Chapter 10

Fixed Point Theorems and Applications

Let $f : X \to X$ be a function mapping a topological space X to itself. If $f(x) = x$ for some x in X, then x maps to itself under f and is known as a fixed point of f. Fixed points and theorems about them play an important role in many areas of pure and applied mathematics. Within topology itself there is a large body of work on fixed point results. One of the most well known is the Brouwer Fixed Point Theorem. It states that every continuous function from an n-ball B^n to itself must have a fixed point. We proved the One-Dimensional Brouwer Fixed Point Theorem as a consequence of the Intermediate Value Theorem in Section 6.3. Here, in Section 10.1, we prove the Two-Dimensional Brouwer Fixed Point Theorem as a consequence of the Two-Dimensional No Retraction Theorem. In Section 10.2, we present an application of the Brouwer Fixed Point Theorem to prove the existence of equilibrium price distributions in a pure exchange economy. In Section 10.3, we present a generalization of the Brouwer Fixed Point Theorem, known as Kakutani's Fixed Point Theorem, which applies to set-valued functions. Finally, in Section 10.4, we apply Kakutani's Fixed Point Theorem to prove the existence of Nash equilibria in game theory. This is one of the most important results in the field of game theory, as it demonstrates that there is a choice of strategies that optimizes the expected outcome for all players of a game.

10.1 The Brouwer Fixed Point Theorem

Imagine taking two pieces of the same-sized paper and laying one piece on top of the other. Every point on the top sheet of paper is associated with some point right below it on the other sheet. Now take the top sheet of paper and crumple it up into a ball without ripping it. Place the crumpled ball back on top of the bottom sheet of paper. Somewhere on the crumpled ball of paper there is a point that is sitting directly above the same point on the bottom sheet of paper that it sat above before the crumpling took place. (See Figure 10.1.) This is an application of the Two-Dimensional Brouwer Fixed Point Theorem, which we prove in this section.

DEFINITION 10.1. *Let $f : X \to X$. A point $x \in X$ is said to be a* ***fixed point*** *of f if $f(x) = x$. A topological space X is said to have the* ***fixed point property*** *if every continuous function $f : X \to X$ has a fixed point.*

EXAMPLE 10.1. The space $[-1, 1]$ has the fixed point property by the One-Dimensional Brouwer Fixed Point Theorem.

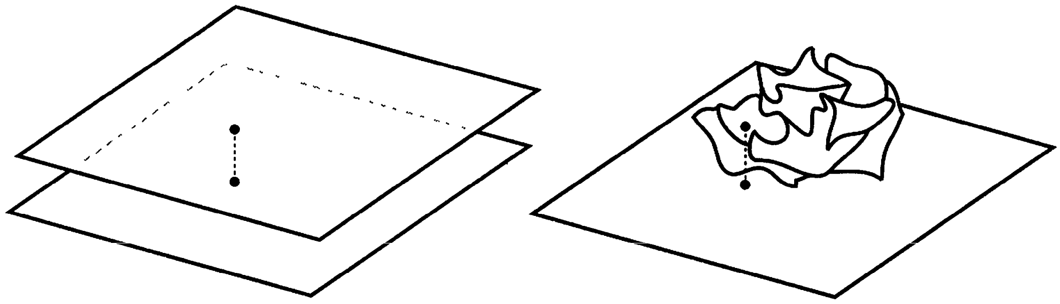

FIGURE 10.1: After the paper crumpling, some point still sits above the same point it sat above prior to crumpling.

EXAMPLE 10.2. The real line $\mathbb{R}$ does not have the fixed point property since there exist continuous functions from $\mathbb{R}$ to $\mathbb{R}$ that do not fix any point. The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x + 1$ is an example.

EXAMPLE 10.3. The circle S^1 does not have the fixed point property since the continuous function $f : S^1 \to S^1$, given by rotating each point on the circle through an angle of π, does not have any fixed points.

The fixed point property is a topological property, meaning that if spaces X and Y are homeomorphic, then X has the fixed point property if and only if Y does. (See Exercise 10.3.) Therefore all closed bounded intervals $[a, b]$ with the standard topology have the fixed point property since $[-1, 1]$ does. Also, all open intervals $(-\infty, b)$, (a, b), and (a, ∞) with the standard topology do not have the fixed point property since $\mathbb{R}$ does not.

One of the most useful theorems in topology is the Brouwer Fixed Point Theorem. It says that for each n, the n-ball B^n has the fixed point property. This theorem is named after the Dutch mathematician L. E. J. Brouwer (1881–1966) who proved the general n-dimensional version of the theorem in 1912. We show here that the Two-Dimensional Brouwer Fixed Point Theorem is equivalent to the Two-Dimensional No Retraction Theorem (with a proof that actually carries over to establish this equivalence for all n). Since we established the Two-Dimensional No Retraction Theorem in Section 9.2, the Two-Dimensional Brouwer Fixed Point Theorem follows.

In a set of supplementary exercises in this section, we present a second proof of the Two-Dimensional Brouwer Fixed Point Theorem, one that involves the degree of circle functions, but does not rely on the No Retraction Theorem.

THEOREM 10.2. ***The Two-Dimensional Brouwer Fixed Point Theorem.*** *Every continuous function $f : D \to D$, mapping the disk to itself, has a fixed point.*

As already indicated, our approach to proving this theorem is to show its equivalence to the Two-Dimensional No Retraction Theorem. That is done via the following theorem:

THEOREM 10.3. *The disk D, as a subspace of $\mathbb{R}^2$, has the fixed point property if and only if there is no retraction from D onto its boundary S^1.*

Proof. To begin, assume that there is a retraction $r : D \to S^1$. Consider the map $q : S^1 \to D$, defined by $q(x) = -x$, where we consider x as a vector in the plane. The function $q \circ r : D \to D$ is continuous and has no fixed point. Therefore, if there is a retraction $r : D \to S^1$, then D does not have the fixed point property.

On the other hand, assume that a continuous function $f : D \to D$ has no fixed point. We show that there is a retraction $r : D \to S^1$. Define $r : D \to S^1$ as follows. First, take the ray in $\mathbb{R}$ running from $f(x)$ through x. Such a ray is well defined since f has no fixed point. Let $r(x)$ be the point where the ray intersects S^1, as illustrated in Figure 10.2. Clearly, r maps D to S^1 and $r(x) = x$ for all $x \in S^1$. It will follow that r is a retraction once the continuity of r is established.

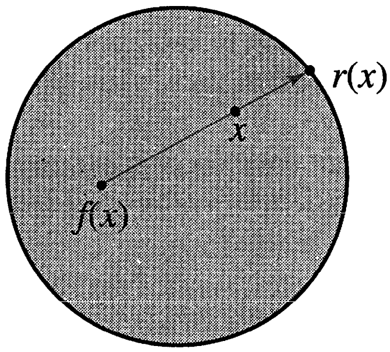

FIGURE 10.2: Defining r from the disk to the bounding circle.

To prove that r is continuous, let U be open in S^1 and x be a point in $r^{-1}(U)$. We show that there is an open set V containing x such that $r(V) \subset U$, and therefore $r^{-1}(U)$ is open. To begin, we choose small open balls O_1 and O_2 centered at $f(x)$ and x, respectively, such that every ray beginning in O_1 and passing through O_2 intersects S^1 in the set U. (See Figure 10.3.) Since f is continuous, we can find an open set V, containing x and contained in O_2, such that $f(V) \subset O_1$. Thus, for all $v \in V$, the ray beginning at $f(v)$ and passing through v intersects S^1

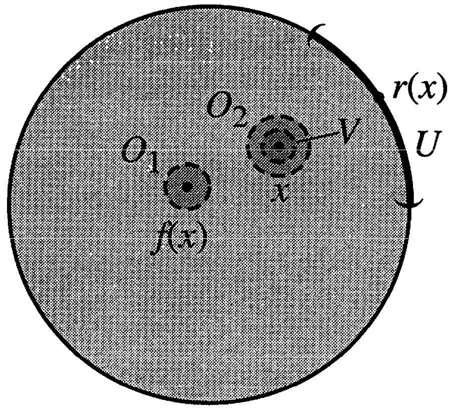

FIGURE 10.3: There exists an open set V containing x such that $f(V) \subset U$.

in U. Therefore $r(v) \in U$, and it follows that r is continuous. Hence we have established that if D does not have the fixed point property, then there is a retraction $r : D \to S^1$, and the proof of the theorem is complete. ■

The Two-Dimensional No Retraction Theorem and Theorem 10.3 immediately imply Theorem 10.2, and therefore we have established the Two-Dimensional Brouwer Fixed Point Theorem.

The approach to the proof of Theorem 10.3 carries through if we replace the disk D and the circle S^1 with the n-ball B^n and the $(n-1)$-sphere S^{n-1}, respectively, and in this way we can show that the n-Dimensional Brouwer Fixed Point Theorem is equivalent to the n-Dimensional No Retraction Theorem.

EXAMPLE 10.4. Suppose we take a map of New England and place it on the ground anywhere within New England, as in Figure 10.4. We assume that New England is topologically equivalent to a disk, and we refer to it as N. Let f be the function assigning to each point in New England the point on the map corresponding to it. We can view f as a continuous function from N to itself. Therefore f must have a fixed point, from which it follows that there must be a point on the map that corresponds exactly to the point on the ground directly beneath it. If, as a guide to travelers, we leave the map where it is, then on the map we would indicate the location of the fixed point with an arrow labeled "You Are Here!"

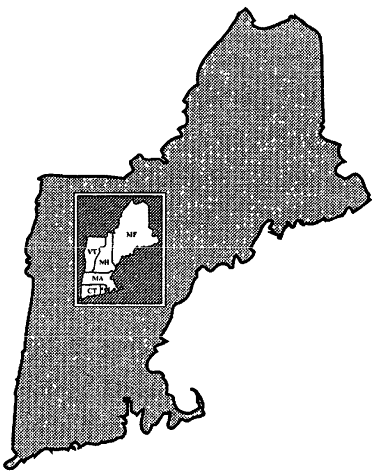

FIGURE 10.4: The function from New England to the map must have a fixed point.

Exercises for Section 10.1

10.1. Show that each of the following spaces does not have the fixed point property:
- **(a)** The interval $(0, 1]$
- **(b)** The torus
- **(c)** The figure-eight space obtained by taking two circles and gluing them together at a point on each
- **(d)** The sphere

10.2. Prove that if a topological space X has the fixed point property, then X is connected.

10.3. Show that if X has the fixed point property and Y is homeomorphic to X, then Y has the fixed point property.

10.4. Show that if X has the fixed point property and A is a retract of X, then A has the fixed point property.

10.5. Show that if X does not have the fixed point property, then for all Y, the product space $X \times Y$ does not have the fixed point property.

10.6. Prove the One-Dimensional Brouwer Fixed Point Theorem from the Two-Dimensional Brouwer Fixed Point Theorem.

10.7. Consider the topological graphs of the letters A–E as illustrated in Figure 10.5. Determine which of these spaces has the fixed point property. (Hint: For the letter E, you may want to use Exercise 10.4.)

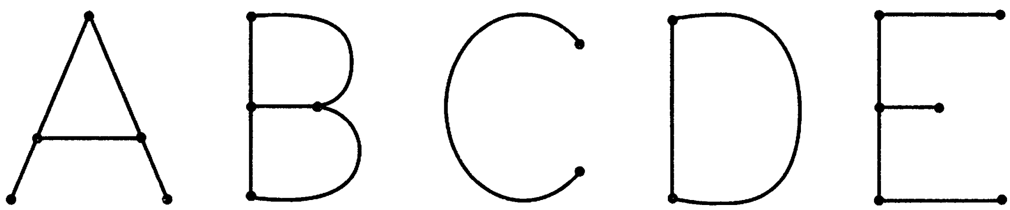

FIGURE 10.5: Which of these letters has the fixed point property?

Supplementary Exercises: Another Approach to the Two-Dimensional Brouwer Fixed Point Theorem

In these exercises, we develop another proof of the Two-Dimensional Brouwer Fixed Point Theorem. In this case the proof is based directly on the degree of circle functions and does not rely on the No Retraction Theorem. In what follows, we assume that points x in S^1, D, and so on, are vectors based at the origin in the plane. We begin with a lemma.

LEMMA 10.4. *Let $g : S^1 \to \mathbb{R}^2 - \{O\}$ be continuous, and assume that the function $h : S^1 \to S^1$, defined by $h(x) = \frac{g(x)}{|g(x)|}$, has degree 1. Then there exists $x \in S^1$ such that $g(x) = ax$ for some $a > 0$.*

The lemma indicates that some $x \in S^1$ (again, considered as a vector based at the origin) maps to a positive scalar multiple of itself under g.

Proof. Define $G : S^1 \times [0, 1] \to \mathbb{R}^2$ by $G(x, t) = (1 - t)g(x) - tx$. The function G defines a straight-line homotopy from g (considered to have range $\mathbb{R}^2$) to the function $a(x) = -x$, which maps each $x \in S^1$ to its antipodal point $-x \in S^1$.

SE 10.8. Show that there exists $(x, t) \in S^1 \times (0, 1)$ such that $G(x, t) = 0$. (Hint: Assume that no such point (x, t) exists, and consider the homotopy of circle functions $H(x, t) : S^1 \times [0, 1] \to S^1$, defined by $H(x, t) = \frac{G(x,t)}{|G(x,t)|}$.)

SE 10.9. Show that if there exists $(x, t) \in S^1 \times (0, 1)$ such that $G(x, t) = 0$, then $g(x) = ax$ for some $a > 0$, thereby proving the lemma. ■

Circle functions do not necessarily have fixed points. A simple rotation of a circle by π leaves no point fixed. This rotation function has degree 1. Interestingly, that is the only degree for which a circle function can have no fixed point. For circle functions whose degree is not equal to 1, we have the following theorem:

THEOREM 10.5. *Let $f : S^1 \to S^1$ be a circle function with $\deg(f) \neq 1$. Then there exist $x', x'' \in S^1$ such that $f(x') = x'$ and $f(x'') = -x''$.*

Thus if the degree of a circle function is not equal to 1, then the function has a fixed point and a point that maps to its antipodal point.

SE 10.10. Prove Theorem 10.5. (Hint: Regard $f : S^1 \to S^1$ as a function mapping into $\mathbb{R}^2 - \{O\}$, and use Lemma 10.4 to prove the existence of the fixed point. Then consider the function $h : S^1 \to S^1$ defined by $h(x) = f(-x)$ and apply the first assertion of the theorem to h.)

Now, we proceed with the Two-Dimensional Brouwer Fixed Point Theorem. Let D be the disk in the plane.

THEOREM 10.6. *Let $f : D \to D$ be continuous. Then there exists $x \in D$ such that $f(x) = x$.*

Proof. Assume that this is not the case. Let $k : D \to \mathbb{R}^2 - \{O\}$ be the function defined by $k(x) = f(x) - x$. The function k maps to $\mathbb{R}^2 - \{O\}$ since we are assuming that f has no fixed point.

SE 10.11. Indicate why Lemma 10.4 applies to the function $k|_{S^1} : S^1 \to \mathbb{R}^2 - \{O\}$. (Hint: Show that the degree of the circle function involved is 0.)

Therefore there exists $x^* \in S^1$ such that $k(x^*) = ax^*$ for some $a > 0$.

SE 10.12. Show that $f(x^*) \in D$, thereby arriving at a contradiction and completing the proof of the theorem. ■

10.2 An Application to Economics

In this section, we show how fixed point theory can demonstrate the existence of economic equilibria. Suppose we have an economy consisting of a finite number of individuals who have certain items to buy and sell. We assume that there is neither consumption of the items nor new production of them. Therefore individuals who wish to have more of a particular item must sell some of their current stock of other items in order to purchase the item they desire. This is called a pure exchange economy.

The price of the items is determined by supply and demand, what in 1776 philosopher and economist Adam Smith (1723–1790) called the "invisible hand." If there is more demand for a product than there is supply, the product's price goes up. If there is excess supply and little demand, the product's price goes down. We would like to know whether there is a choice of prices that balances supply and demand, an equilibrium point where prices stabilize and where all consumers are happy with their particular bundle of goods.

In 1932, John Von Neumann (1903–1957) gave a seminar at Princeton entitled "On a System of Economic Equations and a Generalization of the Brouwer Fixed Point Theorem." In it, he outlined how fixed point theory could be utilized to prove the existence of equilibria in economic models. Generalizations and applications of this concept have resulted in Nobel Prizes in economics for Kenneth Arrow in 1972 and Gerard Debreu (1921–2004) in 1983. Applications to game theory, as discussed in Section 10.4, also led to a Nobel Prize in economics for the mathematician John Nash in 1994.

Let us look at a particularly simple example as a starting point. Suppose we have an economy with only three items available. These are cashmere, butter, and gunpowder. The total quantity of each, in pounds, is called the supply, and the three supplies are denoted S_C, S_B, and S_G, respectively. We keep track of the supply of each item in a vector called the supply vector $\mathbf{S} = (S_C, S_B, S_G)$.

There are n individuals trading in this economy. We denote them by the integers $1, 2, \ldots, n$. Each individual starts with some number of each item, collectively called his or her bundle of goods. Individual i has a bundle $b = (b^i_C, b^i_B, b^i_G)$, a three-dimensional vector giving the quantity of each item in the corresponding component. At any given time, the sum of all of the bundle vectors over the individuals in the economy gives the supply vector,

$$\mathbf{S} = \sum_i \mathbf{b}^i.$$

We are assuming that none of the items are consumed or destroyed over time, so $\mathbf{S}$ is constant. Everyone is simply stockpiling their goods and desirous of a particular mix, depending on the going prices. If an individual wants more butter and less cashmere, she can sell some of her cashmere to someone else at the going price and purchase additional butter. However, if everyone is trying to get rid of cashmere, the price of cashmere is driven down.

We denote the going prices per pound for cashmere, butter, and gunpowder by p_C, p_B, and p_G respectively. We then represent this set of prices by a single price vector $\mathbf{p} = (p_C, p_B, p_G)$.

In fact, it is only the relative price of these items that matters. The particular cost of any item is irrelevant, since it is the ratio of costs that determines, for instance, how much butter can be purchased when a pound of cashmere is sold.

Therefore we divide each of the individual prices by the sum of the prices $p_C + p_B + p_G$ to "normalize" the situation. We denote the resultant prices by

$$p'_C = \frac{p_C}{p_C + p_B + p_G},$$
$$p'_B = \frac{p_B}{p_C + p_B + p_G}, \text{ and}$$
$$p'_G = \frac{p_G}{p_C + p_B + p_G}.$$

These normalized prices have the advantage of summing to 1. To simplify notation, we drop the primes on the price vectors, but keep in mind our assumption that the prices have been normalized. We then keep track of these vectors in our overall price vector $\mathbf{p} = (p_C, p_B, p_G)$ such that

$$p_C + p_B + p_G = 1.$$

In addition, we assume that none of these items is so despised by a consumer that he might pay you to take it away. In other words, the prices are never negative, so

$$p_C \geq 0, \ p_B \geq 0, \text{ and } p_G \geq 0.$$

When the price of an item is 0, there is no demand for that item. Each consumer would rather have the other two items, no matter how expensive they are.

We can graph the set of possible price vectors in 3-space. The equation $p_C + p_B + p_G = 1$ defines a plane. That $p_C \geq 0$, $p_B \geq 0$, and $p_G \geq 0$ limits us to that part of the plane that lies in the first octant of 3-space, yielding a triangle T, as shown in Figure 10.6. Each point in the triangle corresponds to a triple of prices, one for each of our three goods. It will be important for us, in applying the Brouwer Fixed Point Theorem, that T is topologically equivalent to a disk.

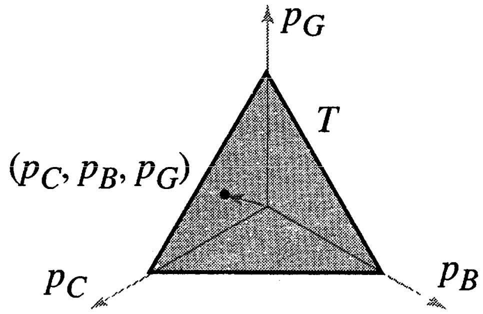

FIGURE 10.6: The triangle T of possible price vectors (p_C, p_B, p_G) for our three goods.

Given a particular price vector $\mathbf{p}$, each individual in the economy has a certain amount of "wealth" in the form of the value of his or her bundle of goods. For the ith consumer, this wealth is given by

$$w^i(\mathbf{p}) = \mathbf{p} \cdot \mathbf{b}^i.$$

An individual's wealth can change depending on the current price vector, so we represent it as a function of $\mathbf{p}$.

At a particular price vector $\mathbf{p}$, individual i's current bundle of goods may not be her optimal choice. She might want to trade certain goods for others. We keep track of her preferences for an optimal bundle at price $\mathbf{p}$ with a demand vector $\mathbf{d}^i(\mathbf{p})$, which gives the combination of items she would like to have at these prices. We assume she spends all of her wealth in exchanging her bundle $\mathbf{b}^i$ to obtain her desired bundle $\mathbf{d}^i(\mathbf{p})$. We express this with the following equation:

$$w^i(\mathbf{p}) = \mathbf{p} \cdot \mathbf{b}^i = \mathbf{p} \cdot \mathbf{d}^i(\mathbf{p}). \tag{10.1}$$

EXAMPLE 10.5. Suppose Wilma has a pound of cashmere, a pound of butter, and three pounds of gunpowder. So she has a supply bundle $\mathbf{b}^W = (1, 1, 3)$. At the price vector $\mathbf{p} = (\frac{1}{6}, \frac{3}{6}, \frac{2}{6})$, she has a net worth of $\mathbf{p} \cdot \mathbf{b}^W = 5/3$. Furthermore, assume that at this price vector, she would prefer a goods distribution as given by the demand vector $\mathbf{d}^W(\mathbf{p}) = (2, 2, 1)$. As required by Equation 10.1, this demand vector is constrained so that the total cost of her bundle at these prices, $\mathbf{p} \cdot \mathbf{d}^W(\mathbf{p})$, is equal to her net worth 5/3.

On the other hand, suppose Dexter has bundle $\mathbf{b}^D = (1, 2, 4)$, giving him a net worth of 5/2. For the same price vector $\mathbf{p}$, he has a demand vector of $\mathbf{d}^D(\mathbf{p}) = (6, 2, \frac{3}{2})$. Notice that Dexter seems to like cashmere more than Wilma since he wishes to convert proportionately more of his wealth to cashmere than she does.

Now, if the prices change, the demand vectors can change as well. For example, if the price vector is $\mathbf{p}' = (\frac{4}{10}, \frac{1}{10}, \frac{5}{10})$, then Wilma's net worth increases to $\mathbf{p}' \cdot \mathbf{b}^W = 2$. Seeing such a low price for butter, she may wish to stock up on it and choose a demand vector $\mathbf{d}^W(\mathbf{p}') = (1, 11, 1)$. Here too, her net worth does not change (it remains at 2), but she has converted much of it into a large supply of low-priced butter.

Each consumer i has a vector-valued demand function $\mathbf{d}^i$ which maps price vectors in T to the consumer's corresponding demand vector. Adding together the individual demand vectors for all of the consumers yields the demand vector $\mathbf{D}(\mathbf{p})$ for the entire economy at the given price vector $\mathbf{p}$. Thus,

$$\mathbf{D}(\mathbf{p}) = \sum_i \mathbf{d}^i(\mathbf{p}).$$

We assume that this vector-valued function $\mathbf{D}$ is continuous in the sense that small changes in the price vector cause small changes in overall demand.

Since $\mathbf{p} \cdot \mathbf{d}^i(\mathbf{p})$ is the net worth of individual i given price vector $\mathbf{p}$, the dot product $\mathbf{p} \cdot \mathbf{D}(\mathbf{p})$ equals the total wealth of the community given those prices. Note that

$$\mathbf{p} \cdot \mathbf{D}(\mathbf{p}) = \sum_i \mathbf{p} \cdot \mathbf{d}^i(\mathbf{p}) = \sum_i \mathbf{p} \cdot \mathbf{b}^i = \mathbf{p} \cdot \mathbf{S}.$$

This yields

$$\textbf{Walras's Law: } \mathbf{p} \cdot \mathbf{D}(\mathbf{p}) = \mathbf{p} \cdot \mathbf{S}.$$

This relationship is named for Leon Walras (1829–1910), one of the first economists to put the field on a mathematical footing.

If the coordinate of $\mathbf{D}(\mathbf{p})$ corresponding to cashmere is greater than the coordinate of the supply vector $\mathbf{S}$ corresponding to cashmere, then at this price there is more demand for cashmere than there is supply. This will drive up the price of cashmere.

However, if at the current prices, everyone can obtain their optimal choice of goods and would not trade any of their current quantity of each item, then the economy is in equilibrium.

Let v_j denote the jth component of a vector $\mathbf{v}$.

DEFINITION 10.7. *A price vector* $\mathbf{p}$ *is an* ***equilibrium price vector*** *if* $D_j(\mathbf{p}) \leq S_j$ *for all* j.

For such a price vector, the supply of every item exceeds the demand. Everyone can obtain their optimal choice of goods. No one desires further exchanges. Although the demand for an item may be strictly less than the total supply of that item, this does not necessarily drive down the price of that item. Since the prices are relative and everyone may have enough of the other products, the price of the item in excess supply need not fall.

Is there an equilibrium price vector? Can the economy be in balance? We will see that the answer is yes using the Brouwer Fixed Point Theorem.

To do so, we define a function $\mathbf{f} : T \to T$ that maps the set of possible price vectors to itself and tells us how the demand for items causes prices to change.

To see how prices change, we consider the following vector:

DEFINITION 10.8. *The* ***excess demand vector*** *is defined by* $\mathbf{E}(\mathbf{p}) = \mathbf{D}(\mathbf{p}) - \mathbf{S}$.

The coordinates of the vector $\mathbf{E}(\mathbf{p})$ tell us whether or not there is more demand or more supply for the items at the prices in the vector $\mathbf{p}$. If a coordinate of $\mathbf{E}(\mathbf{p})$ is positive, we expect the price of the corresponding item to rise since there is more demand than supply. People want more of this product at this price than is available. If the coordinate is negative, however, conditions could drive down the price of this item, as there is more supply than demand at this price.

Using the excess demand vector, Walras's Law can now be rewritten:

$$\textbf{Second form of Walras's Law: } \mathbf{p} \cdot \mathbf{E}(\mathbf{p}) = 0. \qquad (10.2)$$

Given a price vector $\mathbf{p}$, the vector $\mathbf{E}(\mathbf{p})$ points in the direction that we expect the prices to move from $\mathbf{p}$. From it we can construct a function $\mathbf{f}$ that takes each price vector $\mathbf{p}$ in T to another price vector $\mathbf{f}(\mathbf{p})$ in T toward which we would expect the vector $\mathbf{p}$ to move. Hence we will have created a continuous map from T back to itself. The Two-Dimensional Brouwer Fixed Point Theorem tells us that this function must have a fixed point. In other words, there is a price vector $\mathbf{p}$ such that there is no incentive to trade goods when the prices are at $\mathbf{p}$. At that price, the entire economy is in equilibrium.

The rest of this section is devoted to making this idea mathematically precise by appropriately defining $\mathbf{f} : T \to T$ and proving that a fixed point of $\mathbf{f}$ corresponds to an equilibrium price vector.

We begin by defining a function $\mathbf{f}^* : T \to \mathbb{R}^3$ by $\mathbf{f}^*(\mathbf{p}) = \mathbf{p} + \mathbf{E}(\mathbf{p})$. This function gives us an idea of how prices should move due to excess supply or demand.

However, there is a fundamental problem with this definition of $\mathbf{f}^*$. It does not necessarily send price vectors in T back to price vectors in T. The vectors that result might have negative entries, and their coordinates do not necessarily sum to 1. We need to remedy these problems so that we can be in a position to apply the Brouwer Fixed Point Theorem.

In order to make all of the entries nonnegative, we define $\mathbf{v}^+$ to be the vector obtained from a vector $\mathbf{v}$ by changing all negative entries into 0 entries. Then we define our new, improved function to be $\mathbf{f}^{**}(\mathbf{p}) = (\mathbf{p} + \mathbf{E}(\mathbf{p}))^+$. The function $\mathbf{f}^{**}$ maps T to the first octant in $\mathbb{R}^3$, but $\mathbf{f}^{**}(\mathbf{p})$ need not lie in T, since its entries do not necessarily sum to 1. However, by "normalizing" $\mathbf{f}^{**}(\mathbf{p})$, which is to say, dividing the resulting vector by the sum of its entries, we obtain a vector in T. It is important to note that at least one coordinate of $\mathbf{p} + \mathbf{E}(\mathbf{p})$ is positive, and therefore in normalizing $\mathbf{f}^{**}(\mathbf{p})$ we are not dividing by 0. (See Exercise 10.15.)

With $(\mathbf{p}+\mathbf{E}(\mathbf{p}))_j^+$ representing the jth entry of $(\mathbf{p}+\mathbf{E}(\mathbf{p}))^+$, we let $\beta(\mathbf{p}) = \sum_{j=1}^{3}(\mathbf{p} + \mathbf{E}(\mathbf{p}))_j^+$, and we define our desired function, mapping T to itself, as follows:

DEFINITION 10.9. *The price change function* $\mathbf{f} : T \to T$ *is defined by*

$$\mathbf{f}(\mathbf{p}) = \frac{(\mathbf{p} + \mathbf{E}(\mathbf{p}))^+}{\beta(\mathbf{p})}.$$

Figure 10.7 depicts an example of the vectors $\mathbf{p}$, $\mathbf{E}(\mathbf{p})$, $\mathbf{p} + \mathbf{E}(\mathbf{p})$, $(\mathbf{p}+\mathbf{E}(\mathbf{p}))^+$, and $\mathbf{f}(\mathbf{p})$ for a two-dimensional situation, as opposed to the three-dimensional one we have been considering. The second form of Walras's Law implies that $\mathbf{p}$ and $\mathbf{E}(\mathbf{p})$ must be perpendicular.

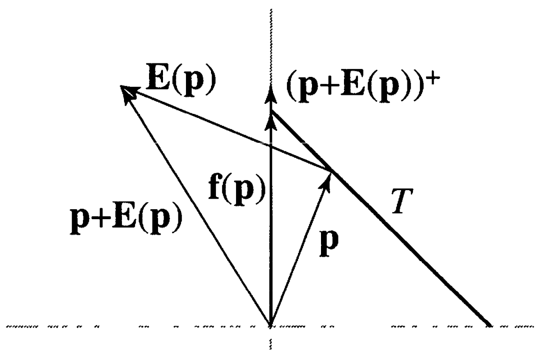

FIGURE 10.7: The vectors $\mathbf{p}$, $\mathbf{E}(\mathbf{p})$, $\mathbf{p} + \mathbf{E}(\mathbf{p})$, $(\mathbf{p} + \mathbf{E}(\mathbf{p}))^+$, and $\mathbf{f}(\mathbf{p})$.

The fact that $\mathbf{D}(\mathbf{p})$ is continuous implies that the price change function $\mathbf{f}(\mathbf{p})$ is also continuous.

Now, we are ready to apply the Two-Dimensional Brouwer Fixed Point Theorem. Since $\mathbf{f} : T \to T$ is a continuous function and T is a disk, there must be a price vector $\mathbf{p}^*$ such that $\mathbf{f}(\mathbf{p}^*) = \mathbf{p}^*$. That is to say, there is some price vector that is fixed by $\mathbf{f}$. We show, in fact, that such a vector is the equilibrium price vector we seek.

THEOREM 10.10. *The fixed points of the price change function* $\mathbf{f}$ *are equilibrium price vectors.*

In order to prove this theorem, we need the following lemma:

LEMMA 10.11. *If* $\mathbf{p}^*$ *is a fixed point of* $\mathbf{f}$, *then* $\beta(\mathbf{p}^*) = 1$.

Proof. Since $\mathbf{p}^*$ is a fixed point of $\mathbf{f}$, we have

$$\mathbf{f}(\mathbf{p}^*) = \frac{(\mathbf{p}^* + \mathbf{E}(\mathbf{p}^*))^+}{\beta(\mathbf{p}^*)} = \mathbf{p}^*. \tag{10.3}$$

This implies that, for $j = 1, 2, 3$,

$$(\mathbf{p}^* + \mathbf{E}(\mathbf{p}^*))_j^+ = \beta(\mathbf{p}^*) p_j^*. \tag{10.4}$$

Let α be an index value (equal to 1, 2, or 3) such that $p_\alpha^* > 0$. Such an α exists since the sum of the entries of $\mathbf{p}^*$ is 1. Since $\beta(\mathbf{p}^*) > 0$ (by Exercise 10.15), it follows from Equation 10.4 that

$$(\mathbf{p}^* + \mathbf{E}(\mathbf{p}^*))_\alpha^+ > 0.$$

Therefore

$$(\mathbf{p}^* + \mathbf{E}(\mathbf{p}^*))_\alpha^+ = (\mathbf{p}^* + \mathbf{E}(\mathbf{p}^*))_\alpha.$$

Plugging this back into Equation 10.4, we have

$$(\mathbf{p}^* + \mathbf{E}(\mathbf{p}^*))_{\alpha'} = \beta(\mathbf{p}^*) p_\alpha^*,$$

which yields

$$E_\alpha(\mathbf{p}^*) = (\beta(\mathbf{p}^*) - 1) p_\alpha^*.$$

Multiplying both sides by p_α^*, we obtain

$$p_\alpha^* E_\alpha(\mathbf{p}^*) = (\beta(\mathbf{p}^*) - 1) p_\alpha^* p_\alpha^*. \tag{10.5}$$

Although we derived Equation 10.5 with the assumption that $p_\alpha^* > 0$, the equation also holds when $p_\alpha^* = 0$ since it then reduces to $0 = 0$. Therefore, since p_j^* is nonnegative, Equation 10.5 holds for all p_j^*. Now, summing Equation 10.5 over all j, we obtain

$$\begin{aligned}
\mathbf{p}^* \cdot \mathbf{E}(\mathbf{p}^*) &= \sum_{j=1}^{3} p_j^* E_j(\mathbf{p}^*) \\
&= \sum_{j=1}^{3} (\beta(\mathbf{p}^*) - 1) p_j^* p_j^* \\
&= (\beta(\mathbf{p}^*) - 1) \sum_{j=1}^{3} p_j^* p_j^* \\
&= (\beta(\mathbf{p}^*) - 1) \mathbf{p}^* \cdot \mathbf{p}^*.
\end{aligned}$$

By the second form of Walras's Law (Equation 10.2), $\mathbf{p}^* \cdot \mathbf{E}(\mathbf{p}^*) = 0$. Therefore, $0 = (\beta(\mathbf{p}^*) - 1)\mathbf{p}^* \cdot \mathbf{p}^*$. Since $\mathbf{p}^* \in T$, it follows that $\mathbf{p}^* \cdot \mathbf{p}^* = 1$. Thus, it must be that $\beta(\mathbf{p}^*) - 1 = 0$, implying that $\beta(\mathbf{p}^*) = 1$, as desired. ■

With the help of Lemma 10.11, we can now prove Theorem 10.10.

Proof of Theorem 10.10. Let $\mathbf{p}^*$ be a fixed point for $\mathbf{f}$. So $\mathbf{f}(\mathbf{p}^*) = \mathbf{p}^*$. Since $\beta(\mathbf{p}^*) = 1$, Equation 10.3 becomes

$$(\mathbf{p}^* + \mathbf{E}(\mathbf{p}^*))^+ = \mathbf{p}^*,$$

which is actually three equations of the form

$$(\mathbf{p}^* + \mathbf{E}(\mathbf{p}^*))_j^+ = p_j^*. \tag{10.6}$$

We consider two possibilities for p_j^*: either $p_j^* = 0$ or $p_j^* > 0$. Keep in mind that all prices must be nonnegative.

If $p_j^* = 0$, then $(\mathbf{p}^* + \mathbf{E}(\mathbf{p}^*))_j^+ = 0$. Therefore $E_j(\mathbf{p}^*)^+ = 0$, and it follows that $E_j(\mathbf{p}^*) \leq 0$. This means that there is no excess demand for item j in this case.

Now, consider the case in which $p_j^* > 0$. Then by Equation 10.6, $(\mathbf{p}^* + \mathbf{E}(\mathbf{p}^*))_j^+ > 0$. Therefore,

$$(\mathbf{p}^* + \mathbf{E}(\mathbf{p}^*))_j^+ = (\mathbf{p}^* + \mathbf{E}(\mathbf{p}^*))_j.$$

Plugging this back into Equation 10.6, we obtain $(\mathbf{p}^* + \mathbf{E}(\mathbf{p}^*))_j = p_j^*$, and therefore $E_j(\mathbf{p}^*) = 0$.

Together, these cases imply that $E_j(\mathbf{p}^*) \leq 0$ for all j. Therefore $D_j(\mathbf{p}^*) \leq S_j$ for all j, and $\mathbf{p}^*$ is an equilibrium price vector. ■

When we have an equilibrium price vector $\mathbf{p}^*$, where $D_j(\mathbf{p}^*) \leq S_j$ for all j, we say the markets clear. Everyone can achieve their demand vector. The Brouwer Fixed Point Theorem tells us there is such a price. So we can all sleep well at night.

Nothing that we have said depended on the fact that we had three items available. The same would hold for an economy with hundreds of thousands of items and millions of individuals. Of course, establishing a corresponding result for a general setting would require the n-Dimensional Brouwer Fixed Point Theorem.

Exercises for Section 10.2

10.13. Describe the topological space of price vectors if the economy consists of only two items. Does the analysis of this section go through in that case?

10.14. In an economy with four goods, the resultant set of normalized price vectors will yield the analog in 4-space of the triangle T we created in 3-space. What is it? How many faces does it have, and what are their equations? How many edges and vertices does it have? What are their equations?

10.15. Show that at least one coordinate of the vector $\mathbf{p} + \mathbf{E}(\mathbf{p})$ is positive.

10.16. Suppose that our entire economy consists of only Carmen and Dexter with initial bundles $\mathbf{b}^C = (1, 1, 2)$ and $\mathbf{b}^D = (3, 4, 2)$ and initial price vector $\mathbf{p}_o = (\frac{1}{6}, \frac{1}{6}, \frac{4}{6})$. Let $\mathbf{f}$ be the price change function.

(a) Assume that when the price vector is $\mathbf{p} = (\frac{1}{6}, \frac{3}{6}, \frac{2}{6})$, Carmen has demand vector $\mathbf{d}^C = (3, 1, 1)$ and Dexter has demand vector $\mathbf{d}^D = (8, 1, 4)$. Determine $\mathbf{f}(\mathbf{p})$ for this $\mathbf{p}$.

(b) Assume that when the price vector is $\mathbf{p}' = (\frac{4}{6}, \frac{1}{6}, \frac{1}{6})$, Carmen has demand vector $\mathbf{d}^C = (\frac{1}{2}, \frac{1}{2}, \frac{9}{2})$ and Dexter has demand vector $\mathbf{d}^D = (2, 3, 7)$. Determine $\mathbf{f}(\mathbf{p}')$ for this $\mathbf{p}'$.

10.17. Let k be a fixed positive real number. Define

$$\mathbf{f}_k(\mathbf{p}) = \frac{(\mathbf{p} + k\mathbf{E}(\mathbf{p}))^+}{\sum_{j=1}^{3}(\mathbf{p} + k\mathbf{E}(\mathbf{p}))_j^+}.$$

Show that a fixed point $\mathbf{p}^*$ of this general price change function also yields an equilibrium price vector.

10.3 Kakutani's Fixed Point Theorem

In 1941, Shizuo Kakutani (1911–2004) proved a generalization of the Brouwer Fixed Point Theorem that has had powerful applications since. Instead of applying to functions from the n-ball B^n to itself, Kakutani's Fixed Point Theorem applies to so-called set-valued functions. Usually, functions associate a point x in a domain X to a point y in the range Y. Here, we will look at functions that take a point x in the domain X and send it to a nonempty subset A of the range Y. We denote such a set-valued function by $f : X \to_S Y$. (In this section we call our usual functions, whose values are points, point-valued functions, in order to distinguish them from set-valued functions.)

EXAMPLE 10.6. Consider the set of all of the people who have ever lived. Let f assign to each person the set of all people that person has ever seen. This is an example of a set-valued function. To each point (person) the function f associates a set of points (the set of people that person has ever seen).

EXAMPLE 10.7. The following are set-valued functions:

(i) Let f assign to each real number x the set of all real numbers greater than x. We write $f(x) = \{y \in \mathbb{R} \mid y \geq x\}$.

(ii) Let g assign to each real number x the set consisting of x and its negative. So $g(x) = \{-x, x\}$.

(iii) Let h assign to each real number x the set $[-1, 4]$ if x is negative or the set $[-4, 1]$ if x is not negative. Therefore,

$$h(x) = \begin{cases} [-1, 4] & \text{if } x < 0, \\ [-4, 1] & \text{if } x \geq 0. \end{cases}$$

For a point-valued function $f : X \to Y$, the graph is defined to be the set $\{(x, y) \mid y = f(x)\}$ in $X \times Y$. We need a similar notion for set-valued functions:

DEFINITION 10.12. *The **graph** of a set-valued function $f : X \to_S Y$ is the subset of $X \times Y$ given by $G_f = \{(x, y) \mid y \in f(x)\}$.*

EXAMPLE 10.8. The graphs of the set-valued functions f, g, and h from Example 10.7 are shown in Figure 10.8.

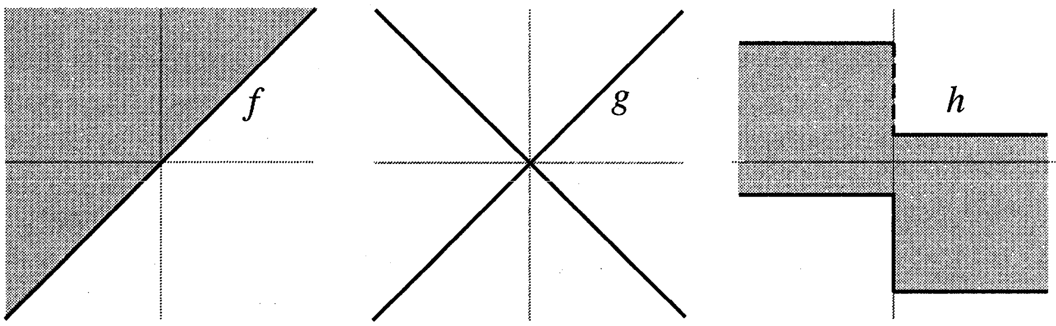

FIGURE 10.8: Graphs of the set-valued functions f, g, and h.

In the hypotheses of the Brouwer Fixed Point Theorem, the continuity of the function is critical. For set-valued functions we do not directly define continuity, but instead we work with a property closely related to it. Combining the results of Exercises 4.10 and 7.13, it follows that if Y is compact and Hausdorff, then a point-valued function $f : X \to Y$ is continuous if and only if the graph of f is a closed subset of $X \times Y$. Therefore we focus on set-valued functions having graphs that are closed sets. In Example 10.8, the set-valued functions f and g have graphs that are closed sets, but the set-valued function h does not.

Having a closed graph is advantageous since it implies that convergent sequences behave reasonably, as the following lemma indicates:

LEMMA 10.13. *Let $f : X \to_S Y$ be a set-valued function whose graph, G_f, is closed in $X \times Y$. If (x_n) is a sequence in X that converges to $x_0 \in X$, and (y_n) is a sequence in Y that converges to $y_0 \in Y$ and satisfies $y_n \in f(x_n)$ for each n, then $y_0 \in f(x_0)$.*

Proof. Form the sequence $((x, y)_n)$ in $X \times Y$ defined by $(x, y)_n = (x_n, y_n)$. Since $y_n \in f(x_n)$, this entire sequence lies in the graph of f. Since (x_n) converges to x_0 and (y_n) converges to y_0, $((x, y)_n)$ converges to (x_0, y_0). Now, either there exists N such that $(x_0, y_0) = (x_N, y_N)$, or (x_0, y_0) is distinct from every (x_n, y_n). In the first case, it directly follows that $(x_0, y_0) \in G_f$. In the second case, (x_0, y_0) must be a limit point of the set $\{(x_n, y_n)\}_{n \in \mathbb{Z}_+}$ and therefore must also be a limit point of G_f. Since G_f is closed, it follows that $(x_0, y_0) \in G_f$ in this case, as well. In either case, (x_0, y_0) is in the graph of f, and therefore $y_0 \in f(x_0)$ as we wanted to show. ∎

Now, what does it mean for a set-valued function to have a fixed point?

DEFINITION 10.14. *Given a set-valued function $F : X \to_S X$, a* ***fixed point*** *of F is a point x^* in X for which $x^* \in F(x^*)$.*

A fixed point of a set-valued function is a point that maps to a set containing the point, in contrast to a fixed point of a point-valued function, which is a point that simply maps to itself.

The functions considered in the Kakutani Fixed Point Theorem are assumed to have a domain that is a polyhedron in $\mathbb{R}^n$, where a polyhedron is a bounded set that can be expressed as a solution set to finitely many inequalities of the form $a_1x_1 + \ldots + a_nx_n \leq b$. (See Definition 0.12.)

The n-Dimensional Kakutani Fixed Point Theorem states that for a polyhedron X in $\mathbb{R}^n$, a set-valued function $F : X \to_S X$ has a fixed point if $F(\mathbf{x})$ is a convex subset of X for each $\mathbf{x}$ in X and if the graph of F is closed in $X \times X$.

The n-Dimensional Kakutani Fixed Point Theorem requires the n-Dimensional Brouwer Fixed Point Theorem in its proof. Here we only address the Kakutani Fixed Point Theorem in dimensions one and two, but the proof directly generalizes assuming the n-Dimensional Brouwer Fixed Point Theorem.

For the one-dimensional case, the possibilities for the polyhedron $X \subset \mathbb{R}$ are limited. In fact, such sets must be closed and bounded intervals $[a, b]$. (See Exercise 10.20.) We ask you to prove the One-Dimensional Kakutani Fixed Point Theorem in Exercise 10.21.

Now we present the Kakutani Fixed Point Theorem in dimension two:

THEOREM 10.15. ***The Two-Dimensional Kakutani Fixed Point Theorem.*** *Let X be a polyhedron in $\mathbb{R}^2$ and $F : X \to_S X$ be a set-valued function such that $F(\mathbf{x})$ is a convex subset of X for each $\mathbf{x}$ in X. If the graph of F is closed in $X \times X$, then there exists $\mathbf{x}^* \in X$ such that $\mathbf{x}^* \in F(\mathbf{x}^*)$.*

> ***Proof.*** A polyhedron in $\mathbb{R}^2$ is either a point, a line segment, or a convex polygon. The cases for a point or line segment fall under the one-dimensional version of the theorem, assigned in Exercise 10.21. Here we address the case for a convex polygon in $\mathbb{R}^2$. For simplicity, we prove the theorem when X is a triangle T in $\mathbb{R}^2$. We then discuss how the method of proof for a triangle carries over to a general convex polygon.
>
> Let T be a triangle in the plane with vertices $\mathbf{v}_0^1$, $\mathbf{v}_0^2$, and $\mathbf{v}_0^3$. (The reason for using the multiple indices will become clear as we progress.) Since T is a triangle, every point in T can be represented as a linear combination of $\mathbf{v}_0^1$, $\mathbf{v}_0^2$, and $\mathbf{v}_0^3$. Specifically, for $\mathbf{x} \in T$ we have $\mathbf{x} = \lambda_0^1\mathbf{v}_0^1 + \lambda_0^2\mathbf{v}_0^2 + \lambda_0^3\mathbf{v}_0^3$, where each $\lambda_0^i \geq 0$ and $\lambda_0^1 + \lambda_0^2 + \lambda_0^3 = 1$. (See Figure 10.9.)
>
> Notice that T is homeomorphic to the disk, and therefore the Two-Dimensional Brouwer Fixed Point Theorem applies to every continuous function from T to T.
>
> To find a fixed point of F, we build a sequence $\mathbf{f}_0, \mathbf{f}_1, \mathbf{f}_2, \ldots$ of point-valued continuous functions that approximate F. By the Two-Dimensional Brouwer Fixed Point Theorem, each $\mathbf{f}_n$ has a fixed point

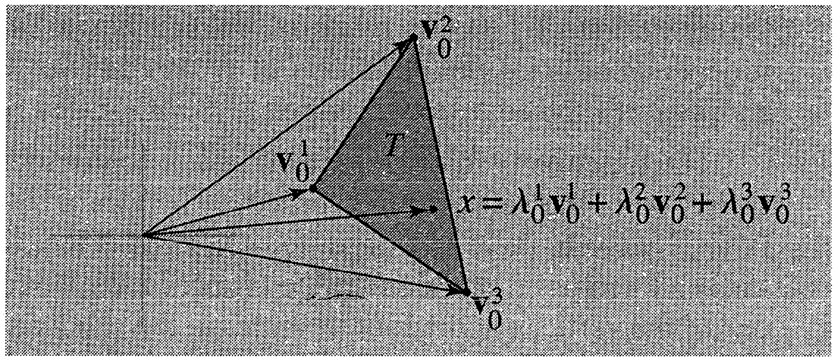

FIGURE 10.9: Every point in T is a linear combination of the vertices.

$\mathbf{x}_n \in T$. We prove that the sequence of fixed points $(\mathbf{x}_n)$ has a subsequence that converges to a fixed point of F.

For each of the three vertices $\mathbf{v}_0^i$, $i = 1, 2, 3$, we pick a particular point $\mathbf{y}_0^i \in f(\mathbf{v}_0^i)$. Then we define a function $\mathbf{f}_0 : T \to T$ so that $\mathbf{f}_0(\mathbf{v}_0^i) = \mathbf{y}_0^i$ on the vertices. We extend this linearly to the triangle by, for each $\mathbf{x} = \lambda_0^1\mathbf{v}_0^1 + \lambda_0^2\mathbf{v}_0^2 + \lambda_0^3\mathbf{v}_0^3$, setting $\mathbf{f}_0(\mathbf{x}) = \lambda_0^1\mathbf{y}_0^1 + \lambda_0^2\mathbf{y}_0^2 + \lambda_0^3\mathbf{y}_0^3$.

Notice that $\mathbf{f}_0$ is not a set-valued function. It is a point-valued function mapping T to itself. Moreover, because $\mathbf{f}_0$ is defined to be the linear combination of its values at the vertices, it is continuous. Therefore the Two-Dimensional Brouwer Fixed Point Theorem applies, and we have a point $\mathbf{x}_0 \in T$ such that $\mathbf{f}_0(\mathbf{x}_0) = \mathbf{x}_0$.

The point $\mathbf{x}_0$ is not necessarily a fixed point of the set-valued function F (but would be, for instance, if it was one of the vertices of T).

Now, to define the next function $\mathbf{f}_1$ in our sequence of functions approximating F, we begin by subdividing T into four smaller triangles, as shown in Figure 10.10.

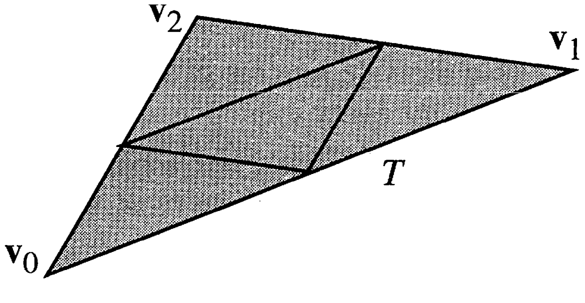

FIGURE 10.10: Subdividing T into smaller triangles.

The vertices of these four triangles are made up of the vertices of T and the midpoints of the edges of T, given by

$$\tfrac{1}{2}\mathbf{v}_0^1 + \tfrac{1}{2}\mathbf{v}_0^2,\ \tfrac{1}{2}\mathbf{v}_0^1 + \tfrac{1}{2}\mathbf{v}_0^3, \text{ and } \tfrac{1}{2}\mathbf{v}_0^2 + \tfrac{1}{2}\mathbf{v}_0^3.$$

We define $\mathbf{f}_1(\mathbf{x})$ in a manner analogous to how we defined $\mathbf{f}_0(\mathbf{x})$. For each vertex $\mathbf{v}$ in the four triangles, we choose a point $\mathbf{y}$ in $F(\mathbf{v})$ and define $\mathbf{f}_1(\mathbf{v}) = \mathbf{y}$. Then, as with $\mathbf{f}_0$, we extend $\mathbf{f}_1$ linearly over each triangle. Notice that if $\mathbf{x}$ lies in the intersection of two different triangles, then the definition of $\mathbf{f}_1(\mathbf{x})$ in terms of either triangle is the same because it depends only on the definition of $\mathbf{f}_1$ on the two vertices that are the endpoints of the edge containing $\mathbf{x}$.

The function $\mathbf{f}_1$ maps T to itself and is continuous since it is a linear extension of the values at the vertices. Therefore, by the Brouwer Fixed Point Theorem, there is a fixed point $\mathbf{x}_1$ of $\mathbf{f}_1$. Here too, $\mathbf{x}_1$ is not necessarily a fixed point of the set-valued function F. But it would be if it was one of the vertices of the four triangles in the subdivision of T. Let T_1 be a subdivision triangle that contains this new fixed point $\mathbf{x}_1$, and assume that the vertices of T_1 are $\mathbf{v}_1^1$, $\mathbf{v}_1^2$, and $\mathbf{v}_1^3$.

We continue this process. Specifically, assume that we have a continuous $\mathbf{f}_{n-1} : T \to T$ with fixed point $\mathbf{x}_{n-1}$ in triangle $T_{n-1} \subset T$ having vertices $\mathbf{v}_{n-1}^1$, $\mathbf{v}_{n-1}^2$, and $\mathbf{v}_{n-1}^3$.

To define $\mathbf{f}_n$, we subdivide each of the triangles used in the definition of $\mathbf{f}_{n-1}$ into four subtriangles as described earlier. Then for each vertex $\mathbf{v}$ in each triangle, we define $\mathbf{f}_n(\mathbf{v})$ to be a point in $F(\mathbf{v})$. Finally, we extend $\mathbf{f}_n$ linearly over each triangle in the subdivision to obtain a continuous function $\mathbf{f}_n : T \to T$. Since $\mathbf{f}_n$ is continuous, it has a fixed point $\mathbf{x}_n \in T$. Let $T_n \subset T$ be a triangle in the subdivision containing $\mathbf{x}_n$, and assume T_n has vertices $\mathbf{v}_n^1$, $\mathbf{v}_n^2$, and $\mathbf{v}_n^3$.

We now have a sequence $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots$ of fixed points of the functions $\mathbf{f}_0, \mathbf{f}_1, \mathbf{f}_2, \ldots$. Note that $\mathbf{x}_n = \lambda_n^1\mathbf{v}_n^1 + \lambda_n^2\mathbf{v}_n^2 + \lambda_n^3\mathbf{v}_n^3$ for some values λ_n^1, λ_n^2, and λ_n^3 in $[0, 1]$. Furthermore, if we have $\mathbf{y}_n^j = \mathbf{f}_n(\mathbf{v}_n^j)$ for each n and j, then $\mathbf{x}_n = \mathbf{f}_n(\mathbf{x}_n) = \lambda_n^1\mathbf{y}_n^1 + \lambda_n^2\mathbf{y}_n^2 + \lambda_n^3\mathbf{y}_n^3$ as well, yielding

$$\mathbf{x}_n = \lambda_n^1\mathbf{v}_n^1 + \lambda_n^2\mathbf{v}_n^2 + \lambda_n^3\mathbf{v}_n^3 = \lambda_n^1\mathbf{y}_n^1 + \lambda_n^2\mathbf{y}_n^2 + \lambda_n^3\mathbf{y}_n^3. \tag{10.7}$$

Since T is a compact subset of the plane, Theorem 7.16 implies that every sequence in T has a convergent subsequence. Let $\mathbf{x}^*$ be the limit of a convergent subsequence $(\mathbf{x}_{j_n})$ of the sequence $(\mathbf{x}_n)$. We show that $\mathbf{x}^*$ is a fixed point of the set-valued function F.

Because the side lengths of the triangles T_n shrink to zero as n approaches infinity, every sequence made up of one point from each triangle T_{j_n} must also converge to $\mathbf{x}^*$. Hence, the sequence of fixed points $(\mathbf{x}_{j_n})$ and the corresponding sequences of vertices $(\mathbf{v}_{j_n}^1)$, $(\mathbf{v}_{j_n}^2)$, and $(\mathbf{v}_{j_n}^3)$ all converge to $\mathbf{x}^*$.

Since the interval $[0, 1]$ is compact in $\mathbb{R}$, Theorem 7.16 implies that every sequence in $[0, 1]$ has a convergent subsequence. Thus the three coefficient sequences $(\lambda_{j_n}^1)$, $(\lambda_{j_n}^2)$, and $(\lambda_{j_n}^3)$ have convergent subsequences. Similarly, the three sequences $(\mathbf{y}_{j_n}^1)$, $(\mathbf{y}_{j_n}^2)$, and $(\mathbf{y}_{j_n}^3)$ are sequences in the compact set T and thus also have convergent subsequences. By taking subsequences, one at a time, of all of these additional sequences, we can thereby choose a single indexing sequence (m_n) such that all ten corresponding subsequences converge: $(\mathbf{x}_{m_n})$ to $\mathbf{x}^*$, $(\mathbf{v}_{m_n}^1)$ to $\mathbf{x}^*$, $(\mathbf{v}_{m_n}^2)$ to $\mathbf{x}^*$, $(\mathbf{v}_{m_n}^3)$ to $\mathbf{x}^*$, $(\lambda_{m_n}^1)$ to a value λ^1, $(\lambda_{m_n}^2)$ to a value λ^2, $(\lambda_{m_n}^3)$ to a value λ^3, $(\mathbf{y}_{m_n}^1)$ to a point $\mathbf{y}^1$, $(\mathbf{y}_{m_n}^2)$ to a point $\mathbf{y}^2$, and $(\mathbf{y}_{m_n}^3)$ to a point $\mathbf{y}^3$.

As m_n approaches infinity, we see from Equation 10.7 that $\mathbf{x}^* = \lambda^1\mathbf{y}^1 + \lambda^2\mathbf{y}^2 + \lambda^3\mathbf{y}^3$. Furthermore $\lambda^1 + \lambda^2 + \lambda^3 = 1$, and $\lambda^1, \lambda^2, \lambda^3 \in [0, 1]$. Therefore $\mathbf{x}^*$ is in the "triangle" with vertices $\mathbf{y}^1$, $\mathbf{y}^2$, and $\mathbf{y}^3$. (We use quotes since the triangle could be a line segment or a point if exactly two of the $\mathbf{y}^i$ are equal or if they all are equal, respectively.)

Since F has a closed graph in $T \times T$, Lemma 10.13 applies. Therefore, since the sequences $(\mathbf{v}^1_{m_n})$ and $(\mathbf{y}^1_{m_n})$ converge to $\mathbf{x}^*$ and $\mathbf{y}^1$, respectively, and since $\mathbf{y}^1_{m_n} \in F(\mathbf{v}^1_{m_n})$ for each m_n, we know that $\mathbf{y}^1 \in F(\mathbf{x}^*)$. Similarly, $\mathbf{y}^2 \in F(\mathbf{x}^*)$ and $\mathbf{y}^3 \in F(\mathbf{x}^*)$.

But $\mathbf{x}^* = \lambda^1\mathbf{y}^1 + \lambda^2\mathbf{y}^2 + \lambda^3\mathbf{y}^3$ is in the triangle with vertices $\mathbf{y}^1$, $\mathbf{y}^2$, and $\mathbf{y}^3$, all of which lie in $F(\mathbf{x}^*)$. Since $F(\mathbf{x}^*)$ is convex, $\mathbf{x}^*$ must also be contained in $F(\mathbf{x}^*)$. In other words, $\mathbf{x}^* \in F(\mathbf{x}^*)$, as we wished to show.

We have now proven the Two-Dimensional Kakutani Fixed Point Theorem assuming that the domain is a triangle in the plane. To prove the result for a general convex polygon in the plane, we use the same approach, but we start by subdividing the polygon into triangles. The initial approximating function $\mathbf{f}_0$ is first defined on the vertices of these triangles and then is extended linearly to each of the triangles, just as was done in the foregoing process. Then, as was done previously, successive approximating functions $\mathbf{f}_n$ are defined by subdividing each triangle considered at the previous stage, defining $\mathbf{f}_n$ on the vertices of the new triangles, and extending the definition linearly to each triangle. The same argument yields a point $\mathbf{x}^*$ such that $\mathbf{x}^* \in F(\mathbf{x}^*)$. ■

This method of proving the Two-Dimensional Kakutani Fixed Point Theorem carries over to the general n-dimensional version, with the n-Dimensional Brouwer Fixed Point Theorem required along the way.

Kakutani's Fixed Point Theorem contains the Brouwer Fixed Point Theorem as a special case, but we need to view the domain of the Brouwer Fixed Point Theorem as a polyhedron to see this. Let

$$P^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid -1 \le x_j \le 1 \text{ for } j = 1, \ldots, n\}.$$

The set P^n is an n-dimensional cube. By definition P^n is a polyhedron, and it is homeomorphic to the n-ball, B^n. Let us consider the Brouwer Fixed Point Theorem on P^n. If $\mathbf{f} : P^n \to P^n$ is a continuous point-valued function, then $\mathbf{f}$ can be considered a set-valued function mapping $\mathbf{x} \in P^n$ to the single-point set $\{\mathbf{f}(\mathbf{x})\}$. The domain of $\mathbf{f}$ is a compact polyhedron, and $\{\mathbf{f}(\mathbf{x})\}$ is a single-point set in P^n and therefore is a nonempty convex subset of the domain. Furthermore, the point-valued function $\mathbf{f}$ is continuous; thus the graph of $\mathbf{f}$ is closed in $P^n \times P^n$ by Exercise 4.10. Kakutani's Fixed Point Theorem then applies, implying that there exists $\mathbf{x} \in P^n$ such that $\mathbf{x} \in \{\mathbf{f}(\mathbf{x})\}$, and therefore $\mathbf{x} = \mathbf{f}(\mathbf{x})$.

It is important to realize, however, that although the Brouwer Fixed Point Theorem is a special case of the Kakutani Fixed Point Theorem, the latter does not supplant the former, since the Brouwer Fixed Point Theorem is needed in the proof of the Kakutani Fixed Point Theorem.

Exercises for Section 10.3

10.18. For each of the following set-valued functions, draw the graph and determine whether or not $F(x)$ is convex for all values of x in the domain:

(a) Let $F : \mathbb{R} \rightarrow_S \mathbb{R}$ be given by $F(x) = \{y \in \mathbb{R} \mid y \leq x\}$.

(b) Let $G : \mathbb{R} \rightarrow_S \mathbb{R}$ be given by $G(x) = \{nx \in \mathbb{R} \mid n \in \mathbb{Z}\}$.

(c) Let $H : \mathbb{R} \rightarrow_S \mathbb{R}^2$ be given by $H(x) = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 y_2 = x\}$.

10.19. Define a relation on $\mathbb{R}$ by $a \sim b$ if there exists an integer c such that $a, b \in [c, c+1)$.

(a) Show that $\sim$ is an equivalence relation.

(b) Let $f : \mathbb{R} \rightarrow_S \mathbb{R}$, be the set-valued function defined by $f(x) = [x]$, where $[x]$ is the equivalence class of x under the equivalence relation $\sim$. Sketch the graph of f.

(c) With f as defined in (b), is $f(x)$ convex for each $x \in \mathbb{R}$?

(In general an equivalence relation on a set X gives rise to a natural set-valued function, sending each point in X to the equivalence class of points in X that contains it.)

10.20. Prove that a polyhedron in $\mathbb{R}$ must be a closed and bounded interval $[a, b]$.

10.21. Explain the proof of the Kakutani Fixed Point Theorem in the case that X is the interval $[0, 1]$. Include pictures of n graphs of F, f_0, and f_1 and the fixed points x_0 and x_1 in your explanation.

10.22. Determine whether or not the Kakutani Fixed Point Theorem applies to each of the following set-valued functions. Note that you need to check that X is a polyhedron, that $F(x)$ is convex for each $x \in X$, and that the graph of F is closed. If Kakutani's Fixed Point Theorem applies, find as many fixed points as you can for F.

(a) $X = [0, 1]$ and $F(x) = \{y \in [0, 1] \mid y \geq 1 - x\}$.

(b) $X = [0, 1]$ and $F(x) = \{y \in [0, 1] \mid \frac{y}{x} \text{ is irrational}\}$.

(c) $X = [0, 1]$ and $F(x) = \{y \in [0, 1] \mid y \leq x^2\}$.

(d) $X = [0, \infty)$ and $F(x) = \{y \in [0, \infty) \mid y \geq 2x\}$.

(e) $X = [0, 1] \cup [2, 3]$ and

$$F(x) = \begin{cases} [2, 3] & \text{if } x \in [0, 1], \\ [0, 1] & \text{if } x \in [2, 3]. \end{cases}$$

(f) $X = [0, 1] \times [0, 1]$ and

$$F(x, y) = \{(x, y) \in [0, 1] \times [0, 1] \mid x \geq y \text{ and } y = 1 - x\}.$$

10.23. For each of the following hypotheses in the Kakutani Fixed Point Theorem, give an example to show that the theorem does not hold if the hypothesis is dropped. That is, find a set-valued function $F : X \rightarrow_S X$ that satisfies each of the other two hypotheses, but does not have a fixed point.

(a) X is a polyhedron.

(b) $F(x)$ is convex for each $x \in X$.

(c) G_F is closed.

10.4 Game Theory and the Nash Equilibrium

In this section we introduce game theory and provide an application of the Kakutani Fixed Point Theorem, using it to prove John Nash's celebrated theorem on the existence of equilibria in n-person games. Although all of what follows applies to n-person games, we will restrict ourselves to at most three players in order to keep the notation straightforward.

Suppose that Elaine, George, and Newman agree to play a game. At a given turn, each of the three has a finite number of moves that she or he can make. For Elaine, these choices are labeled 1 through n_E. For George, they are labeled 1 through n_G. And for Newman, they are labeled 1 through n_N. These moves are called pure strategies.

We set the rules of our game so that all three players make a move without knowing what the other players have chosen to do. Then they each receive some payoff. When Elaine makes move i, George makes move j, and Newman makes move k. Consequently, Elaine receives payoff E_{ijk}, George receives payoff G_{ijk}, and Newman receives payoff N_{ijk}. Each player has an associated $n_E \times n_G \times n_N$ three-dimensional payoff array and all three are familiar with the three payoff arrays.

Of course, if Elaine always made the same move, it would not take George and Newman long to decide how best to maximize their own payoffs relative to hers. So instead of consistently making move i, Elaine may choose to play a different strategy. She may choose to play each move with a certain probability.

It makes good sense to do this. For instance, in a poker game, a player does not want to bluff every time certain cards are in her hand, as the other players may soon pick up on the fact that she consistently does this. Rather, she should bluff with a certain probability that she has predetermined. This will be her strategy. Then no particular card pattern will be associated with her bluffing.

We denote the probability that Elaine plays move i by p_i, that George plays move j by q_j, and that Newman plays move k by r_k. We make the basic probability assumptions that

$$p_i \geq 0, \ \ q_j \geq 0, \ \text{ and } r_k \geq 0 \text{ for all } i, j, k, \tag{10.8}$$

and

$$\sum_i p_i = 1, \ \ \sum_j q_j = 1, \ \text{ and } \sum_k r_k = 1. \tag{10.9}$$

DEFINITION 10.16. *The corresponding probability vectors*

$$\mathbf{p} = (p_1, p_2, \dots, p_{n_E}), \ \mathbf{q} = (q_1, q_2, \dots, q_{n_G}), \\ \text{and } \mathbf{r} = (r_1, r_2, \dots, r_{n_N})$$

are called ***mixed strategies****.*

Now, the payoff for each player becomes an expected value. For Elaine, it is given by

$$E(\mathbf{p}, \mathbf{q}, \mathbf{r}) = \sum_{i,j,k} E_{ijk} p_i q_j r_k.$$

This expected value is the sum of the possible payoffs multiplied by the probability that such a payoff occurs. We similarly define the expected values $G(\mathbf{p}, \mathbf{q}, \mathbf{r})$ and $N(\mathbf{p}, \mathbf{q}, \mathbf{r})$ for George and Newman, respectively. If Elaine, George, and Newman play their games enough times, we expect their payoffs to average out to approximately $E(\mathbf{p}, \mathbf{q}, \mathbf{r})$, $G(\mathbf{p}, \mathbf{q}, \mathbf{r})$, and $N(\mathbf{p}, \mathbf{q}, \mathbf{r})$, respectively.

EXAMPLE 10.9. Suppose Elaine, George, and Newman are deciding where to go for dinner. They can choose either the Happy Står Chinese Restaurant or the New Yorker Diner. All three agree to simultaneously yell out either "Chinese" or "diner." If two of them agree and the third does not, they all go to the restaurant chosen by the two who agreed, and the third has to pay 10 dollars to each of the others for their dinner. If all three agree, they go to the restaurant that they all chose, and everyone pays for their own dinner.

Elaine's payoff array then has entries of the form

$$E_{ijk} = \begin{cases} -20 & \text{if } i \neq j = k, \\ 0 & \text{if } i = j = k, \\ 10 & \text{otherwise.} \end{cases}$$

The payoff arrays for George and Newman are defined similarly.

Let us assume that Elaine, George, and Newman are equally likely to pick the Chinese restaurant or to pick the diner on any given evening. Then their mixed strategies are all the same, namely $\mathbf{p} = \mathbf{q} = \mathbf{r} = (\frac{1}{2}, \frac{1}{2})$. From this, we can compute the expected value of Elaine's payoff. We find that

$$E = -20(\tfrac{1}{4}) + 0(\tfrac{1}{4}) + 10(\tfrac{1}{2}) = 0.$$

Similarly, the expected values for George and Newman are also both 0.

Now, suppose that in a particular game, Elaine knows that George and Newman are using mixed strategies $\mathbf{q}$ and $\mathbf{r}$, respectively. Elaine's goal is to obtain the largest possible payoff; that is, she wants to maximize her expected value, given $\mathbf{q}$ and $\mathbf{r}$. Thus, she wants to find a strategy $\mathbf{p}$ so that $E(\mathbf{p}, \mathbf{q}, \mathbf{r}) \geq E(\mathbf{p}', \mathbf{q}, \mathbf{r})$ over all possible probability vectors $\mathbf{p}'$. There may be more than one such strategy for Elaine, and therefore we have the following definition:

DEFINITION 10.17. *Let $P(\mathbf{q}, \mathbf{r})$ be the collection of all probability vectors $\mathbf{p}$ that satisfy $E(\mathbf{p}, \mathbf{q}, \mathbf{r}) \geq E(\mathbf{p}', q, \mathbf{r})$ over all mixed strategies $\mathbf{p}'$. This is called the **set of optimal mixed strategies associated to mixed strategies** $\mathbf{q}$ **and** $\mathbf{r}$.*

Similarly, $Q(\mathbf{p}, \mathbf{r})$ is the set of optimal mixed strategies for George, maximizing his payoff when $\mathbf{p}$ and $\mathbf{r}$ are given, and $R(\mathbf{p}, \mathbf{q})$ is the corresponding set of optimal mixed strategies for Newman, given $\mathbf{p}$ and $\mathbf{q}$.

Now, how does Elaine choose vectors to maximize the expected value $E(\mathbf{p}, \mathbf{q}, \mathbf{r})$? For a fixed $\mathbf{q}$ and $\mathbf{r}$, the expected value is a linear equation in the components of $\mathbf{p}$, as follows:

$$E(\mathbf{p}, \mathbf{q}, \mathbf{r}) = \sum_{i,j,k} E_{ijk} p_i q_j r_k = \sum_i (\sum_{j,k} E_{ijk} q_j r_k) p_i. \tag{10.10}$$

Letting $a_i = \sum_{j,k} E_{ijk} q_j r_k$, we can rewrite Equation 10.10 as

$$E(\mathbf{p}, \mathbf{q}, \mathbf{r}) = a_1 p_1 + a_2 p_2 + \ldots + a_{n_E} p_{n_E}.$$

Once we have rewritten the expected value in this way, it becomes clear how to maximize it, as in the following example:

EXAMPLE 10.10. Suppose that in a particular game played by Elaine, George, and Newman, there are five possible moves from which Elaine can choose. Further suppose that for the fixed mixed strategies $\mathbf{q}$ and $\mathbf{r}$ of George and Newman, respectively, The expected value of Elaine's payoff is

$$E(\mathbf{p}, \mathbf{q}, \mathbf{r}) = 3p_1 + 5p_2 + 4p_3 + 5p_4 + p_5.$$

She wants to find $\mathbf{p} = (p_1, p_2, p_3, p_4, p_5)$ to maximize $E(\mathbf{p}, \mathbf{q}, \mathbf{r})$. Clearly, the best choice for Elaine is to make the components of $\mathbf{p}$ with smaller coefficients as tiny as possible and those with the largest coefficients as big as possible. In this case, the largest coefficient is 5. So p_1, p_3, and p_5 should be chosen to be 0. Since the coefficients of p_2 and p_4 are both 5, she could choose $p_2 = 1$ and $p_4 = 0$. Or she could choose $p_2 = 0$ and $p_4 = 1$. In fact, she could choose p_2 and p_4 to be any combination that adds to 1. She will still have a probability vector that maximizes the payoff. Therefore she chooses her set of optimal mixed strategies to be

$$P(\mathbf{q}, \mathbf{r}) = \{(0, p_2, 0, p_4, 0) \mid p_2 + p_4 = 1, p_2 = 0, p_4 = 0\}.$$

In general, we see that the set of optimal mixed strategies $P(\mathbf{q}, \mathbf{r})$ is given by

$$P(\mathbf{q}, \mathbf{r}) = \{\mathbf{p} \mid \sum_i p_i = 1 \text{ and } p_i = 0 \text{ if } a_i < \max_j\{a_j\}\}. \tag{10.11}$$

It follows that $P(\mathbf{q}, \mathbf{r})$ is a closed, bounded, and convex subset of $\mathbb{R}^{n_E}$. (See Exercise 10.26.)

Now, if Elaine chooses an optimal mixed strategy $\mathbf{p}$ in response to the mixed strategies $\mathbf{q}$ and $\mathbf{r}$ of George and Newman, then George might want to change his mixed strategy to an optimal one, given that Elaine and Newman have chosen mixed strategies $\mathbf{p}$ and $\mathbf{r}$. And, of course, Newman might want to change his mixed strategy as well.

We would like to know if there is a choice of mixed strategies for all of the players such that every player is using an optimal mixed strategy at the same time. Under that circumstance, there would be no incentive for the players to change strategies.

DEFINITION 10.18. *Mixed strategy vectors* **p**, **q**, *and* **r** *are said to* ***solve the game*** *if* $\mathbf{p} \in P(\mathbf{q}, \mathbf{r})$, $\mathbf{q} \in Q(\mathbf{p}, \mathbf{r})$, *and* $\mathbf{r} \in R(\mathbf{p}, \mathbf{q})$. *We then say that* **p**, **q**, *and* **r** *are a* ***Nash equilibrium***.

In a Nash equilibrium, no player has an incentive to change his or her strategy, assuming no one else changes theirs. Each player is making as much as he or she possibly can, given the current strategies of the other players.

In his doctoral dissertation at Princeton University in 1950, John Nash proved that every game has at least one equilibrium (which later became known as a Nash equilibrium). His results revolutionized the field of game theory and subsequently had a significant impact on economics and the social sciences. In 1994, the Nobel Prize in economics was awarded to John Nash, Reinhard Selten, and John C. Harsanyi, with the latter two receiving the award for their refinements to the Nash equilibrium theory for situations where multiple equilibria exist and situations where each player does not have complete information about the other players' strategies in a game.

We present and prove Nash's Theorem momentarily, but first we look at an example.

EXAMPLE 10.11. Let us determine the Nash equilibria for the restaurant game of Example 10.9. We consider the situation from Elaine's point of view. Let $\mathbf{p} = (a, 1-a)$ be her mixed strategy. So a is the probability that she picks the Chinese restaurant, and $1-a$ is the probability that she picks the diner. Similarly, let George's mixed strategy be $\mathbf{q} = (b, 1-b)$, and let Newman's be $\mathbf{r} = (c, 1-c)$.

Then the expected value for Elaine's payoff is

$$E = 10[ab(1-c) + a(1-b)c + (1-a)(1-b)c + (1-a)b(1-c)]$$
$$-20[a(1-b)(1-c) + (1-a)(bc)],$$

which simplifies to

$$E = 10[2a(b+c-1) + c + b - 3cb].$$

For the time being, assume that b and c are fixed. Let us see how Elaine can maximize E by choosing an appropriate a. We consider three different possibilities for the sum $b+c$. First, if $b+c > 1$, then Elaine obtains maximum E by choosing $a = 1$. Note that with $a = 1$, both $a+b$ and $a+c$ are greater than 1, since both b and c must be nonzero for $b+c > 1$ to hold. Then, with $a+c > 1$, by the same argument we used for Elaine, it follows that George can maximize his expected value by choosing $b = 1$. Similarly, by choosing $c = 1$, Newman can maximize his expected value. So $a = b = c = 1$ is a Nash equilibrium. This equilibrium corresponds to having everyone always choose the Chinese restaurant.

In like fashion, it can be shown that if $b + c < 1$, then all three players will maximize their expected values with the choices $a = b = c = 0$. (See Exercise 10.27.) Therefore everyone always choosing the diner is a second Nash equilibrium.

The last possibility to consider is $b + c = 1$. In this case it does not matter what probability Elaine picks, as far as her expected value goes. But from the perspective of George and Newman, it does matter. If either $a + b$ or $a + c$ does not equal 1, then, as we saw earlier, the corresponding player will change his probability to 0 or 1, forcing the other players to change, and having everyone settle on one of the two equilibria already mentioned. The only case where this does not occur is if $a + b = 1$, $a + c = 1$, and $b + c = 1$. In this case we have $a = \frac{1}{2}$, $b = \frac{1}{2}$, and $c = \frac{1}{2}$. So this is a third Nash equilibrium; it corresponds to having each player choose each restaurant half of the time.

Before we proceed with Nash's Theorem, we review the components that lead up to it:

(i) In an n-person game each player has a choice of moves to make, and each move is made without knowing the moves of the others.

(ii) Associated to each player is a payoff array giving the payoffs the player receives for all of the possible outcomes associated with each player making a choice of move.

(iii) Each player can choose a mixed strategy, and the combined choices of mixed strategies determine the average payoff, or expected value, for each player.

(iv) Each player has a set of optimal mixed strategies for each choice of mixed strategies by the other players. Each optimal mixed strategy results in the maximum possible expected value for the player given the mixed strategies of the others.

(v) A Nash equilibrium is a choice of mixed strategies that yields, for each player, the maximum possible expected value relative to the mixed strategies of the other players.

THEOREM 10.19. ***Nash's Theorem.*** *There exists a Nash equilibrium for every n-person game.*

Proof. We present the proof for three-person games to keep the notation simple, but the same proof applies to n-person games. Given the three payoff arrays E_{ijk}, G_{ijk}, and N_{ijk}, we need to demonstrate that there is a set of mixed strategies $\mathbf{p}^*$, $\mathbf{q}^*$, and $\mathbf{r}^*$ that solves the game.

Let $m = n_E + n_G + n_N$. For a choice of mixed strategy vectors $\mathbf{p} = (p_1, \ldots, p_{n_E})$, $\mathbf{q} = (q_1, \ldots, q_{n_G})$, and $\mathbf{r} = (r_1, \ldots, r_{n_N})$, we define an m-vector by concatenating the components of these vectors, as follows:

$$\mathbf{w} = (\mathbf{p}, \mathbf{q}, \mathbf{r}) = (p_1, \ldots, p_{n_E}, q_1, \ldots, q_{n_G}, r_1, \ldots, r_{n_N}).$$

Each such vector $\mathbf{w}$ represents a combined choice of individual mixed strategies from each player. Its components must satisfy Inequalities 10.8 and Equations 10.9. It follows that the set of possible vectors $\mathbf{w}$ is a polyhedron X in $\mathbb{R}^m$.

We define a set-valued function on X by

$$F(\mathbf{p}, \mathbf{q}, \mathbf{r}) = \{(\mathbf{p}', \mathbf{q}', \mathbf{r}') \mid \mathbf{p}' \in P(\mathbf{q}, \mathbf{r}), \mathbf{q}' \in Q(\mathbf{p}, \mathbf{r}), \mathbf{r}' \in R(\mathbf{p}, \mathbf{q})\}.$$

In other words, the vector $\mathbf{w} = (\mathbf{p}, \mathbf{q}, \mathbf{r})$ is sent to the collection of vectors that have the property that their first n_E components are an optimal strategy for Elaine, given that George and Newman stick with strategies $\mathbf{q}$ and $\mathbf{r}$; their second n_G components are an optimal strategy for George, given that Elaine and Newman stick with strategies $\mathbf{p}$ and $\mathbf{r}$; and their last n_N components are an optimal strategy for Newman, given that Elaine and George stick with strategies $\mathbf{p}$ and $\mathbf{q}$.

If we show that there is a $\mathbf{w}^*$ such that $\mathbf{w}^* \in F(\mathbf{w}^*)$, then we will have proven the theorem, because such a vector $\mathbf{w}^* = (\mathbf{p}^*, \mathbf{q}^*, \mathbf{r}^*)$ is made up of three vectors, $\mathbf{p}^*, \mathbf{q}^*$, and $\mathbf{r}^*$ such that $\mathbf{p}^* \in P(\mathbf{q}^*, \mathbf{r}^*)$, $\mathbf{q}^* \in Q(\mathbf{p}^*, \mathbf{r}^*)$, and $\mathbf{r}^* \in R(\mathbf{p}^*, \mathbf{q}^*)$. Thus, we need to prove that there is a fixed point for the set-valued function $F : X \to_S X$.

We show that Kakutani's Fixed Point Theorem applies to F. It was previously observed that X is a polyhedron in $\mathbb{R}^m$. Since each of $P(\mathbf{q}, \mathbf{r})$, $Q(\mathbf{p}, \mathbf{r})$, and $R(\mathbf{p}, \mathbf{q})$ is convex, it must be that $F(\mathbf{w})$ is convex as well. (See Exercise 10.25.) Hence, we need only show that the graph G_F is a closed subset of $X \times X \subset \mathbb{R}^{2m}$. Let $(\mathbf{x}_0, \mathbf{y}_0)$ be a limit point of G_F. For each positive integer i, pick a point $(\mathbf{x}_i, \mathbf{y}_i)$ in the intersection of G_F with the ball of radius $1/i$ centered at $(\mathbf{x}_0, \mathbf{y}_0)$. We obtain a sequence of points $(\mathbf{x}_i, \mathbf{y}_i)$ in G_F converging to $(\mathbf{x}_0, \mathbf{y}_0)$. Let $\mathbf{x}_i = (\mathbf{p}_i, \mathbf{q}_i, \mathbf{r}_i)$, $\mathbf{y}_i = (\mathbf{s}_i, \mathbf{t}_i, \mathbf{u}_i)$, $\mathbf{x}_0 = (\mathbf{p}_0, \mathbf{q}_0, \mathbf{r}_0)$, and $\mathbf{y}_0 = (\mathbf{s}_0, \mathbf{t}_0, \mathbf{u}_0)$. We then have the following convergent sequences: $(\mathbf{p}_i)$ to $\mathbf{p}_0$, $(\mathbf{q}_i)$ to $\mathbf{q}_0$, $(\mathbf{r}_i)$ to $\mathbf{r}_0$, $(\mathbf{s}_i)$ to $\mathbf{s}_0$, $(\mathbf{t}_i)$ to $\mathbf{t}_0$, and $(\mathbf{u}_i)$ to $\mathbf{u}_0$.

Note that $\mathbf{y}_i \in F(\mathbf{x}_i)$ for all i. Therefore $\mathbf{s}_i \in P(\mathbf{q}_i, \mathbf{r}_i)$, $\mathbf{t}_i \in Q(\mathbf{p}_i, \mathbf{r}_i)$, and $\mathbf{u}_i \in R(\mathbf{p}_i, \mathbf{q}_i)$. Hence, for all $\mathbf{p}'$, $\mathbf{q}'$, and $\mathbf{r}'$, we have

$$E(\mathbf{s}_i, \mathbf{q}_i, \mathbf{r}_i) \geq E(\mathbf{p}', \mathbf{q}_i, \mathbf{r}_i),$$

$$G(\mathbf{p}_i, \mathbf{t}_i, \mathbf{r}_i) \geq G(\mathbf{p}_i, \mathbf{q}', \mathbf{r}_i), \text{ and}$$

$$N(\mathbf{p}_i, \mathbf{q}_i, \mathbf{u}_i) \geq N(\mathbf{p}_i, \mathbf{q}_i, \mathbf{r}').$$

Since E, G, and N are continuous functions, these inequalities hold as we take the limit as i approaches infinity. (See Exercise 10.24.) This implies that

$$E(\mathbf{s}_0, \mathbf{q}_0, \mathbf{r}_0) \geq E(\mathbf{p}', \mathbf{q}_0, \mathbf{r}_0),$$

$$G(\mathbf{p}_0, \mathbf{t}_0, \mathbf{r}_0) \geq G(\mathbf{p}_0, \mathbf{q}', \mathbf{r}_0), \text{ and}$$

$$N(\mathbf{p}_0, \mathbf{q}_0, \mathbf{u}_0) \geq N(\mathbf{p}_0, \mathbf{q}_0, \mathbf{r}').$$

Therefore $\mathbf{s}_0 \in P(\mathbf{q}_0, \mathbf{r}_0)$, $\mathbf{t}_0 \in Q(\mathbf{p}_0, \mathbf{r}_0)$, and $\mathbf{u}_0 \in R(\mathbf{p}_0, \mathbf{q}_0)$. Thus, $\mathbf{y}_0 \in F(\mathbf{x}_0)$, implying that $(\mathbf{x}_0, \mathbf{y}_0)$ is in the graph of F. It follows that the graph of F is closed. The Kakutani Fixed Point Theorem now applies, and therefore there exists a $\mathbf{w}^*$ such that $\mathbf{w}^* \in F(\mathbf{w}^*)$, as we wished to show. ■

Thus, for our three-player game, we are assured that a Nash equilibrium exists, and therefore it is possible for each player to choose a mixed strategy that results in maximum expected value relative to the others' choices of mixed strategy.

Even though the previous proof involves a three-person game, we need the general version of the Kakutani Fixed Point Theorem to establish the existence of the Nash equilibrium. This is because it is the total number of plays that the players can make, rather than the total number of players, that determines the dimension of the space in which we are working.

As we indicated earlier, these results carry over to n-person games for any positive integer n. The proof for the general case is essentially the same as the proof presented here, with the Kakutani Fixed Point Theorem establishing the existence of the Nash equilibrium.

EXAMPLE 10.12. Here we look at an example of the classic prisoner's dilemma two-person game. George and Newman have been charged with a crime, and each of them is interviewed separately about it. Since the prosecutor does not have sufficient evidence for a conviction, George and Newman are each offered a minimal sentence for telling on the other. So, George and Newman can each either remain silent or tell on the other.

If both George and Newman choose to be silent, each will receive a six-month sentence for a lesser crime. If they each tell on the other, then they each receive a reduced 24-month sentence because they assisted in the prosecution. If one tells while the other is silent, then the one who told receives a three-month sentence while the one who remained silent receives the full 72-month sentence.

For each player, the four possible outcomes and the resulting consequences are displayed in the payoff matrices in Figure 10.11. The payoffs $G_{i,j}$ and $N_{i,j}$ in the matrices represent the sentences (in months) received by George and Newman, respectively, in each circumstance. Here, of course, the desire of each is to minimize their sentence.

Newman \ George	SILENT	TELL
SILENT	6	3
TELL	72	24

G

Newman \ George	SILENT	TELL
SILENT	6	72
TELL	3	24

N

FIGURE 10.11: The prisoner's dilemma payoff matrices.

It is straightforward to show that the only Nash equilibrium in this game corresponds to having both players always tell on the other. (See Exercise 10.28.) Note, however, that the Nash equilibrium does not yield the best cooperative outcome of the game. If George and Newman agreed in advance to be silent, they would receive a better result than the one in the Nash equilibrium. However, since the game is not played cooperatively, for each player the threat of having the other player tell on them while they remain silent leads the player away from silence into the equilibrium.

Exercises for Section 10.4

10.24. Assume that $f : X \to \mathbb{R}$ is a continuous function, (x_i) converges to x_0, and (y_i) converges to y_0. Show that if $f(x_i) \geq f(y_i)$ for all i, then $f(x_0) \geq f(y_0)$.

10.25. Prove that if A is a convex subset of $\mathbb{R}^n$ and B is a convex subset of $\mathbb{R}^m$, then $A \times B$ is a convex subset of $\mathbb{R}^{n+m}$.

10.26. Given that the set of optimal mixed strategies $P(\mathbf{q}, \mathbf{r})$ satisfies Equation 10.11, prove that $P(\mathbf{q}, \mathbf{r})$ is a closed, bounded, and convex subset of $\mathbb{R}^{n_E}$.

10.27. Show that in Example 10.11, if we have a Nash equilibrium with $b + c < 1$, then $a = b = c = 0$.

10.28. Show that the only Nash equilibrium in the prisoner's dilemma game in Example 10.12 corresponds to having both players always tell on the other.

10.29. Elaine, George, and Newman are playing the same restaurant game as in Example 10.9. Suppose, however, that they have payoffs as follows:

- **(i)** If George and Newman agree, regardless of what Elaine calls out, they then go to the restaurant that George and Newman chose, and Elaine has to pay them three dollars each for coffee.
- **(ii)** If Elaine agrees with either George or Newman, but not both, they then go to the restaurant chosen by the two who agreed, and the third person pays 10 dollars to each of the others for dinner.
- **(iii)** If all three agree, they then go to the restaurant that they all chose, and George and Newman each pay Elaine five dollars toward her dinner.

Find all Nash equilibria for this game.

10.30. Elaine, George, and Newman can choose between the Happy Star Chinese Restaurant, Bobo's Burgers, or Pat's Pizza in the same game as in Example 10.9. Suppose that when two or three of them agree, the payout is the same as in the game in Example 10.9, and when all three disagree they all eat at home and there is no payout.

(a) If all three use a mixed strategy vector $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ to determine which restaurant to yell out, what are the resulting expected values?

(b) Find all Nash equilibria for this game.