subseteq [0,1]$)

(Step 2: Verify $inf L = 0$ and $sup L = 1$ by picking 2 specific subsequences)

By picking $\{a_{n_k}\} = \{a_1, a_3, a_5, \ldots\}$, we have $\lim_{k\to\infty} a_{n_k} = 1$

By picking $\{a'_{n_k}\} = \{a_2, a_4, a_6, \ldots\}$, we have $\lim_{k\to\infty} a'_{n_k} = 0$

When computing $\mathrm{limsup}_{n\to\infty} a_n$ and $\mathrm{liminf}_{n\to\infty} a_n$, using definition may be a bit messy since we need to consider all combinations of subsequences. So to make our computation easier, we have the following theorem

Theorem: (M_k, m_k theorem)

$$\mathrm{limsup}_{n\to\infty} a_n = \lim_{k\to\infty} M_k = \lim_{k\to\infty} \sup\{a_k, a_{k+1}, \ldots.\}$$

$$\mathrm{liminf}_{n\to\infty} a_n = \lim_{k\to\infty} m_k = \lim_{k\to\infty} \inf\{a_k, a_{k+1}, \ldots.\}$$

Example 3

Find $\mathrm{limsup}_{n\to\infty} x_n$ and $\mathrm{liminf}_{n\to\infty} x_n$ of

$$x_n = \sin\left(\frac{n\pi}{3}\right)$$

Solution:

$$\left\{\sin\left(\frac{\pi}{3}\right), \sin\left(\frac{2\pi}{3}\right), \sin\left(\frac{3\pi}{3}\right), \sin\left(\frac{4\pi}{3}\right), \sin\left(\frac{5\pi}{3}\right), \sin\left(\frac{6\pi}{3}\right), \sin\left(\frac{7\pi}{3}\right), \sin\left(\frac{8\pi}{3}\right), \ldots\right\}$$

$$= \left\{\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0, \ldots.\right\}$$

We see the sequence is repeated every 6 terms.

$$\text{Hence } \sup\{x_k, x_{k+1}, x_{k+2}, \ldots\} = \sup\left\{\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0\right\} = \frac{\sqrt{3}}{2}$$

Thus $\mathrm{limsup}_{n\to\infty} x_n = \lim_{k\to\infty} \sup\{x_k, x_{k+1}, \dots\} = \lim_{k\to\infty} \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$

Hence $\inf\{x_k, x_{k+1n}, x_{k+2}, \dots\} = \inf\left\{\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0\right\} = -\frac{\sqrt{3}}{2}$

Thus $\mathrm{liminf}_{n\to\infty} x_n = \lim_{k\to\infty} \inf\{x_k, x_{k+1}, \dots\} = \lim_{k\to\infty} -\frac{\sqrt{3}}{2} = -\frac{\sqrt{3}}{2}$

Example 4

Find $\mathrm{limsup}_{n\to\infty} x_n$ and $\mathrm{liminf}_{n\to\infty} x_n$ of

$$x_n = \begin{cases} 2^{\frac{1}{n+1}} & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$$

IDEA: We first write out a few terms, we have

$$\{x_1. x_2, x_3, x_4, \dots\dots\} = \left\{2^{\frac{1}{2}}, 1, 2^{\frac{1}{4}}, 1, 2^{\frac{1}{6}}, 1, \dots..\right\}$$

Solution:

For some complicated sequence, we can compute our M_k and m_k as follows:

$k =$	1	2	3	4	5	6	7	…….
M_k	$2^{\frac{1}{2}}$	$2^{\frac{1}{4}}$	$2^{\frac{1}{4}}$	$2^{\frac{1}{6}}$	$2^{\frac{1}{6}}$	$2^{\frac{1}{8}}$	$2^{\frac{1}{8}}$	…….
m_k	1	1	1	1	1	1	1	……

We can get

$$M_k = \begin{cases} 2^{\frac{1}{k+1}} & \text{if } k \text{ is odd} \\ 2^{\frac{1}{k+2}} & \text{if } k \text{ is even} \end{cases} \quad and \quad m_k = 1$$

Hence $limsup_{n\to\infty} x_n = \lim_{k\to\infty} M_k = \begin{cases} \lim_{k\to\infty} 2^{\frac{1}{k+1}} & \text{if } k \text{ is odd} \\ \lim_{k\to\infty} 2^{\frac{1}{k+2}} & \text{if } k \text{ is even} \end{cases} = 1$

and $liminf_{n\to\infty} x_n = \lim_{k\to\infty} m_k = 1$

*Remark: Since $\mathrm{limsup}_{n\to\infty} x_n = \mathrm{liminf}_{n\to\infty} x_n$, we see $\lim_{n\to\infty} x_n$ exists and equal to 1.

Next, we try to look at some abstract problems

Example 5 (Practice Exercise #33)

Let $\{x_n\}$ and $\{y_n\}$ be 2 sequences of real numbers, prove that

$$\limsup_{n\to\infty}(x_n + y_n) \le \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$$

Provided that the R.H.S. is not the form $\infty - \infty$

Solution:

Note that $\limsup_{n\to\infty}(x_n + y_n) = \lim_{k\to\infty} M_k$ (where $M_k = \sup\{x_k + y_k, x_{k+1} + y_{k+1}, \ldots\}$)

Note that

$$M_k = \sup\{x_k + y_k, x_{k+1} + y_{k+1}, \ldots\} = \sup_{n\geq k}\{x_n + y_n\} \leq \sup_{n\geq k}\{x_n\} + \sup_{n\geq k}\{y_n\}$$

Taking limit to both sides, we have

$$\lim_{k\to\infty} M_k \leq \lim_{k\to\infty}\sup_{n\geq k}\{x_n\} + \lim_{k\to\infty}\sup_{n\geq k}\{y_n\}$$
$$\Leftrightarrow \limsup_{n\to\infty}(x_n + y_n) \leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$$

Application of limit superior and limit inferior

To check the convergence of $\sum a_n$, in original root test and ratio test, we need to check the value $\lim_{n\to\infty}\sqrt[n]{|a_n|}$ and $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$. But in some cases, the limit of a_n does not exist. Then the root test and ratio test does not have any conclusion. Now with limit superior and limit inferior, we can derive a stronger test.

Theorem: (Strong Root Test)

Given $\sum_{n=1}^{\infty} a_n$, if

a) $\limsup_{n\to\infty}\sqrt[n]{|a_n|} < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely
b) $\limsup_{n\to\infty}\sqrt[n]{|a_n|} > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges
c) $\limsup_{n\to\infty}\sqrt[n]{|a_n|} = 1$, then NO information is given

Theorem: (Strong Ratio Test)

Given $\sum_{n=1}^{\infty} a_n$, if

a) $\limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, $\sum_{n=1}^{\infty} a_n$ converges absolutely
b) $\liminf_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$, $\sum_{n=1}^{\infty} a_n$ diverges

For Example, please refer to Tutorial Note #25 in MATH202 which is available either http://www.math.ust.hk/~makyli

Here we shall go through a proof of root test, since there is important technique involved here.

Proof of Root Test:

a) For $\limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1$, say $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{k\to\infty} M_k = M < 1$

(where $M_k = \sup\{\sqrt[k]{|a_k|}, \sqrt[k+1]{|a_{k+1}|}, \sqrt[k+2]{|a_{k+2}|}, \ldots\}$)

Pick r such that $M < r < 1$, there is N such that

$M_N < r \Leftrightarrow \sup\{\sqrt[N]{|a_N|}, \sqrt[N+1]{|a_{N+1}|}, \sqrt[N+2]{|a_{N+2}|}, \ldots\} < r$

$\Leftrightarrow \sqrt[k]{|a_k|} < r \Leftrightarrow |a_k| < r^k$ for all $k \geq N$

Important Technique!!

Then

$$\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{N} |a_k| + \sum_{k=N+1}^{\infty} |a_k| < \sum_{k=1}^{N} |a_k| + \sum_{k=N+1}^{\infty} r^k$$

The first part is finite and therefore converges and second part is a geometric series with $r < 1$ and hence converges. Thus $\sum_{k=1}^{\infty} |a_k|$ converges and $\sum_{k=1}^{\infty} a_k$ converges absolutely.

b) For $\limsup_{n\to\infty} \sqrt[n]{|a_n|} > 1$, let $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{k\to\infty} M_k = M > 1$

There is N such that for $k > N$, $M_k > 1$

Then there are infinitely many $\sqrt[k]{|a_k|} > 1$ (or $|a_k| > 1$), thus $\lim_{k\to\infty} a_k \neq 0$, by term test, the series diverges.

The Technique used in part a) is particularly useful in handling many problems about limit superior and limit inferior. We shall see it later.

In the following, there are some suggested exercises, you should try to do them in order to understand the material. If you have any questions about them, you are welcome to find me during office hours. You are also welcome to submit your work (complete or incomplete) to me and I can give some comments to your work.

☺Exercise 1

Find the limit superior and limit inferior of the following sequence by

1) Original Definition and 2) M_k, m_k-theorem

a) $x_n = (-1)^n n$

b) $y_n = \{0,1,0,1,2,0,1,2,3,0,1,2,3,4, \ldots \ldots\}$

c) $z_n = 2^{n\cos(\frac{2n\pi}{3})}$

d) $w_n = \left(1 + \frac{(-1)^n}{n}\right)^n$

(Hint: $e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$ and $e^{-1} = \lim_{n\to\infty}\left(1 - \frac{1}{n}\right)^n$)

☺Exercise 2

Check whether the following series converges

$$a + 1 + a^3 + a^2 + a^5 + a^4 + a^7 + \cdots ...$$

for $a < 1$

☺Exercise 3

For any sequence $\{x_n\}$ and $c \geq 0$, prove that

$$\limsup_{n\to\infty} cx_n = c \limsup_{n\to\infty} x_n \quad and \quad \liminf_{n\to\infty} cx_n = c \liminf_{n\to\infty} x_n$$

☺Exercise 4 (Practice Exercise #35)

Let $\{a_n\}$ and $\{b_n\}$ be sequences of non-negative real numbers. Show that

$$\limsup_{n\to\infty} a_n b_n \leq \left(\limsup_{n\to\infty} a_n\right)\left(\limsup_{n\to\infty} b_n\right)$$

Provided that right side is not of the form $0 \cdot \infty$

☺Exercise 5 (Practice Exercise #34)

a) Let $\{a_n\}$ and $\{b_n\}$ be sequences of non-negative real numbers. Show that

$$\liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n \leq \liminf_{n\to\infty}(a_n + b_n) \leq \liminf_{n\to\infty} a_n + \limsup_{n\to\infty} b_n \leq \limsup_{n\to\infty}(a_n + b_n)$$

(Hint: To prove the 2[nd] inequality, i.e. $liminf_{n\to\infty}(a_n + b_n) \leq liminf_{n\to\infty} a_n + limsup_{n\to\infty} b_n$

Use the fact that there is subsequence a_{n_k} *such that* $lim_{k\to\infty} a_{n_k} = liminf_{n\to\infty} a_n$ *and let* M_k *to be* $sup\{b_k, b_{k+1}, \ldots.\}$ *and* $lim_{k\to\infty} M_k = limsup_{n\to\infty} b_n$*. Verify the inequality*

$$inf\{a_{n_k} + b_{n_k}, a_{n_k+1} + b_{n_k+1}, a_{n_k+2} + b_{n_k+2}, \ldots\} \leq a_{n_k} + b_{n_k} \leq a_{n_k} + M_{n_k}$$

and complete the proof by taking limlt. The 3[rd] inequality is similar.)

b) If $\liminf_{n\to\infty} a_n \geq \frac{1}{2}$ and $\liminf_{n\to\infty} b_n \geq \frac{1}{2}$ and $\lim_{n\to\infty}(a_n + b_n) = 1$. Show that

$$\lim_{n\to\infty} a_n = \frac{1}{2} = \lim_{n\to\infty} b_n$$

☺Exercise 6 (Part a of strong ratio test)

Prove that $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(Hint: The method is similar to the one in root test, try to follow that, the following equality may be useful: for $n \geq N$

$$|a_n| = \left|\frac{a_n}{a_{n-1}}\right| \left|\frac{a_{n-1}}{a_{n-2}}\right| \ldots . \left|\frac{a_{N+1}}{a_N}\right| |a_N|$$