1. Complex Sequences and Series

Let $\mathbb{C}$ denote the set $\{(x,y)\colon x,y \text{ real}\}$ of complex numbers and i denote the number $(0,1)$. For any real number t, identify t with $(t,0)$. For $z=(x,y)=x+iy$, let $\operatorname{Re} z = x$, $\operatorname{Im} z = y$, $\overline{z} = x-iy$ and $|z| = \sqrt{x^2+y^2}$. The distance between z and w is then given by $|z-w|$. For $z \neq 0$, $\arg z$ denotes the polar angle of (x,y) in radian (modulo 2π). Every nonzero complex number has a *polar representation* $z = r \operatorname{cis}\theta$, where $r=|z|$, $\theta = \arg z$ and $\operatorname{cis}\theta = \cos\theta + i\sin\theta$.

Properties. Let z, w be complex numbers and n be an integer. Then

(1) $z+\overline{z} = 2\operatorname{Re} z$, $z-\overline{z} = 2i\operatorname{Im} z$, $z\overline{z} = |z|^2$, $\dfrac{z}{\overline{z}} = \operatorname{cis}(2\arg z)$ for $z \neq 0$,

(2) $\overline{z+w} = \overline{z}+\overline{w}$, $\overline{z-w} = \overline{z}-\overline{w}$, $\overline{zw} = \overline{z}\,\overline{w}$, $\overline{z/w} = \overline{z}/\overline{w}$ for $w \neq 0$,

(3) $|z+w| \leq |z|+|w|$, $|z-w| \geq \big||z|-|w|\big|$, $|zw| = |z||w|$, $|z/w| = |z|/|w|$ for $w \neq 0$,

(4) for z, $w \neq 0$, $\arg(zw) = \arg z + \arg w$, $\arg(\dfrac{z}{w}) = \arg z - \arg w$, $\arg(z^n) = n\arg z$.

In analysis, reasoning involving limits are very common and important. We will begin with the concept of the limit of a sequence.

(For convenience, we will abbreviate "if and only if" by "iff" or " $\Longleftrightarrow$ " in the sequel.)

Definition. A sequence $\{z_1, z_2, z_3, \ldots\}$ (or in short, $\{z_n\}$) *converges* to $z \in \mathbb{C}$ (denoted by $z_n \to z$) iff for each $\varepsilon > 0$, there is N_ε such that $n \geq N_\varepsilon$ implies $|z_n - z| \leq \varepsilon$ (in short, $\lim\limits_{n\to\infty} |z_n - z| = 0$.) Otherwise, the sequense is said to *diverge*.

Examples. (1) $z_n = z^n \begin{cases} \text{converges to } 0 & \text{if } |z|<1, \\ \text{converges to } 1 & \text{if } z=1, \\ \text{diverges} & \text{otherwise.} \end{cases}$ (For $|z|<1$, $|z_n - 0| = |z|^n \to 0$ as $n\to\infty$. For $|z|>1, |z_n| = |z|^n \to \infty$. For $|z|=1$ and $z \neq 1$, we have $z = \operatorname{cis}\theta$ with $\theta \neq 2k\pi$, $z^n = \operatorname{cis}(n\theta)$ "spins" around the unit circle.)

(2) $\lim\limits_{n\to\infty} \dfrac{n}{n+z} = 1$, since $\left|\dfrac{n}{n+z} - 1\right| = \dfrac{|z|}{|n+z|} = \dfrac{\sqrt{x^2+y^2}}{\sqrt{(n+x)^2+y^2}} \to 0$ as $n \to \infty$.

Often the limit of a sequence is difficult or impossible to find. We now introduce a criterion that allows us to conclude a sequence is convergent without having to identify the limit explicitly.

Definition. A sequence $\{z_n\}$ is a *Cauchy* sequence iff for each $\varepsilon > 0$, there is N_ε such that $m,n \geq N_\varepsilon$ implies $|z_m - z_n| \leq \varepsilon$ (in short, $\lim\limits_{m,n\to\infty} |z_n - z_m| = 0$).

Lemma. A Cauchy sequence $\{a_n\}$ of real numbers must converge to some real number.

Proof. For $\varepsilon = 1$, there is N_1 such that $m,n \geq N_1$ implies $|a_m - a_n| \leq 1$ (i.e. $a_n - 1 \leq a_m \leq a_n + 1$). Taking $n = N_1$ and letting $p = \min\{a_1, \ldots, a_{N_1}\} - 1$ and $q = \max\{a_1, \ldots, a_{N_1}\} + 1$, we get $p \leq a_m \leq q$ for all m. Next, let $S = \{x \in \mathbb{R} : x \leq a_k \text{ for infinitely many } k\}$. Note $p \in S$, but $q \notin S$. Also, if $x' \leq x$ and $x \in S$, then $x' \in S$. This implies S is an interval of the form $(-\infty, a)$ or $(-\infty, a]$, where a is the right endpoint of S. We will show $a_n \to a$.

Given $\varepsilon > 0$, there is N_ε such that $m, n \ge N_\varepsilon$ implies $|a_m - a_n| \le \frac{\varepsilon}{2}$. Since $a + \frac{\varepsilon}{2} \notin S$, $a + \frac{\varepsilon}{2} \le a_k$ for only finitely many k. However, $a - \frac{\varepsilon}{2} \in S$, so $a - \frac{\varepsilon}{2} \le a_k$ for infinitely many k. Hence, we can find some $m \ge N_\varepsilon$ such that $a - \frac{\varepsilon}{2} \le a_m \le a + \frac{\varepsilon}{2}$ (i.e. $|a_m - a| \le \frac{\varepsilon}{2}$). Then for all $n \ge N_\varepsilon$, we have

$$|a_n - a| \le |a_n - a_m| + |a_m - a| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

QED.

Theorem. A sequence $\{z_n\}$ converges if and only if it is a Cauchy sequence.

Proof. Suppose $z_n \to z$, then for any $\varepsilon > 0$, the closed disk with z as center and radius $\frac{\varepsilon}{2}$ contains $z_K, z_{K+1}, z_{K+2}, \ldots$ for some K. Let $N_\varepsilon = K$, then $m, n \ge N_\varepsilon$ implies $|z_m - z_n| = |(z_m - z) + (z - z_n)| \le |z_m - z| + |z - z_n| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Conversely, suppose $\{z_n\}$ is a Cauchy sequence. Since $|\operatorname{Re} z_m - \operatorname{Re} z_n| \le |z_m - z_n|$, $\{\operatorname{Re} z_n\}$ is also a Cauchy sequence. Similarly, $\{\operatorname{Im} z_n\}$ is also a Cauchy sequence. By the lemma above, $\{\operatorname{Re} z_n\}$ converges to some real x and $\{\operatorname{Im} z_n\}$ converges to some real y. Then $\{z_n\}$ converges to $x + iy$. **QED.**

Definitions. A series $\sum_{k=1}^{\infty} z_k = z_1 + z_2 + z_3 + \ldots$ *converges* iff the sequence $z_1, z_1+z_2, z_1+z_2+z_3, \ldots$ converges (iff $S_n = z_1 + z_2 + \ldots + z_n$ is a Cauchy sequence). Otherwise, the series is said to *diverge*.

There are two simple tests for checking convergence of series, namely the term test and the absolute convergence test. The former provides a necessary condition for convergence and the latter provides a sufficient condition for convergence.

Term Test. If $\sum_{k=1}^{\infty} z_k$ converges, then $\lim_{n\to\infty} z_n = \lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = 0$.

Absolute Convergence Test. If $\sum_{k=1}^{\infty} |z_k|$ converges, then $\sum_{k=1}^{\infty} z_k$ converges (because $T_n = |z_1|+|z_2|+\ldots+|z_n|$ is a Cauchy sequence and for $m > n$, $|S_m - S_n| = |z_{n+1}+z_{n+2}+\ldots+z_m| \le |z_{n+1}|+|z_{n+2}|+\ldots+|z_m| = T_m - T_n$, forcing S_n to be a Cauchy sequence.)

Examples. (1) $\sum_{k=1}^{\infty} \frac{1}{k^2+i}$ converges because $\sum_{k=1}^{\infty} \left|\frac{1}{k^2+i}\right| = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k^4+1}} \left(\approx \sum_{k=1}^{\infty} \frac{1}{k^2}\right)$ converges.

(2) $\sum_{k=1}^{\infty} \frac{1}{k+i}$ diverges because $\operatorname{Re}\left(\sum_{k=1}^{\infty} \frac{1}{k+i}\right) = \operatorname{Re}\left(\sum_{k=1}^{\infty} \frac{k-i}{k^2+1}\right) = \sum_{k=1}^{\infty} \frac{k}{k^2+1} \left(\approx \sum_{k=1}^{\infty} \frac{1}{k}\right)$ diverges.

Exercises

1. For what complex values z will the following series converge

   (a) $\sum_{n=0}^{\infty} \left(\frac{z}{1+z}\right)^n$; (b) $\sum_{n=0}^{\infty} \frac{z^n}{1+z^{2n}}$?

2. When will equality occur in the triangle inequality? That is, under what conditions on w and z will $|w+z| = |w| + |z|$?

3. Establish the identity $\left|\sum_{k=1}^{n} \alpha_k \beta_k\right|^2 = \sum_{k=1}^{n} |\alpha_k|^2 \sum_{k=1}^{n} |\beta_k|^2 - \sum_{1 \le k < j \le n} |\alpha_k \overline{\beta_j} - \alpha_j \overline{\beta_k}|^2$ for the case $n = 2$. (This implies the Cauchy-Schwarz inequality

$$\left|\sum_{k=1}^{n} \alpha_k \beta_k\right| \le \sqrt{\sum_{k=1}^{n} |\alpha_k|^2} \sqrt{\sum_{k=1}^{n} |\beta_k|^2}.)$$

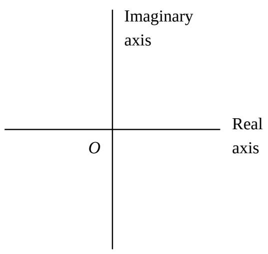
4. Suppose $0 < a_0 \le a_1 \le \cdots \le a_n$. Prove that the polynomial $P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$ has no root in the open unit disk $D = \{z : |z| < 1\}$. [*Hint*: Consider $(1 - z)P(z)$.]

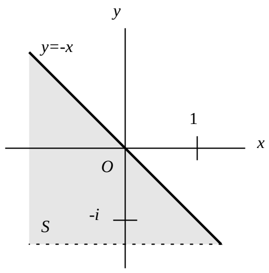
5. Prove that if $11z^{10} + 10iz^9 + 10iz - 11 = 0$, then $|z| = 1$. [*Hint*: Solve for z^9.] (This problem came from the 1989 William Lowell Putnam Mathematical Competition.)

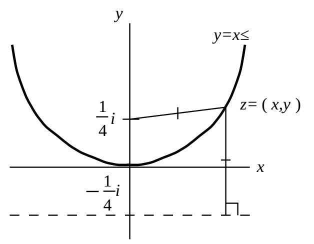
6. Let $P(z) = z^n + c_1 z^{n-1} + \cdots + c_n$ with $c_1, \cdots, c_n$ real. Suppose $|P(i)| < 1$. Prove that there is a root $x + iy$ of $P(z)$ satisfying $(x^2 + y^2 + 1)^2 - 4y^2 < 1$. (This problem came from the 1989 USA Mathematical Olympiad.)

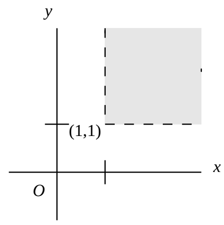
2. Set Descriptions and Terminologies

For $a, b \in \mathbb{C}$, the line L through a in the direction b consists of all $z = a + tb$ $(-\infty < t < \infty)$, so $L = \left\{z\colon \mathrm{Im}\left(\dfrac{z-a}{b}\right) = 0\right\}$. The circle C with center $c \in \mathbb{C}$ and radius r is given by $C = \{z\colon |z - c| = r\}$.









Examples. (1) $S = \{z\colon |z + i| \le |z - 1|\}$ consists of complex numbers that are closer to $-i$ than 1, i.e. the closed half plane below $y = -x$.

(2) To describe the parabola $y = x^2$ in complex variable, we set $x = \dfrac{z + \overline{z}}{2}$, $y = \dfrac{z - \overline{z}}{2i}$ and get $\dfrac{z - \overline{z}}{2i} = \left(\dfrac{z + \overline{z}}{2}\right)^2$. Alternatively, using the fact that the focus is at $(0, \dfrac{1}{4}) = \dfrac{1}{4}i$ and the directrix is $y = -\dfrac{1}{4}$, the equation of the parabola is $\left|z - \dfrac{i}{4}\right| = \left|\mathrm{Im}\, z + \dfrac{1}{4}\right|$.

(3) $\left\{z\colon 0 < \arg(z - 1 - i) < \dfrac{\pi}{2}\right\}$ is the open quarter-plane lying in the first quadrant with vertex at $(1, 1)$ and edges parallel to the x, y axes.

Next, we will introduce some notations and terminologies often used in the sequel.

The open disk $\{z\colon |z - z_0| < r\}$ is denoted by $B(z_0, r)$ and is referred to as the *r-neighborhood of* z_0. The closed disk $\{z\colon |z - z_0| \le r\}$ is denoted by $\overline{B(z_0, r)}$. The circle $\{z\colon |z - z_0| = r\}$ is denoted by $C(z_0, r)$.

The *boundary* of a set S is denoted by ∂S and consists of all points z such that every neighborhood of z contains a point in S and a point not in S. In particular, we note that a set S and its complement $\mathbb{C} \setminus S$ have the same boundary.

A set is *open* iff it does not contain any boundary point. A set is *closed* iff it contains all boundary points. A set containing some, but not all, boundary points is neither open nor closed. The *interior* of a set S is $S \setminus \partial S$ and the *closure* of S is $S \cup \partial S$.

A set is *bounded* iff it is contained inside a neighborhood of O. A set is *compact* iff it is closed and bounded. A set S is *disconnected* iff it is contained in the union of two disjoint, open sets A, B each of which contains at least one point of S, i.e. there are open sets A and B such that $S \subset A \cup B$, $A \cap B = \emptyset$, $S \cap A \neq \emptyset$ and $S \cap B \neq \emptyset$. A set is *connected* iff it is not disconnected. A set is a *region* or a *domain* iff it is open and connected.

Remarks. (1) S is open $\iff$ every point in S has a neighborhood containing in S.

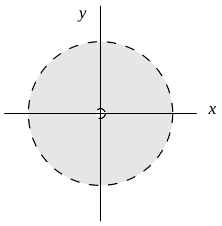
(2) S is closed $\iff$ $\mathbb{C} \setminus S$ is open $\iff$ $z_n \in S, z_n \to z$ implies $z \in S$.

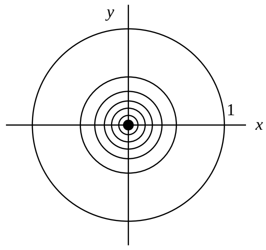
Theorem. Every region S is *polygonally connected*, i.e. any two points in S can be joined by a polygonal line lying totally in S. In fact, the polygonal line can be chosen to consist of horizontal or vertical segments.

Proof. Suppose $p \in S$. Let $A = \{z \in S\colon z \text{ can be connected to } p \text{ by a polygonal line lying totally in } S\}$ and $B = S \setminus A$. Then A must be open because for each $z \in A$, there is $B(z,r) \subset S$ and every point in $B(z,r)$ can be connected to z (hence to p) by such a polygonal line, i.e. $B(z,r) \subset A$. Similarly B is open because for each $w \in B$, there is $B(w,t) \subset S$ and every point in this $B(w,t)$ can be connected to w (hence not to p) by such a polygonal line, i.e. $B(w,t) \subset B$.

Since S is connected, $S = A \cup B$ and A is nonempty, B must be empty. Therefore $A = S$. **QED.**

The theorem requires S open! A circle is connected, but not polygonally connected. In advanced courses, it will be showed that a polygonally connected set is connected. So an open and polygonally connected set is a region.





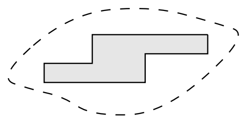
Examples. (1) Let $S = B(0,1) \setminus \{0\}$. Then $\partial S = C(0,1) \cup \{0\}$ (0 is a boundary point because every neighborhood of 0 contains points in S and 0, which is <u>not</u> in S). So S is open, bounded, connected, not closed and not compact.

(2) Let $S = \{z\colon \operatorname{Re} z \geq 0\}$, the right half plane. Then ∂S is the imaginary axis. So S is closed, connected, not open, not bounded and not compact.

(3) Let $S = C(0,1)$, the unit circle. Then $\partial S = S$. So S is closed, bounded, compact, connected and not open.

(4) Let $S = \left\{z\colon |z| = \dfrac{1}{n} \text{ for } n = 1,2,3,\ldots\right\}$. Observe that 0 is not in S, but it is a boundary point of S. So $\partial S = S \cup \{0\}$. S is not connected because for instance, $S \subset \underbrace{\left\{z\colon |z| > \frac{3}{4}\right\}}_{\text{open}} \cup \underbrace{\left\{z\colon |z| < \frac{3}{4}\right\}}_{\text{open}}$, where the two sets are disjoint.

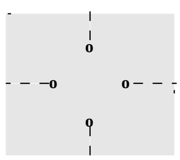
Definitions. (1) A *right-angled polygon* is a polygon (without self intersections) whose edges consist of horizontal and vertical segments.

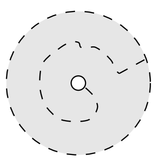


(2) A *simply connected region* is a region such that if a right-angled polygon is in the region, then the inside of the polygon is also in the region.

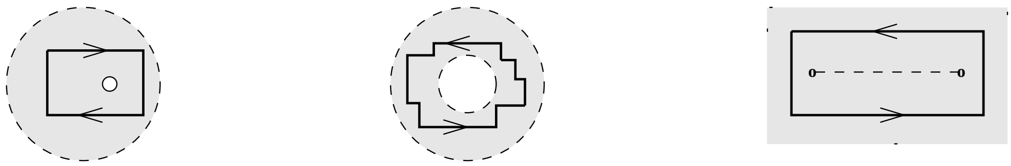
Remarks. Simply connected regions have no "holes". A region is simply connected iff every polygon can be "shrunk" to a point without leaving the region.

Examples. (1) The following are examples of simply connected regions: (i) open convex sets (in particular, disks, half-planes, infinite strips), (ii) complex plane with nonintersecting infinite slits removed, (iii) open unit disk with a spiral joining the center to the boundary removed.





(2) The following are examples of non-simply connected regions: (i) punctured disk, (ii) annulus, (iii) complement of a segment in $\mathbb{C}$.



Exercises

1. Describe the sets whose points satisfy the following relations

   (a) $\left|\dfrac{z-1}{z+1}\right| = 2;$ (b) $|z+1| - |z-1| < 2;$

   (c) $|z^2 - 1| = 1;$ (d) $\arg \dfrac{z-1}{z+i} = \dfrac{\pi}{3}.$

2. Given distinct complex numbers α, β, γ. Show that they are collinear if and only if $\operatorname{Im}(\alpha\overline{\beta} + \beta\overline{\gamma} + \gamma\overline{\alpha}) = 0$.

3. Show that the numbers α, β, γ are the vertices of an equilateral triangle if and only if $\alpha + \omega\beta + \omega^2\gamma = 0$, where $\omega \neq 1$ is a cube root of unity.

4. Prove that if z_1, z_2, z_3 are distinct complex numbers and $|z_1| = |z_2| = |z_3|$, then $\arg \dfrac{z_2 - z_3}{z_1 - z_3} = \dfrac{1}{2} \arg \dfrac{z_2}{z_1}$.

5. What is the boundary of the set $\{z : \operatorname{Re} z \text{ and } \operatorname{Im} z \text{ are rational}\}$?

3. Continuity and Uniform Convergence

Definitions. A function $f\colon D \subset \mathbb{C} \to \mathbb{C}$ has a *limit* c *at* $z \in D \cup \partial D$ (i.e. $\lim_{w\to z} f(w) = c$) iff for every sequence $z_n \in D \to z$, we have $f(z_n) \to c$ (iff for each $\varepsilon > 0$, there is $\delta > 0$ such that $0 < |w - z| < \delta$ implies $|f(w) - c| < \varepsilon$.) We say f is *continuous at* $z \in D$ iff $\lim_{w\to z} f(w) = f(z)$; f is *continuous on* D iff f is continuous at every point of D.

The ε-δ definition of limit at z can be rephrased as "for each ε-neighborhood $B(c, \varepsilon)$ of c, there is a δ-neighborhood $B(z, \delta)$ of z such that $w \in B(z, \delta) \setminus \{z\}$ implies $f(w) \in B(c, \varepsilon)$." Later we will discuss about "neighborhoods of ∞" and use this version to extend the notion of limit to the case at ∞.

Of course, as usual, the sum, difference, product, quotient (where the denomiator is nonzero) and composition of continuous functions are continuous.

Examples. (1) Polynomials $P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0$ are continuous everywhere on $\mathbb{C}$.

(2) Rational functions $\dfrac{P(z)}{Q(z)}$ (where P and Q are polynomials) are continuous at z, where $Q(z) \neq 0$.

(3) If $P(x, y) = \displaystyle\sum_{m,n=1}^{N} a_{mn} x^m y^n$, then $f(z) = P(z, \overline{z})$ is continuous everywhere on $\mathbb{C}$. In fact, compositions of continuous functions are continuous.

For functions defined by taking limits or summations, to check continuity we usually rely on uniform convergence.

Definition. A sequence of functions $S_n\colon D \subset \mathbb{C} \to \mathbb{C}$ *converges uniformly* to a function $S\colon D \to \mathbb{C}$ iff for every $\varepsilon > 0$, there is N_ε such that $m \geq N_\varepsilon$ implies $|S(z) - S_n(z)| < \varepsilon$ for all $z \in D$. (For series $\displaystyle\sum_{k=1}^{\infty} f_k(z)$, consider $S_n = \displaystyle\sum_{k=1}^{n} f_k(z)$.)

Weierstrass M-test. If for each k, there is a number M_k such that $|f_k(z)| \leq M_k$ for all z in D and $\displaystyle\sum_{k=1}^{\infty} M_k$ converges, then $\displaystyle\sum_{k=1}^{\infty} f_k(z)$ converges (absolutely and) uniformly on D.

Proof. (Absolute convergence follows by comparing $\displaystyle\sum_{k=1}^{\infty} |f_k(z)|$ with $\displaystyle\sum_{k=1}^{\infty} M_k$.) For uniform convergence, let $\varepsilon > 0$. Since $\displaystyle\sum_{k=1}^{\infty} M_k$ converges, there is N_ε such that $n \geq N_\varepsilon$ implies $\displaystyle\sum_{k=n+1}^{\infty} M_k < \varepsilon$. Then for every z in D,

$$\left|\sum_{k=1}^{\infty} f_k(z) - \sum_{k=1}^{n} f_k(z)\right| = \left|\sum_{k=n+1}^{\infty} f_k(z)\right| \leq \sum_{k=n+1}^{\infty} |f_k(z)| \leq \sum_{k=n+1}^{\infty} M_k < \varepsilon.$$

QED.

Properties of Uniform Convergence.

(1) If S_k's are continuous on D and S_k converges uniformly to S on D, then S is continuous on D and $\lim_{w\to z}\lim_{k\to\infty} S_k(w) = \lim_{w\to z} S(w) = S(z) = \lim_{k\to\infty} S_k(z) = \lim_{k\to\infty}\lim_{w\to z} S_k(w)$. (For series, if f_k's are continuous on D and $\sum_{k=1}^{\infty} f_k$ converges uniformly to f on D, then f is continuous on D and $\lim_{w\to z}\sum_{k=1}^{\infty} f_k(w) = \lim_{w\to z} f(w) = f(z) = \sum_{k=1}^{\infty} f_k(z) = \sum_{k=1}^{\infty}\lim_{w\to z} f_k(w)$.)

(2) If S_k's are integrable and converges uniformly, then $\int \lim_{k\to\infty} S_k = \lim_{k\to\infty}\int S_k$. (For series, if f_k's are integrable and $\sum_{k=1}^{\infty} f_k$ converges uniformly, then $\int \sum_{k=1}^{\infty} f_k = \sum_{k=1}^{\infty}\int f_k$.)

(**Proof.** We will prove (1) and postpone (2) until complex integration is defined. (In (1), the parenthetical statement follows from the first statement by taking $S_n = \sum_{k=1}^{n} f_k$.) Given $\varepsilon > 0$. Since S_k converges uniformly to S, there is $N = N_{\varepsilon/3}$ such that $k \ge N_{\varepsilon/3}$ implies $|S(z) - S_k(z)| < \frac{\varepsilon}{3}$ for all $z \in D$. For a fixed $w \in D$, since S_N is continuous at w, there is $\delta > 0$ such that $|w - z| < \delta \Rightarrow |S_N(w) - S_N(z)| < \frac{\varepsilon}{3}$. Then

$$\begin{aligned} |S(w) - S(z)| &= |S(w) - S_N(w) + S_N(w) - S_N(z) + S_N(z) - S(z)| \\ &\le |S(w) - S_N(w)| + |S_N(w) - S_N(z)| + |S_N(z) - S(z)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

So S is continuous at any arbitrary $w \in D$.)

Example. Consider $f(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!}$. Suppose $z \in B(0, R)$. The series $\sum_{k=0}^{\infty} \frac{R^k}{k!}$ converges by the ratio test because $\lim_{k\to\infty} \frac{R^{k+1}}{(k+1)!} \bigg/ \frac{R^k}{k!} = R \lim_{k\to\infty} \frac{1}{k+1} = 0$. Since $\left|\frac{z^k}{k!}\right| \le \frac{R^k}{k!} = M_k$, by Weierstrass M-test, $\sum_{k=0}^{\infty} \frac{z^k}{k!}$ converges uniformly on $B(0, R)$. Since $\frac{z^k}{k!}$ are continuous on $B(0, R)$, f is continuous on $B(0, R)$. Finally, since R is arbitrary, f is continuous everywhere on the complex plane.

Exercises

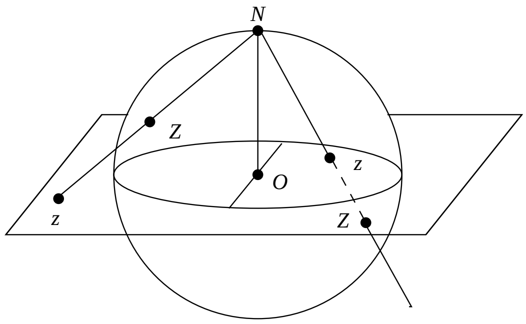
1. Show that $f(z) = \sum_{k=1}^{\infty} \frac{1}{k^2 + z}$ is continuous on the right half plane $H = \{z : \operatorname{Re} z > 0\}$.

2. Show that $f(z) = \sum_{k=0}^{\infty} k z^k$ is continuous on $D = \{z : |z| < 1\}$.

4. Stereographic Projection

In many situations, it is best to treat infinity as a point. Stereographic projection explains how this can be done.

Let $S^2 = \{(\xi,\eta,\zeta)\colon \xi^2+\eta^2+\zeta^2=1\}$ be the unit sphere and $N=(0,0,1)$ be the north pole. There is a one-to-one correspondence between $S^2 \setminus N$ and the complex plane $\mathbb{C}$ given by $Z=(\xi,\eta,\zeta) \leftrightarrow z = x+iy = (x,y,0)$ as in the figure. The equation of the line through N, z implies $\dfrac{x}{\xi} = \dfrac{y}{\eta} = \dfrac{1}{1-\zeta}$. It follows that $x = \dfrac{\xi}{1-\zeta}$, $y = \dfrac{\eta}{1-\zeta}$. Conversely, $1 = \xi^2+\eta^2+\zeta^2 = (x^2+y^2)(1-\zeta)^2+\zeta^2 \Rightarrow x^2+y^2 = \dfrac{1+\zeta}{1-\zeta} = \dfrac{2}{1-\zeta} - 1$. So $\xi = x(1-\zeta) = \dfrac{2x}{x^2+y^2+1}$, $\eta = y(1-\zeta) = \dfrac{2y}{x^2+y^2+1}$, $\zeta = \dfrac{x^2+y^2-1}{x^2+y^2+1}$.



We see that if $z = x+iy \to \infty$, then $x^2+y^2 = |z|^2 \to \infty$, so $(\xi,\eta,\zeta) \to (0,0,1)$. Thus $N=(0,0,1) \leftrightarrow \infty$.

The set $\mathbb{C} \cup \{\infty\}$ is usually referred to as the *extended complex plane*. Because of the correspondence, some like to use S^2 to denote the same set. If S^2 is used, then the extended complex plane is sometimes also referred to as the *Riemann sphere*.

Now to analyze how the neighborhoods of a point are transformed under stereographic projection, we study the correspondence of the boundary circles. A circle on S^2 is the intersection of S^2 with a plane $A\xi + B\eta + C\zeta = D$. This corresponds to $(C-D)(x^2+y^2) + 2Ax + 2By = C+D$, which is a $\begin{cases} \text{circle on } \mathbb{C} \text{ if } C \neq D \\ \text{line on } \mathbb{C} \text{ if } C = D \end{cases}$. Conversely, a circle on $\mathbb{C}$ is given by $x^2+y^2+ax+by=c$, which corresponds to $1+\zeta+a\xi+b\eta = c(1-\zeta)$ on S^2, i.e. a circle on S^2.

In particular, the circle on S^2 coming from the plane $\zeta = \zeta_0 < 1$ corresponds to $x^2+y^2 = \dfrac{1+\zeta_0}{1-\zeta_0}$, i.e. the boundary of a neighborhood of N corresponds to a circle centered at the origin. Then neighborhoods of $N \leftrightarrow \infty$ correspond to sets of the form $\{z\colon |z| > r\}$. By abuse of notation, we will let $B(\infty, r) = \{z\colon |z|>r\}$. The remark following the definition of $\lim\limits_{w\to z} f(w)$ suggests how limits involving ∞ can be defined.

Definitions. (1) $z_n \to \infty$ iff $|z_n| \to \infty$ (iff for any $\varepsilon > 0$, there is N_ε such that $n \geq N_\varepsilon \Rightarrow |z_n| > \varepsilon$.)

(2) $\lim\limits_{z\to\infty} f(z) = c \in \mathbb{C}$ iff for any $\varepsilon > 0$, there is $\delta > 0$ such that $|z| > \delta \Rightarrow |f(z)-c| < \varepsilon$ (iff $\lim\limits_{z\to\infty} f(z) = \lim\limits_{w\to 0} f(\dfrac{1}{w}) = c$.)

(3) $\lim\limits_{z\to c\in\mathbb{C}} f(z) = \infty$ iff for any $\varepsilon > 0$, there is $\delta > 0$ such that $0 < |z-c| < \delta \Rightarrow |f(z)| > \varepsilon$ (iff $\lim\limits_{z\to c} \dfrac{1}{f(z)} = 0$.)

(4) $\lim\limits_{z\to\infty} f(z) = \infty$ iff for any $\varepsilon > 0$, there is $\delta > 0$ such that $|z| > \delta \Rightarrow |f(z)| > \varepsilon$.

(5) $f(z)$ is *continuous at* ∞ iff $g(z) = f(\frac{1}{z})$ is continuous at 0.

Definitions. (1) Let f be defined in a neighborhood of $z \in \mathbb{C}$. Then f is *differentiable at* z iff

$$f'(z) = \lim_{h\to 0, h\in\mathbb{C}} \frac{f(z+h) - f(z)}{h}$$

exists; f is *differentiable at* ∞ iff $f(\frac{1}{z})$ is differentiable at 0.

(2) f is *holomorphic* or *analytic* (or *regular*) on a set S iff f is differentiable at every point of some open set containing S.

(3) f is *univalent* (or *schlicht*) on an open set iff f is differentiable and one-to-one there.

(4) f is *entire* (or *integral*) iff f is differentiable on the complex plane.

Of course, as usual, the sum, difference, product, quotient and chain rules are valid in (complex) differentiation. Some common functions are differentiable, *e.g.* polynomials are entire functions and $(z^n)' = nz^{n-1}$ for integer n. However, some are not, *e.g.* the conjugate function $\overline{z}$ is not differentiable anywhere because $\displaystyle\lim_{h\to 0, h\in\mathbb{C}} \frac{\overline{z+h} - \overline{z}}{h} = \lim_{h\to 0, h\in\mathbb{C}} \frac{\overline{h}}{h}$ doesn't exist, as can be seen by the following computations :

$$\lim_{h\to 0, h\in\mathbb{R}} \frac{\overline{h}}{h} = \lim_{h\to 0, h\in\mathbb{R}} \frac{h}{h} = 1 \text{ and } \lim_{h=it\to 0, t\in\mathbb{R}} \frac{\overline{h}}{h} = \lim_{t\to 0} \frac{-it}{it} = -1.$$

Exercises

1. The *chordal metric* $d(w, z)$ of two complex numbers w and z is the distance in $\mathbb{R}^3$ between the points corresponded to w and z under stereographic projection.
   (a) Express $d(w, z)$ in terms of w and z only.
   (b) Express $d(w, \infty) = \lim_{z\to\infty} d(w, z)$ in terms of w only.
   (c) Show that $z_n \to z \in \mathbb{C}\cup\{\infty\}$ if and only if $d(z_n, z) \to 0$.

2. Prove that $f(z) = |z|$ is not differentiable anywhere, but $g(z) = |z|^2$ is differentiable at 0 only.

5. Power Series

Definition. A *power series centered at* c is a function of the form $f(z) = \sum_{n=0}^{\infty} a_n(z-c)^n$. (Note that $f(c) = a_0$, so the series converges for at least the point $z = c$.)

To understand what the domains of power series (as functions) look like, we introduce the concept of the upper limit of a sequence and recall the root test.

Definition. Let $\{x_n\}$ be a sequence of real numbers. The *upper limit* or *limit superior* of $\{x_n\}$ (denoted by $\overline{\lim\limits_{n\to\infty}}\, x_n$ or $\limsup\limits_{n\to\infty} x_n$ is

(a) the number L if L has the properties that for each $\varepsilon > 0$,

  (i) $x_n < L + \varepsilon$ for all except finitely many n and

  (ii) $x_n > L - \varepsilon$ for infinitely many n.

(b) $+\infty$ if for each real number r, there is $x_n > r$.

(c) $-\infty$ if for each real number r, only a finite number of the x_n's are greater than r.

Essentially, by taking $\varepsilon \to 0$, items (i) and (ii) imply L is the limit of some subsequence. Furthermore, item (i) implies no larger number is also a limit of a subsequence. So the upper limit of $\{x_n\}$ is the largest limit of any subsequence of $\{x_n\}$. Now if $\lim\limits_{n\to\infty} x_n$ exists, then all subsequences converge to the same limit. In that case, $\overline{\lim\limits_{n\to\infty}}\, x_n = \lim\limits_{n\to\infty} x_n$.

Examples. (1) $\overline{\lim\limits_{n\to\infty}}\, \dfrac{1}{n} = 0$.

(2) $\overline{\lim\limits_{n\to\infty}}\,[n + (-1)^n n] = \overline{\lim\limits_{n\to\infty}}\,\{0, 4, 0, 8, 0, 12, \ldots\} = \lim\{4, 8, 12, \ldots\} = +\infty$.

(3) $\overline{\lim\limits_{n\to\infty}}\, \sin\dfrac{n\pi}{2} = \overline{\lim\limits_{n\to\infty}}\,\{1, 0, -1, 0, 1, 0, \ldots\} = \lim\{1, 1, 1, \ldots\} = 1$.

Root Test. Given a series $\sum a_n$, let $\rho = \overline{\lim\limits_{n\to\infty}}\, \sqrt[n]{|a_n|}$, then $\sum a_n \begin{cases} \text{converges absolutely} & \text{if } \rho < 1, \\ \text{diverges} & \text{if } \rho > 1, \\ \text{is inconclusive} & \text{if } \rho = 1. \end{cases}$

Proof. (Case $\rho < 1$) Take x such that $\rho < x < 1$. Let $\varepsilon = x - \rho > 0$. Then by property (i), all but finitely many n satisfy $\sqrt[n]{|a_n|} < \rho + \varepsilon = x$, then $|a_n| < x^n$. Since $x < 1$, $\sum x^n$ converges, which implies $\sum |a_n|$ converges.

(Case $\rho > 1$) Take x such that $1 < x < \rho$. Let $\varepsilon = \rho - x > 0$. Then by property (ii) $\sqrt[n]{|a_n|} > \rho - \varepsilon = x > 1$ for infinitely many n, then $\lim\limits_{n\to\infty} |a_n| \neq 0$, so $\sum a_n$ diverges.

(Case $\rho = 1$) Consider $\sum_{n=1}^{\infty} \dfrac{1}{n}$ and $\sum_{n=1}^{\infty} \dfrac{1}{n^2}$. For both cases, $\rho = 1$, but the first series diverges and the second series converges. **QED.**

Theorem 1. The power series $f(z) = \sum_{n=0}^{\infty} a_n(z-c)^n$ converges absolutely for $|z - c| < R$ and diverges for $|z - c| > R$, where $R = \dfrac{1}{\overline{\lim\limits_{n\to\infty}}\, \sqrt[n]{|a_n|}}$. (In case $R = 0$, the series converges at $z = c$ only.) R is called the *radius of convergence* and the disk $|z-c|<R$ is called the *disk* or *domain of convergence*. Furthermore, $\displaystyle\sum_{n=0}^{\infty} a_n(z-c)^n$ converges uniformly on any smaller disk $|z-c| \le R' < R$.

Proof. The first statement follows from the root test. The last statement follows by Weierstrass M-test. Since $|a_n(z-c)^n| \le |a_n|\, R'^n$ for all $|z-c| \le R'$, so we set $M_n = |a_n|\, R'^n$ and use the absolute convergence of the power series at $z = R' + c$. **QED.**

In a course on real analysis, it is usually shown that if $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ exists, then $\displaystyle\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$. This can be used in computing the radius of convergence.

The convergence on the boundary circle $|z-c| = R$ can be arbitrary as the following examples show.

Examples. (1) For $\displaystyle\sum_{n=1}^{\infty}\frac{z^n}{n^2}$, $R=1$. If $|z|=1$, then $\displaystyle\sum_{n=1}^{\infty}\left|\frac{z^n}{n^2}\right| = \sum_{n=1}^{\infty}\frac{1}{n^2}$ converges by p-test.

(2) For $\displaystyle\sum_{n=0}^{\infty} z^n$, $R=1$. If $|z|=1$, then $\displaystyle\sum_{n=0}^{\infty} z^n$ diverges because $\displaystyle\lim_{n\to\infty} z^n \ne 0$.

(3) For $\displaystyle\sum_{n=1}^{\infty}\frac{z^n}{n}$, $R=1$. If $z=1$, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges. If $z=-1$, $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ converges by alternating series test.

Later we will show that every holomorphic function can be locally represented by power series (i.e. f holomorphic at $c \Rightarrow f(z) = \displaystyle\sum_{n=0}^{\infty} a_n(z-c)^n$ in a neighborhood of c). Therefore, the local properties of holomorphic functions can be understood by studying power series. First, we will see that power series can be differentiated (term-by-term) in their domains of convergence.

Theorem 2. If $f(z) = \displaystyle\sum_{n=0}^{\infty} a_n(z-c)^n$ converges on $|z-c|<R$, then $f'(z) = \displaystyle\sum_{n=1}^{\infty} n a_n (z-c)^{n-1}$ converges on $|z-c|<R$, too. Consequently, power series are infinitely differentiable on their disk of convergence.

Proof. The proof is divided into two steps. The first step is to observe that

$$\overline{\lim_{n\to\infty}}\sqrt[n-1]{n\,|a_n|} = \overline{\lim_{n\to\infty}}\left((n\,|a_n|)^{\frac{1}{n}}\right)^{\frac{n}{n-1}} = \overline{\lim_{n\to\infty}}\left(n^{\frac{1}{n}}\,|a_n|^{\frac{1}{n}}\right) = \overline{\lim_{n\to\infty}}\sqrt[n]{|a_n|},$$

so the power series have the same disk of convergence. The second step is to obtain an inequality of the form $\displaystyle\left|\frac{f(z+h)-f(z)}{h} - \sum_{n=1}^{\infty} n a_n (z-c)^{n-1}\right| \le A|h|$.

Fix z such that $|z-c|<R$. Suffices to consider small h (say $|h| < \delta < R - |z-c|$.) Then

$$\frac{f(z+h)-f(z)}{h} - \sum_{n=1}^{\infty} n a_n (z-c)^{n-1} = \sum_{n=2}^{\infty} a_n \underbrace{\left(\sum_{k=2}^{n}\binom{n}{k} h^{k-1}(z-c)^{n-k}\right)}_{b_n}.$$

Observe that for $k \ge 2$,

$$\binom{n}{k} = \binom{n}{k-2}\frac{(n-k+2)}{k-1}\frac{(n-k+1)}{k} \le \binom{n}{k-2} n(n-1).$$

So, for $z \neq c$,

$$\begin{aligned}|b_n| &\le \sum_{k=2}^{\infty} \binom{n}{k} |h|^{k-1}|z-c|^{n-k} \le \frac{n(n-1)|h|}{|z-c|^2} \sum_{k=2}^{n} \binom{n}{k-2} |h|^{k-2}|z-c|^{n-k+2} \\ &\le \frac{n(n-1)|h|}{|z-c|^2} \sum_{j=0}^{n} \binom{n}{j} |h|^j|z-c|^{n-j} = \frac{n(n-1)|h|}{|z-c|^2}\,(|z-c|+|h|)^n.\end{aligned}$$

Therefore, as $h \to 0$,

$$\begin{aligned}\left|\frac{f(z+h)-f(z)}{h} - \sum_{n=1}^{\infty} n a_n (z-c)^{n-1}\right| &= \left|\sum_{n=2}^{\infty} a_n b_n\right| \le \sum_{n=2}^{\infty} |a_n||b_n| \\ &\le |h| \underbrace{\frac{1}{|z-c|^2} \sum_{n=2}^{\infty} n(n-1)|a_n|(|z-c|+\delta)^n}_{A_z} \to 0.\end{aligned}$$

(Note that the series in A_z converges because the first step above implies $f''(w) = \sum_{n=2}^{\infty} n(n-1)a_n(w-c)^{n-2}$ converges absolutely on $|w-c| < R$, in particular at $w = c + |z-c| + \delta$ by the condition on δ.) **QED.**

Taylor's Theorem for Power Series. If $f(z) = \sum_{n=0}^{\infty} a_n(z-c)^n$ has a nonzero radius of convergence, then the coefficient $a_n = \frac{f^{(n)}(c)}{n!}$, i.e. $f(z)$ equals its Taylor series $\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(z-c)^n$.

Proof. Apply theorem 2 repeatedly (i.e. inductively) and evaluate at $z = c$. **QED.**

Identity Theorem for Power Series. If $f(z) = \sum_{n=0}^{\infty} a_n(z-c)^n = 0$ for $z = z_k \neq c$, $(k = 1, 2, \ldots)$ and $\{z_k\}$ converges to c, then $a_n = 0$ for all n.

Proof. We are given a power series $f(z) = a_0 + a_1(z-c) + a_2(z-c)^2 + \cdots$. Now $a_0 = f(c) = \lim_{k\to\infty} f(z_k) = 0$, $a_1 = \lim_{z\to c} \frac{f(z)}{z-c} = \lim_{k\to\infty} \frac{f(z_k)}{z_k - c} = 0$, $\ldots$, $a_n = \lim_{z\to c} \frac{f(z)}{(z-c)^n} = \lim_{k\to\infty} \frac{f(z_k)}{(z_k-c)^n} = 0, \ldots$. **QED.**

Uniqueness Theorem for Power Series. If $\sum_{n=0}^{\infty} a_n(z-c)^n = \sum_{n=0}^{\infty} b_n(z-c)^n$ for $z = z_k \neq c$, $(k = 1, 2, \ldots)$ and $\{z_k\}$ converges to c, then $a_n = b_n$ for all n.

Proof. Consider $f(z) = \sum_{n=0}^{\infty}(a_n - b_n)(z-c)^n$ and apply the identity theorem for power series. **QED.**

Exercises

1. Find $\overline{\lim_{n\to\infty}}\, x_n$, where

   (a) $x_n = n \sin\left(\frac{1}{n}\right)$;

   (b) $x_n = \cos \frac{n\pi}{4}$;

   (c) $x_n = 1 + (-1)^n \frac{2n}{n+1}$.

2. Find the radius of convergence for the following power series:

(a) $\sum_{n=1}^{\infty} z^{n!}$;

(b) $\sum_{n=0}^{\infty} (n+2^n)z^n$.

3. Give an example of two power series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ with radii of convergence R_1 and R_2, respectively, such that the power series $\sum_{n=0}^{\infty} (a_n + b_n) z^n$ has a radius of convergence greater than $R_1 + R_2$.

4. Show that if $|z| < 1$, then $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$. Find a power series for $\frac{1}{(1-z)^2}$ with centered at 0. What is the radius of convergence for this series?

5. Explain why there is no power series $f(z) = \sum_{n=0}^{\infty} c_n z^n$ such that $f\left(\frac{1}{k}\right) = 1$ for $k = 2, 3, 4, \ldots$ and $f'(0) > 0$.

6. Does there exist a power series $f(z) = \sum_{n=0}^{\infty} c_n z^n$ such that $f\left(\frac{1}{k}\right) = \frac{1}{k^2}$ and $f\left(-\frac{1}{k}\right) = \frac{1}{k^3}$ for $k = 1, 2, 3, \ldots$?

7. If $f(z) = \sum_{n=0}^{\infty} c_n z^n$ satisfies $f\left(\frac{1}{k}\right) = \frac{k^2}{k^2+1}$, $k = 1, 2, 3, \ldots$, compute the values of the derivatives $f^{(n)}(0)$, $n = 1, 2, 3 \ldots$.

6. Cauchy–Riemann Equations

Recall f is differentiable at z_0 means $\displaystyle\lim_{h\to 0, h\in\mathbb{C}} \frac{f(z_0+h)-f(z_0)}{h} = f'(z_0)$ exists and f is holomorphic at z_0 means f is differentiable in a neighborhood of z_0 in which case $f'(w)$ exists in some disk $B(z_0, r)$ about z_0.

Question. Given a function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ with two real variables, say $f(x,y) = (u(x,y), v(x,y)) = u(x,y) + iv(x,y)$, how can we tell if it is a differentiable function of $z = x+iy$?

Example. If $f(z) = z^2, z = x+iy$, then $f(x,y) = (x+iy)^2 = (x^2-y^2) + i(2xy) = (x^2-y^2, 2xy)$.

Theorem. If $f(z) = f(x,y) = (\overbrace{u(x,y)}^{\text{real}}, \overbrace{v(x,y)}^{\text{real}}) = u(x,y) + iv(x,y) = u(z) + iv(z)$ is differentiable at z_0, then $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ exist at z_0, $\dfrac{\partial u}{\partial x}(z_0) = \dfrac{\partial v}{\partial y}(z_0)$ and $\dfrac{\partial u}{\partial y}(z_0) = -\dfrac{\partial v}{\partial x}(z_0)$.

[**Notation.** If we define $f_x = \dfrac{\partial f}{\partial x} = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$ and $f_y = \dfrac{\partial f}{\partial y} = \dfrac{\partial u}{\partial y} + i\dfrac{\partial v}{\partial y}$, then the above *Cauchy-Riemann equations* $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ are equivalent to $f_y = if_x$.]

Proof. Taking $h \to 0$ along the real axis, we have

$$f'(z_0) = \lim_{h\to 0} \frac{f(x_0+iy_0+h) - f(x_0+iy_0)}{h} = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h} = f_x(x_0, y_0).$$

Taking $h \to 0$ along the imaginary axis, say $h = it$, we have

$$f'(z_0) = \lim_{t\to 0} \frac{f(x_0+iy_0+it) - f(x_0+iy_0)}{it} = \frac{1}{i}\lim_{t\to 0} \frac{f(x_0, y_0+t) - f(x_0, y_0)}{t} = \frac{1}{i} f_y(x_0, y_0).$$

QED.

Theorem. If f_x and f_y exist in a neighborhood of z_0, are continuous at z_0 and $f_y(z_0) = if_x(z_0)$, then f is differentiable at z_0.

Proof. Write $f(z_0) = u(x_0, y_0) + iv(x_0, y_0), h = s+it$. Then

$$\begin{aligned}
\frac{u(z_0+h) - u(z_0)}{h} &= \frac{u(x_0+s, y_0+t) - u(x_0, y_0)}{s+it} \\
&= \frac{u(x_0+s, y_0+t) - u(x_0+s, y_0)}{s+it} + \frac{u(x_0+s, y_0) - u(x_0, y_0)}{s+it} \\
&= \frac{t}{s+it}\left[\frac{\partial u}{\partial y}(x_0+s, y_0+\alpha t)\right] + \frac{s}{s+it}\left[\frac{\partial u}{\partial x}(x_0+\beta t, y_0)\right],\ 0 < \alpha, \beta < 1
\end{aligned}$$

(by the mean value theorem).
Similarly,

$$\frac{v(z_0+h) - v(z_0)}{h} = \frac{t}{s+it}\left[\frac{\partial v}{\partial y}(x_0+s, y_0+\gamma t)\right] + \frac{s}{s+it}\left[\frac{\partial v}{\partial x}(x_0+\delta t, y_0)\right], \quad 0 < \gamma, \delta < 1.$$

Now, $f_y(z_0) = if_x(z_0)$ implies $f_x(z_0) = \dfrac{t}{s+it} f_y(z_0) + \dfrac{s}{s+it} f_x(z_0)$. Using these equations and the fact $\left|\dfrac{t}{s+it}\right|, \left|\dfrac{s}{s+it}\right| \le 1$, it follows that as $h = s+it \to 0$,

$$\begin{aligned}\left|\frac{f(z_0+h)-f(z_0)}{h} - f_x(z_0)\right| &= \left|\frac{t}{s+it}\left[\frac{\partial u}{\partial y}(x_0+s, y_0+\alpha t) + i\frac{\partial v}{\partial y}(x_0+s, y_0+\gamma t) - f_y(z_0)\right]\right. \\ &\quad \left. + \frac{s}{s+it}\left[\frac{\partial u}{\partial x}(x_0+\beta t, y_0) + i\frac{\partial v}{\partial x}(x_0+\delta t, y_0) - f_x(z_0)\right]\right|\end{aligned}$$

$$\le \left|\frac{\partial u}{\partial y}(x_0+s, y_0+\alpha t) + i\frac{\partial v}{\partial y}(x_0+s, y_0+\gamma t) - f_y(z_0)\right| + \left|\frac{\partial u}{\partial x}(x_0+\beta t, y_0) + i\frac{\partial v}{\partial x}(x_0+\delta t, y_0) - f_x(z_0)\right| \to 0.$$

So f is differentiable at z_0 and $f'(z_0) = f_x(z_0)$. **QED.**

Examples. (1) If a holomorphic function $G(z) = u(z) + iv(z) = u(x,y) + iv(x,y)$ has the real part $u(x,y) = x^2 - y^2$, what can $v(z)$ be?

Solution.

$$\left.\begin{aligned}\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 2y \Rightarrow v(x,y) = 2xy + C_1(y) \\ \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x \Rightarrow v(x,y) = 2xy + C_2(x)\end{aligned}\right\} \Rightarrow v(x,y) = 2xy + \text{constant}$$

(2) If $f = u + iv$ is holomorphic on a region D and $u \equiv$ constant, show that $f \equiv$ constant.

Solution.

$$\left.\begin{aligned}\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 0 \Rightarrow v(x,y) \equiv C_1(y) \\ \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 0 \Rightarrow v(x,y) = C_2(x)\end{aligned}\right\} \Rightarrow v(x,y) \equiv \text{constant} \Rightarrow \text{f} \equiv \text{constant}$$

(3) If f is holomorphic on a region D and $|f(z)| \equiv$ constant, then $f \equiv$ constant.

Solution. If $|f| \equiv 0$, then $f \equiv 0$. Otherwise $u^2 + v^2 \equiv$ constant, so taking partial derivatives, we get $2u\dfrac{\partial u}{\partial x} + 2v\dfrac{\partial v}{\partial x} \equiv 0$, $2u\dfrac{\partial u}{\partial y} + 2v\dfrac{\partial v}{\partial y} \equiv 0$. Since $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y} = \dfrac{\partial u}{\partial x}$, we get $\dfrac{\partial u}{\partial x}\left(= \dfrac{\partial v}{\partial y}\right) \equiv 0$, $\dfrac{\partial v}{\partial x}\left(= -\dfrac{\partial u}{\partial y}\right) \equiv 0$. Then $u \equiv$ constant, $v \equiv$ constant, so $f \equiv$ constant.

Examples (2) and (3) are special cases of the open mapping theorem, to be proved later.

Exercises

1. If $f = u + iv$ is holomorphic on some domain, given $u(x,y)$ below, find the possibilities of $v(x,y)$:

   (a) $u(x,y) = x^3 - 3xy^2$;
   (b) $u(x,y) = e^{-y}\cos x$;
   (c) $u(x,y) = \ln(x^2+y^2)$;
   (d) $u(x,y) = \dfrac{y}{(1-x)^2 + y^2}$.

2. Show that there is no holomorphic function $f = u + iv$ with $u(x,y) = x^2 + y^2$.

3. Suppose f is an entire function of the form $f(x,y) = u(x) + iv(y)$. Prove that f is a polynomial of degree at most one.

4. Suppose that f and $\overline{f}$ are holomorphic on a region D. Show that f is a constant function on D.

5. Let G be a region and $G^* = \{z : \overline{z} \in G\}$ is the mirror image of G across the x-axis. If $f : G \to \mathbb{C}$ is holomorphic, show that $f^* : G^* \to \mathbb{C}$ defined by $f^*(z) = \overline{f(\overline{z})}$ is holomorphic.

6. Write z in polar coordinates. Then $f(z) = u(z) + iv(z) = u(r, \theta) + iv(r, \theta)$. Establish the polar form of the Cauchy-Riemann equations
$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

7. Definitions of Common Functions

In this section, we will enlarge our collection of complex-valued functions by extending the common functions, such as exponential, logarithm and trigonometric functions, to complex domains.

To define e^z, we want it to satisfy (1) $e^{w+z} = e^w e^z$, (2) e^x is the same as before for x real, and (3) e^z is entire.

Suppose $f(w+z) = f(w)f(z)$ and $f(x) = e^x$ for x real. Then $f(z) = f(x+iy) = e^x f(iy) = e^x(A(y) + iB(y))$. For f to be entire (differentiable everywhere), the Cauchy-Riemann equations require $f_y = if_x$, i.e. $e^x(A'(y) + iB'(y)) = ie^x(A(y) + iB(y))$. Then $A'(y) = -B(y)$ and $B'(y) = A(y)$. So $A''(y) = -A(y)$ and $B(y) = -A'(y)$. Now $f(0) = 1$, so $A(0) = 1$ and $B(0) = 0$. Therefore, $A(y) = \cos y$ and $B(y) = \sin y$.

Definition. For $z = x + iy$, $e^z = e^x(\cos y + i \sin y) = e^x \operatorname{cis} y$. (In particular, $e^{iy} = \operatorname{cis} y$.)

In passing, we mention the famous *Euler's equation* $e^{i\pi} + 1 = 0$, which relates five of the most important constants $e, i, \pi, 1, 0$ in mathematics.

Properties.

(1) $e^{w+z} = e^w e^z$ (2) $|e^z| = e^x$ (3) $e^z \neq 0$ for all z (4) $\dfrac{d}{dz}e^z = e^z$

Since $e^{iy} = \cos y + i \sin y, e^{-iy} = \cos y - i \sin y$, so $\cos y = \dfrac{e^{iy} + e^{-iy}}{2}$ and $\sin y = \dfrac{e^{iy} - e^{-iy}}{2i}$. We will use these identities to define trigonometric functions of complex numbers.

Definitions. $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}, \sin z = \dfrac{e^{iz} - e^{-iz}}{2i}, \tan z = \dfrac{\sin z}{\cos z}, \sec z = \dfrac{1}{\cos z}, \csc z = \dfrac{1}{\sin z}, \cot z = \dfrac{\cos z}{\sin z}.$

The derivatives of these functions are the same as before. Also, $\cos^2 z + \sin^2 z \equiv 1$ for all z and the usual trigonometric identities are still valid. However, it is *not* true that $|\cos z| \leq 1$ and $|\sin z| \leq 1$ for all z, *e.g.* $\cos 10i = \dfrac{e^{-10} + e^{10}}{2} > 1000$.

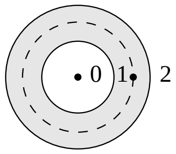
Now we turn our attention to defining logarithm. This is done by inverting exponentiation.

Definitions. If U, V are open sets, $f\colon U \to V$ is one-to-one and onto, then the *inverse of* f is $f^{-1}\colon V \to U$ defined by $f^{-1}(v) = u$ whenever $f(u) = v$. Also we say g is the *inverse of* f *at* $z \in V$ if g is the inverse of f in some neighborhood of z in V.

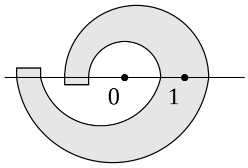
Consider the equation $e^z = w$ with $w \neq 0$. Let $z = x + iy$ be a solution of $e^z = w$. Then $e^z = e^x \operatorname{cis} y = w \Rightarrow e^x = |w|, y = \arg w \Rightarrow z = \ln|w| + i \arg w$. However, $\arg w$ is multivalued. Let $\operatorname{Arg} w$ denote the so-called "principal branch" of argument, which is defined by $-\pi < \operatorname{Arg} w \leq \pi$. The general solution of $e^z = w$ is $z = \ln|w| + i(\operatorname{Arg} w + 2k\pi)$, where k is any integer.

Definitions. For $w \neq 0$, $\log w = \ln|w| + i \arg w$, where $\arg w = \operatorname{Arg} w + 2k\pi$, k any integer. The choice $\operatorname{Log} w = \ln|w| + i \operatorname{Arg} w$ is the *principal branch of the logarithm function.*

To make $\log w$ continuous, we need to choose $\arg w$ continuously.



Examples. (1) On the annulus $A = \{w\colon 1 < |w| < 2\}$, $\arg w$ **cannot** be defined continuously. This can be seen by looking at the values of $\arg w$ on the circle $|w| = \dfrac{3}{2}$. At $w = \dfrac{3}{2}$, $\arg w = 2k\pi$. As w moves on the circle $|w| = \dfrac{3}{2}$ counterclockwise, the value of $\arg w$ increases. When it get back to $w = \dfrac{3}{2}$ at the end, the value of $\arg w$ have increased by 2π from the starting $2k\pi$. So $\arg w$ cannot be made continuously.



(2) For the region on the left, $\arg w$ **can** be defined continuously. We can define $\arg w$ by choosing $\arg 1 = \operatorname{Arg} 1 = 0$ and for any other w in the region, draw a polygonal path joining 1 to w and define $\arg w$ continuously along the path by starting at 1. It is visually clear that $\arg w$ will be well-defined (even if different paths are used) and continuous. It is bad to simply define $\arg w$ by taking the principal branch $\operatorname{Arg} w$ because the region "wraps" around the origin a little more than once.

On the word "branch" If $\arg w$ is defined continuously on a region, then $\arg w + 2k\pi$ (for any integer k) is also continuous on the same region and gives also the argument of points on the region. We refer to both of these functions as different "branches" of the argument function.

Similarly, if $\log z$ is defined continuously on a domain, then $\log z + 2\pi i$ is also a continuous inverse of e^z (since $e^{\log z + 2\pi i} = z = e^{\log z}$). We simply say that both are different "branches" of the logarithm function.

To differentiate the logarithm function, we will apply the inverse rule "$\dfrac{dw}{dz} = 1 \Big/ \dfrac{dz}{dw}$".

Inverse Rule. Let g be the inverse of f at z_0 and g be continuous near z_0. If f is differentiable at $g(z_0)$ and if $f'(g(z_0)) \neq 0$, then g is differentiable at z_0, $g'(z_0) = \dfrac{1}{f'(g(z_0))}$.

Proof. Since $f(g(z)) = z$ in a neighborhood of z_0, and $g(z) \to g(z_0)$ as $z \to z_0$ in that neighborhood, we have $\dfrac{g(z) - g(z_0)}{z - z_0} = \dfrac{1}{\dfrac{f(g(z)) - f(g(z_0))}{g(z) - g(z_0)}} \to \dfrac{1}{f'(g(z_0))}$ by the existence of $f'(g(z_0)) \neq 0$.

QED.

From the theorem, we see that if $\arg w$ can be defined continuously on a region, then $\log w$ will be continuous and even differentiable. Just take $g(z) = \log z$, $f(z) = e^z$, then $\dfrac{d}{dz} \log z = g'(z) = \dfrac{1}{e^{\log z}} = \dfrac{1}{z}$.

Definitions. For $\alpha \neq 0$, define $\alpha^\beta = e^{(\beta \log \alpha)}$ and the *principal value* of α^β is $e^{(\beta \operatorname{Log} \alpha)}$.

Example. $i^i = e^{(i \log i)} = e^{i(\ln|i| + i \arg i)} = e^{i(0 + i(\frac{\pi}{2} + 2k\pi))} = e^{-\frac{\pi}{2} - 2k\pi}$ (which is amazingly all real-valued!). The principal value of i^i is $e^{-\frac{\pi}{2}}$.

Remarks. (1) $\dfrac{d}{dz}\alpha^z = \dfrac{d}{dz} e^{(z \log \alpha)} = \alpha^z \log \alpha$ (for each fixed value of $\log \alpha$.)

(2) If $\arg z$ can be defined continuously on a region, then $\dfrac{d}{dz} z^\alpha = \dfrac{d}{dz} e^{(\alpha \log z)} = e^{(\alpha \log z)} \dfrac{\alpha}{z} = \alpha z^{\alpha - 1}$.

Example. We will show that, on $\Omega = \mathbb{C} \setminus ((-\infty, -1] \cup [1, +\infty))$, there is a continuous branch of $\sqrt{z^2 - 1}$.

Observe that

$$\sqrt{z^2 - 1} = (z^2 - 1)^{1/2} = e^{\frac{1}{2}\log(z^2 - 1)} = e^{\frac{1}{2}\left(\ln|z^2 - 1| + i\ \arg(z^2 - 1)\right)}.$$

Since $z^2 - 1 \neq 0$ on Ω, $\ln|z^2 - 1|$ is continuous on Ω. Next we will consider $\arg(z^2 - 1)$ as the composition of $w = f(z) = z^2 - 1$ on Ω with $g(w) = \arg w$ on $f(\Omega)$, where the argument is to be chosen later. Now f is clearly continuous on Ω and $f(\Omega) = \mathbb{C} \setminus [0, +\infty)$. Next we will choose $0 < \arg w < 2\pi$ on $f(\Omega)$ so as to make

g continuous on $f(\Omega)$. Therefore, $\arg(z^2-1) = g(f(z))$ is continuous on Ω, $\log(z^2-1)$ is continuous on Ω and $\sqrt{z^2-1}$ is continuous on Ω.

Exercises

1. Find all $z = x+iy$ such that $e^z = 1$. Use this to find all the roots of $\cos z$ and $\sin z$.

2. Prove that $\sin 2z = 2\sin z\cos z$ for all complex number z.

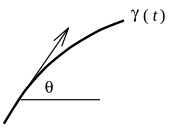
3. Discuss if it is possible to define $\log(z-1)$ continuously on $\mathbb{C}\setminus[-1,1]$. Also, discuss the possibility for $\log\left(\dfrac{z+1}{z-1}\right)$ to be continuously defined on $\mathbb{C}\setminus[-1,1]$.

8. Conformal Mappings

Definitions. (1) f is *conformal at* $z_0 \in \mathbb{C}$ iff f is holomorphic in a neighborhood of z_0 and $f'(z_0) \neq 0$.

(2) f is a *conformal mapping from a region* U *onto a region* V iff f is conformal at each point of U, one-to-one on U and onto V. In that case, we also say U and V are *conformally equivalent.*

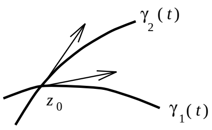
Observation. If f is conformal at z_0, then f is angle preserving (in direction and magnitude).

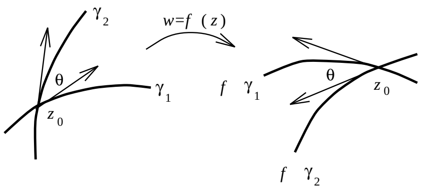


(For $\gamma(t) = (x(t), y(t)), \gamma'(t) = (x'(t), y'(t))$. In the figure, we have $\tan\theta = \dfrac{dy}{dx} = \dfrac{y'(t)}{x'(t)}$. So, $\theta = \tan^{-1}\dfrac{y'(t)}{x'(t)} = \arg\gamma'(t)$.

Suppose $\gamma_1(t_0) = \gamma_2(t_0) = z_0$.

Then the angle between γ_1, γ_2 at z_0 is given by $\arg\gamma_2'(t_0) - \arg\gamma_1'(t_0)$.





So, angle between $f\circ\gamma_1, f\circ\gamma_2$ at $w_0 = f(z_0)$

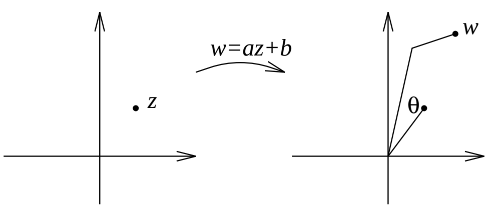
$$\begin{aligned}
&= \arg(f\circ\gamma_2)'(t_0) - \arg(f\circ\gamma_1)'(t_0)\\
&= \arg(f'(z_0)\gamma_2'(t_0)) - \arg(f'(z_0)\gamma_1'(t_0))\\
&= \arg f'(z_0) + \arg\gamma_2'(t_0) - \arg f'(z_0) - \arg\gamma_1'(t_0)\\
&= \text{angle between } \gamma_1, \gamma_2 \text{ at } z_0.)
\end{aligned}$$

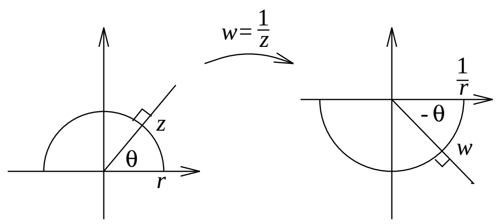
The observation has an interesting interpretation: if you lives at z_0 and f is a typhoon that blows your home from z_0 to $f(z_0)$, then you would not know that you have been moved because all the streets near your home intersect at the same angles as before!

Examples. [It will be convenient to treat lines as circles of infinite radius and a "circle" will mean either a line or a circle.]

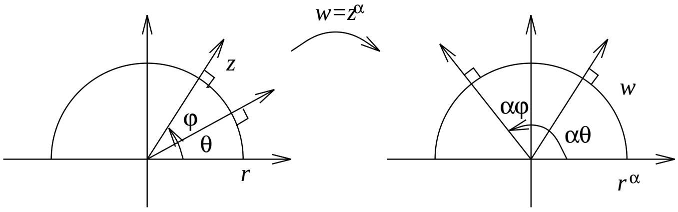
(1) $w = az + b, a \neq 0$ is a conformal mapping from $\mathbb{C}$ to $\mathbb{C}$. Writing $a = Re^{i\alpha}, b = u + iv$), we see that $az + b = R(e^{i\alpha}z) + u + iv$ rotates z (with respect to the origin) by angle α, then expands (or contracts) by a factor of R, then translate by (u, v). This mapping takes a line to a line and a circle to a circle, i.e. takes "circles" to "circles".

(2) $w = \dfrac{1}{z}$ is a conformal mapping of $\mathbb{C}\setminus\{0\}$ onto $\mathbb{C}\setminus\{0\}$. The equation $\alpha(x^2+y^2)+\beta x+\gamma y+\delta = 0$ (line if $\alpha = 0$, circle if $\alpha \neq 0$) is equivalent to $\alpha z\overline{z} + Dz + \overline{D}\overline{z} + \delta = 0$, where $D = \dfrac{\beta - i\gamma}{2}$. Under the mapping it becomes $\delta w\overline{w} + \overline{D}w + D\overline{w} + \alpha = 0$. So the map $w = \dfrac{1}{z}$ sends "circles" to "circles".

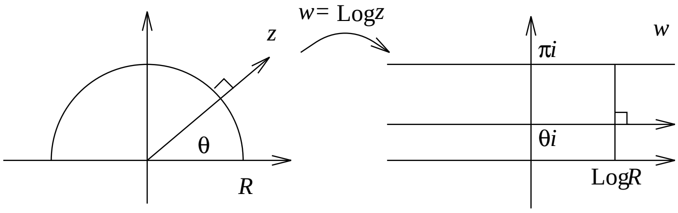




(3) $w = z^\alpha, \alpha > 0$ is conformal at all $z \neq 0$.



(4) $w = \operatorname{Log} z$ is conformal at $z \in \mathbb{C} \setminus (-\infty, 0]$.



(5) *Möbius transformations* (or *linear fractional transformations* or *bilinear transformations*) are functions of the form $w = T(z) = \dfrac{az+b}{cz+d}$, where $ad - bc \neq 0$. (Note if $ad - bc = 0$, then $T(z) \equiv$ constant.)

Properties of Möbius Transformations.

(1) (Basic Property) $T(z) = \dfrac{az+b}{cz+d}$ $(ad - bc \neq 0)$ is a one-to-one map from $\mathbb{C} \cup \{\infty\}$ onto $\mathbb{C} \cup \{\infty\}$, where $T(\infty) = \begin{cases} \dfrac{a}{c} & \text{if } c \neq 0 \\ \infty & \text{if } c = 0 \end{cases}$, $\infty = \begin{cases} T(\dfrac{-d}{c}) & \text{if } c \neq 0 \\ T(\infty) & \text{if } c = 0 \end{cases}$; $T^{-1}(z) = \dfrac{dz - b}{-cz + a}, T(T^{-1}(z)) = z = T^{-1}(T(z))$; $T'(z) = \dfrac{ad - bc}{(cz+d)^2} \neq 0$ for $z \neq \dfrac{-d}{c}$. So for $c = 0, T(z)$ is a conformal mapping from $\mathbb{C}$ onto $\mathbb{C}$ and for $c \neq 0, T(z)$ is a conformal mapping from $\mathbb{C} \setminus \left\{-\dfrac{d}{c}\right\}$ onto $\mathbb{C} \setminus \left\{\dfrac{a}{c}\right\}$.

(2) (Algebraic Property) If $T_1(z) = \dfrac{a_1 z + b_1}{c_1 z + d_1}$ $(a_1 d_1 - b_1 c_1 \neq 0)$ and $T_2(z) = \dfrac{a_2 z + b_2}{c_2 z + d_2}$ $(a_2 d_2 - b_2 c_2 \neq 0)$, then $T_1 \circ T_2(x) = T_1(T_2(z)) = \dfrac{(a_1 a_2 + b_1 c_2) z + (a_1 b_2 + b_1 d_2)}{(c_1 a_2 + d_1 c_2) z + (c_1 b_2 + d_1 d_2)}$.

(Observe that $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{pmatrix}$. This shows that there is a group homomorphism from the Möbius transformations to the two-by-two matrices.)

(3) (Geometric Property) $T(z) = \dfrac{az+b}{cz+d} = \begin{cases} \dfrac{a}{c} - \left(\dfrac{ad - bc}{c}\right) \dfrac{1}{cz+d} & \text{if } c \neq 0 \\ \dfrac{a}{d} z + \dfrac{b}{d} & \text{if } c = 0 \end{cases}$ is a composition of mappings of the form $Az + B$ and $\dfrac{1}{z}$. From examples (1) and (2), we conclude that Möbius transformations take "circles" to "circles".

(4) (Analytic Property) A *fixed point* of $T(z)$ is a point z_0 such that $T(z_0) = z_0$. The only Möbius transformation having more than two fixed points is the identity mapping.

(**Proof.** $T(z) = \dfrac{az+b}{cz+d} = z \iff cz^2 + (d - a)z - b = 0$, which has at most two roots, unless $c = 0$, $d - a = 0$, $b = 0$.)

Hence, if two Möbius transformations take the same values on three points of $\mathbb{C} \cup \{\infty\}$, then they must be identical.

(**Proof.** $S(z_1) = T(z_1)$, $S(z_2) = T(z_2)$, $S(z_3) = T(z_3) \Rightarrow S \circ T^{-1}$ has three fixed points $T(z_1)$, $T(z_2)$, $T(z_3) \Rightarrow S \circ T^{-1}$ is the identity mapping $\iff S(z) \equiv T(z)$.)

Definitions. For distinct $z_2, z_3, z_4 \in \mathbb{C} \cup \{\infty\}$, let $S_{z_2,z_3,z_4}(z) = \dfrac{z - z_3}{z - z_4} \Big/ \dfrac{z_2 - z_3}{z_2 - z_4}$. (Then S_{z_2,z_3,z_4} is a Möbius transformation that takes the "circle" through z_2, z_3, z_4 to the real axis because S_{z_2,z_3,z_4} sends z_2, z_3, z_4 to $1, 0, \infty$, respectively.)

The *cross ratio* of z_1, z_2, z_3, z_4 is $(z_1, z_2, z_3, z_4) = S_{z_2,z_3,z_4}(z_1) = \dfrac{z_1 - z_3}{z_1 - z_4} \times \dfrac{z_2 - z_4}{z_2 - z_3}$. (If one of z_1, z_2, z_3, z_4 is ∞, then we take limit to get the cross ratio.)

Properties of Cross Ratio.

(1) $(z, 1, 0, \infty) = \lim\limits_{w \to \infty} (z, 1, 0, w) = \lim\limits_{w \to \infty} \dfrac{z(1 - w)}{(z - w)} = z$.

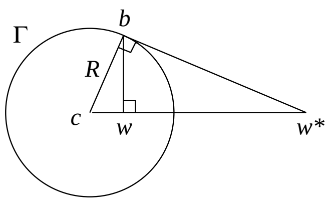
(2) $(\overline{z}_1, \overline{z}_2, \overline{z}_3, \overline{z}_4) = \overline{(z_1, z_2, z_3, z_4)}$.

(3) $(Tz_1, Tz_2, Tz_3, Tz_4) = (z_1, z_2, z_3, z_4)$ for any Möbius transformation T.

(**Proof.** $S_{z_2,z_3,z_4} \circ T^{-1}$: $\begin{array}{l} Tz_2 \to 1 \\ Tz_3 \to 0 \\ Tz_4 \to \infty \end{array}$. So by the holomorphic property, $S_{z_2,z_3,z_4} \circ T^{-1} = S_{Tz_2,Tz_3,Tz_4}$. Hence, evaluating both sides at $T(z_1)$, we get $S_{z_2,z_3,z_4}(z_1) = S_{Tz_2,Tz_3,Tz_4}(Tz_1)$.)

(4) z_1, z_2, z_3, z_4 are distinct points on a "circle" if and only if (z_1, z_2, z_3, z_4) is real. (This is because S_{z_2,z_3,z_4} takes the circle through z_2, z_3, z_4 to the circle through $1, 0, \infty$, which is the real axis.)

Below we shall show that cross ratio is a useful device that allows us to transform geometrical informations (such as, symmetry of points and inside/outside of circles) into algebraic quantities for computation and derivation of important principles.



Definition. For a given circle Γ: $|z - c| = R$, points w, $w^* \in \mathbb{C} \cup \{\infty\}$ are *symmetric with respect to* Γ iff $(w^* - c)(\overline{w} - \overline{c}) = R^2$. This is denoted by $w \sim^{\Gamma} w^*$. Equivalently,$w \sim^{\Gamma} w^* \iff w^* = \dfrac{R^2}{\overline{w} - \overline{c}} + c$. (Observe that c, w, w^* are collinear because $w^* - c = \dfrac{R^2}{|w - c|^2}(w - c)$. Also, $\dfrac{|w - c|}{R} = \dfrac{R}{|w^* - c|}$ implies $\angle cbw^* = 90°$.) In the case Γ is a line, w^* is taken to be the mirror image of w with respect to Γ.

Properties of Symmetric Points.

(1) The line ww^* is orthogonal to Γ (because c, w, w^* are collinear).

(2) A point is the center of Γ if and only if it is symmetric to ∞ with respect to Γ, i.e. $w = c \iff w^* = \infty$.

(3) $w \sim^{\Gamma} w^* \iff$ for any distinct z_2, z_3, z_4 on Γ, $(w^*, z_2, z_3, z_4) = \overline{(w, z_2, z_3, z_4)}$.

(**Proof.** Consider the Möbius transformation $T(z) = \dfrac{R^2}{z - c} + \overline{c}$, then $T(\dfrac{R^2}{\overline{w} - \overline{c}} + c) = \overline{w}$. Since z_2 on Γ implies $(z_2 - c)(\overline{z}_2 - \overline{c}) = R^2$, it follows that $T(z_2) = \overline{z}_2$. Similarly, $T(z_3) = \overline{z}_3$ and $T(z_4) = \overline{z}_4$.

($\Rightarrow$) If $w \sim^{\Gamma} w^*$, then $(w^*, z_2, z_3, z_4) = (T(w^*), T(z_2), T(z_3), T(z_4)) = (\overline{w}, \overline{z}_2, \overline{z}_3, \overline{z}_4)$.
($\Leftarrow$) We have

$$(w^*, z_2, z_3, z_4) = \overline{(w, z_2, z_3, z_4)} = (T(\frac{R^2}{\overline{w} - \overline{c}} + c), T(z_2), T(z_3), T(z_4)) = (\frac{R^2}{\overline{w} - \overline{c}} + c, z_2, z_3, z_4).$$

Since S_{z_2,z_3,z_4} is one-to-one, the definition of cross ratio implies $w^* = \dfrac{R^2}{\overline{w} - \overline{c}} + c$, i.e. $w \sim^\Gamma w^*$.)

Möbius transformations send pairs of symmetric points to pairs of symmetric points as the following shows.

Symmetry Principle. $w \sim^\Gamma w^* \iff T(w) \sim^{T(\Gamma)} T(w^*)$ for every Möbius transformation T.

Proof. This follows easily by property (3) above because

$$(w^*, z_2, z_3, z_4) = \overline{(w, z_2, z_3, z_4)} \quad \text{if and only if} \quad (T(w^*), T(z_2), T(z_3), T(z_4)) = \overline{(T(w), T(z_2), T(z_3), T(z_4))}.$$

QED.

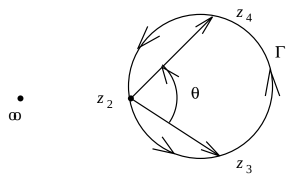
Möbius transformations need not send the inside of "circles" to the inside of the image "circles" (*e.g.* consider $w = \dfrac{1}{z}$ inside the unit circle). However, there is an orientation principle which says Möbius transformations *do* send the right sides of oriented "circles" to the right sides of their image "circles".

Definition. An *orientation* of a circle Γ is an ordered triple points (z_2, z_3, z_4) on Γ.

By property (4) of cross ratio, $\Gamma = \{z\colon \operatorname{Im}(z, z_2, z_3, z_4) = 0\}$. Now, the *right side* of Γ with respect to (z_2, z_3, z_4) (i.e. the side the right hand "touches" as one "walks" on Γ from z_2 to z_3 to z_4) is given by the set $R_{z_2,z_3,z_4}\Gamma = \{z\colon \operatorname{Im}(z, z_2, z_3, z_4) > 0\}$.

To see this, take the case (z_2, z_3, z_4) is counterclockwise, then the right side is the outside of Γ, which includes ∞. Since $S_{z_2,z_3,z_4}(z) = (z, z_2, z_3, z_4)$ is one-to-one and onto and sends Γ to the real axis, it must send the outside of Γ onto either the upper or the lower half plane because of connectivity. (To be precise, suppose p_1, p_2 are outside Γ such that S_{z_2,z_3,z_4} sends them to q_1, q_2, one on the upper half plane and the other on the lower half plane. Take a (continuous) curve joining p_1 and p_2 without intersecting Γ, then the image of the curve under S_{z_2,z_3,z_4} will not intersect the real axis. However, all curves joining q_1 and q_2 must intersect the real axis, which yields a contradiction.)

By definition, $R_{z_2,z_3,z_4}\Gamma$ is the side of Γ that is sent to the upper half plane. We intend to show, for the counterclockwise case, that this is the outside of Γ. For this, it suffices to see $\infty \in R_{z_2,z_3,z_4}\Gamma$.



Now observe that $(\infty, z_2, z_3, z_4) = \dfrac{z_2 - z_4}{z_2 - z_3} = \dfrac{z_4 - z_2}{z_3 - z_2} = Re^{i\theta}, 0 < \theta < \pi$. So, $\operatorname{Im}(\infty, z_2, z_3, z_4) > 0$, which implies $\infty \in R_{z_2,z_3,z_4}\Gamma$.
The clockwise case is similar. Also, the case Γ is a line can be checked, *e.g.* $R_{1,0,\infty}\mathbb{R} = \{z\colon \operatorname{Im} z > 0\}$ is the upper half plane)

Similarly, the *left side* of Γ with respect to (z_2, z_3, z_4) is given by $L_{z_2,z_3,z_4}\Gamma = \{z\colon \operatorname{Im}(z, z_2, z_3, z_4) < 0\}$.

Orientation Principle. For an orientation (z_2, z_3, z_4) on a circle Γ and T a Möbius transformation, we have $T(R_{z_2,z_3,z_4}\Gamma) = R_{T(z_2),T(z_3),T(z_4)}T(\Gamma)$ and $T(L_{z_2,z_3,z_4}\Gamma) = L_{T(z_2),T(z_3),T(z_4)}T(\Gamma)$. In particular, T must send each side of Γ onto a side of $T\Gamma$.

Proof. $z \in R_{z_2,z_3,z_4}\Gamma \iff \operatorname{Im}(z, z_2, z_3, z_4) = \operatorname{Im}(Tz, Tz_2, Tz_3, Tz_4) > 0 \iff Tz \in R_{Tz_2,Tz_3,Tz_4}T(\Gamma)$. Similarly, $z \in L_{z_2,z_3,z_4}\Gamma \iff Tz \in L_{Tz_2,Tz_3,Tz_4}T(\Gamma)$. **QED.**

Theorem. For real θ, $|a| < 1$, $T(z) = e^{i\theta}\left(\dfrac{z - a}{1 - \overline{a}z}\right)$ is a conformal map of the open unit disk D onto itself. (Later we will show these are the **only** conformal maps from D onto D.)

Proof. Observe that T takes the unit circle ∂D to itself because for $|z| = 1$, $|T(z)| = \left|e^{i\theta}\dfrac{z - a}{1 - \overline{a}z}\right| = \dfrac{|z - a|}{|\overline{z}z - \overline{a}z|} = \dfrac{|z - a|}{|\overline{z} - \overline{a}||z|} = 1$. Since $T(a) = 0$, the orientation principle implies T maps D onto D. **QED.**

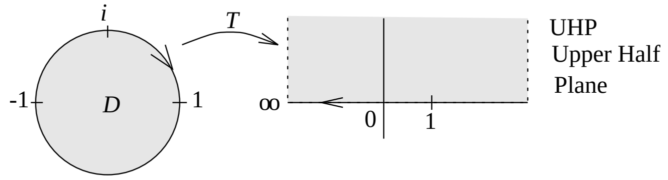
In an advanced course, a theorem called the *Riemann Mapping Theorem* is usually proved, which asserts that for every simply connected region $\Omega \neq \mathbb{C}$ (i.e. Ω has no "holes" in its interior), there is a conformal mapping f from the open unit disk D onto Ω. That is, every proper simply connected region is conformally equivalent to D. The mapping between D and Ω is called a *Riemann mapping.*

Remarks. By composing with another conformal mapping from D onto D, we see that Riemann mappings are not unique. In fact, if there is one Riemann mapping, there are infinitely many. However, if the values $f(0) \in \Omega$ and $f'(0) > 0$ are prescribed, then there is a second half of the Riemann mapping theorem, which asserts that there is exactly one Riemann mapping satisfying these additional conditions.

Later on we will prove many theorems about the open unit disk or functions on it. Because of the Riemann mapping theorem, they then can be viewed as theorems about arbitrary proper simply connected regions or functions on them. So, the open unit disk is a "canonical object" in complex analysis.

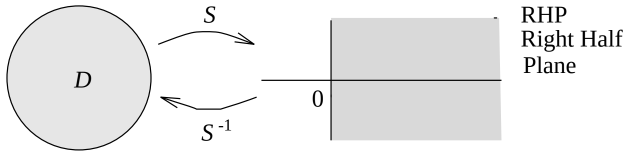
Let us compute some conformal mappings between D and some regions. Examples (1),(2),(3),(4) and (6) are Riemmann mappings between the open unit disk D and some common regions. Example (8) shows an interesting application of the symmetry principle and the orientation principle in conformal mappings.

Examples.



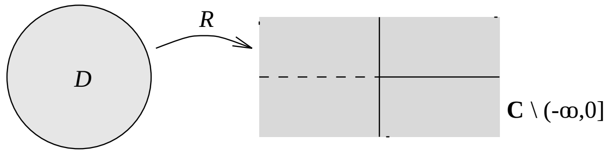
(1) $T(z) = S_{i,1,-1}(z) = i\left(\dfrac{1-z}{1+z}\right),$

$$T^{-1}(z) = \frac{i-z}{i+z}.$$

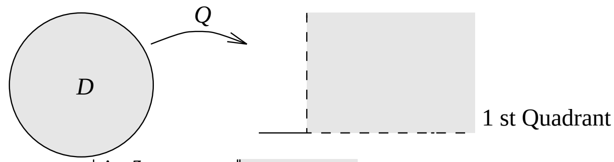


(2) $S(z) = \dfrac{1}{i}T(z) = \dfrac{1-z}{1+z},$

$$S^{-1}(z) = \frac{1-z}{1+z}.$$

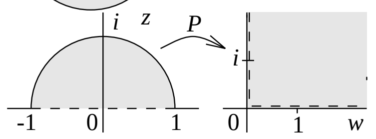


(3) $R(z) = (S(z))^2 = \left(\dfrac{1-z}{1+z}\right)^2.$



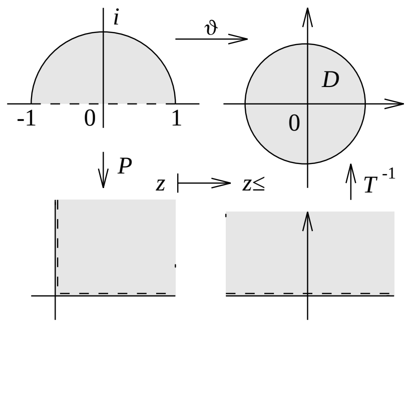
(4) $Q(z) = \sqrt{T(z)} = \sqrt{i\left(\dfrac{1-z}{1+z}\right)}.$

(the principal square root)



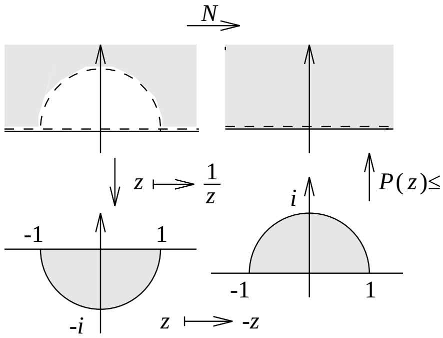
(5) $P(z) = i\left(\dfrac{1-z}{1+z}\right)$, P: $\begin{array}{r} i \mapsto 1 \\ 1 \mapsto 0 \\ 0 \mapsto i \\ -1 \mapsto \infty \end{array}$,

$P^{-1}(w) = \dfrac{i-w}{i+w}$. (*c.f.* Example (1))

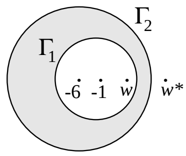


(6) $\vartheta(z) = T^{-1}(P(z)^2) = T^{-1}\left(-\left(\dfrac{1-z}{1+z}\right)^2\right)$

$$= \frac{-\left(\frac{1-z}{1+z}\right)^2 - i}{\left(\frac{1-z}{1+z}\right)^2 - i} = -i\frac{z^2+2iz+1}{z^2-2iz+1}$$



(7) $N(z) = P\left(-\frac{1}{z}\right)^2 = -\left(\frac{1-(-\frac{1}{z})}{1+(-\frac{1}{z})}\right)^2 = -\frac{(z+1)^2}{(z-1)^2}.$



(8) Let U be the interior of the annulus bounded by $\Gamma_1\colon |z+1| = 9, \Gamma_2\colon |z+6| = 16$. Find a conformal map from U onto a concentric annulus V centred at the origin.

Key. Find a pair of points w and w^* symmetric with respect to Γ_1 and Γ_2. (Such a pair must be collinear with the centers, so they must be real.)

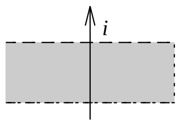
For such a pair, $(w+1)(w^*+1) = 81, (w+6)(w^*+6) = 256$. Solving we get $w = 2, w^* = 26$ (or converse). Consider $T(z) = \dfrac{z-2}{z-26}$. This is a Möbius transformation, hence takes circles to circles and right sides to right sides.

Since $w = 2 \overset{\Gamma_1,\Gamma_2}{\sim} w^* = 26$, the symmetry principle implies $T(w) = 0 \overset{T\Gamma_1,T\Gamma_2}{\sim} T(w^*) = \infty$. It follows $T(w) = 0$ is the center of $T\Gamma_1, T\Gamma_2$. Since $T(8) = -\dfrac{1}{3}$ and $T(10) = -\dfrac{1}{2}$, the orientation principle implies that T sends the inside of Γ_2 to the inside of $T\Gamma_2$ and the outside of Γ_1 to the outside of $T\Gamma_1$. So, $V = \left\{w\colon \dfrac{1}{3} < |w| < \dfrac{1}{2}\right\}$.

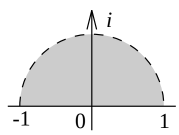
Exercises

1. Find conformal mappings from the open unit disk $D = \{z : |z| < 1\}$ onto the following regions:

   (a) the infinite strip $\{z : 0 < \operatorname{Im} z < 1\}$;

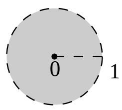
   

   (b) the upper semidisk $\{z\colon |z| < 1 \text{ and } \operatorname{Im} z > 0\}$.

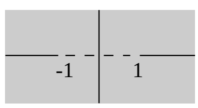
   

   [*Hint*: Find a map from 1st quadrant onto the upper half semidisk.]

   (c) the slit disk $D \setminus [0, 1)$

   (d) the finite-slit plane $\mathbb{C} \cup \{\infty\} \setminus [-1, 1]$

   [*Hint*: Find a map from $\mathbb{C} \setminus (-\infty, 0]$ to $(\mathbb{C} \cup \{\infty\} \setminus [-1, 1]$.]

2. What is the range of e^z if we take z to lie in the infinite strip $|\operatorname{Im} z| < \dfrac{\pi}{2}$? What are the images of the horizontal lines and vertical segments in $|\operatorname{Im} z| < \dfrac{\pi}{2}$ under the e^z mapping? Give an example of a holomorphic function from the open unit disk D onto $\mathbb{C} \setminus \{0\}$ with derivative never equal to zero.

3. Find r such that there is a one-to-one conformal mapping from $\{z : \operatorname{Im} z < 0 \text{ and } |z + 2i| > 1\}$ onto $\{w : r < |w| < 1\}$.

4. Let $a < b$ and $T(z) = \dfrac{z - ia}{z - ib}$. Define $L_1 = \{z\colon \operatorname{Im} z = b\}$, $L_2 = \{z\colon \operatorname{Im} z = a\}, L_3 = \{z\colon \operatorname{Re} z = 0\}$. Determine which of the regions A, B, C, D, E, F in Figure 1, are mapped by T onto the regions U, V, W, X, Y, Z in Figure 2.

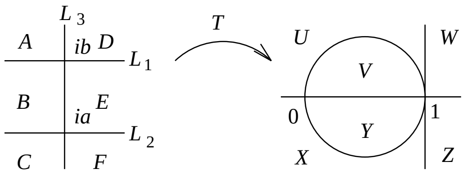


Figure 1 **Figure 2**

[*Hint*: Orient L_3 by (∞, ia, ib).]

9. Contour Integrals

Definition. If $\gamma\colon [a,b] \to \mathbb{C}$ is continuous and is given by $\gamma(t) = u(t) + iv(t)$, then we define

$$\int_a^b \gamma(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt.$$

Example. For $\gamma(t) = e^{it}, 0 \le t \le \dfrac{\pi}{2}$, $\displaystyle\int_0^{\frac{\pi}{2}} \gamma(t)\,dt = \int_0^{\frac{\pi}{2}} \cos t\,dt + i\int_0^{\frac{\pi}{2}} \sin t\,dt = 1 + i.$

Definitions. A continuous curve $z\colon [a,b] \to \mathbb{C}, z(t) = x(t) + iy(t)$, is *piecewise smooth* iff except for finitely many points, $x'(t)$ and $y'(t)$ are continuous on $[a,b]$ and $z'(t) = x'(t) + iy'(t) \ne 0$ on $[a,b]$. The curve is *closed* iff $z(a) = z(b)$. A closed curve is *simple* iff the function is one-to-one on $[a,b)$, i.e. it has no self-intersection.

Henceforth all curves will be assumed piecewise smooth and have finite lengths.

Definitions. If C is a smooth curve given by $z\colon [a,b] \to \mathbb{C}$ and f is continuous on the curve C, then we define $\displaystyle\int_C f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt$ and $\displaystyle\int_C |f(z)|\,|dz| = \int_a^b |f(z(t))|\,|z'(t)|\,dt.$

Examples. (1) Let C be the unit circle given by $z(t) = e^{it} (0 \le t \le 2\pi)$. Now $f(z) = \dfrac{1}{z}$ is continuous on C. So $\displaystyle\int_C \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{e^{it}}(ie^{it})\,dt = 2\pi i$ and $\displaystyle\int_C \left|\frac{1}{z}\right| |dz| = \int_0^{2\pi} dt = 2\pi.$

(2) Let C be the curve $z(t) = 1 + it^2$ $(0 \le t \le 1)$ and $f(z) = z^2$. Then $\displaystyle\int_C f(z)\,dz = \int_0^1 (1+it^2)^2(2it\,dt) = \int_0^1 (2it - 4t^3 - 8it^5)\,dt = (it^2 - t^4 - \frac{4i}{3}t^6)\Big|_0^1 = -1 - \frac{i}{3}.$

Properties of Contour Integrals.

(1) If C_1, C_2 given by $z_1\colon [a,b] \to \mathbb{C}, z_2\colon [c,d] \to \mathbb{C}$ are smoothly equivalent (in the sense that $z_2 = z_1 \circ \lambda$, where $\lambda\colon [c,d] \to [a,b]$ is one-to-one with continuous derivative and $\lambda(c) = a, \lambda(d) = b$), then by the change of variable $s = \lambda(t)$, we have

$$\int_{C_2} f(z)\,dz = \int_{t=c}^{t=d} f(z_2(t))z_2'(t)\,dt = \int_{s=a}^{s=b} f(z_1(s))z_1'(s)\,ds = \int_{C_1} f(z)\,dz.$$

(2) $\displaystyle\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$, where $-C$ is given by $\tilde{z}\colon [a,b] \to \mathbb{C}, \tilde{z}(t) = z(a+b-t)$.

(3) $\displaystyle\int_C [f(z) + g(z)]\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz, \int_C \alpha f(z)\,dz = \alpha\int_C f(z)\,dz.$

(4) **Fundamental Theorem of Calculus.** $\displaystyle\int_C f'(z)\,dz = f(z(b)) - f(z(a))$, where C is given by $z\colon [a,b] \to \mathbb{C}$.

Closed Curve Theorem. If C is a closed curve and f has an antiderivative (say $F(z)$) in a region containing C, then $\int_C f(z)\,dz = \int_C F'(z)\,dz = F(z(b)) - F(z(a)) = 0$. (In particular, $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$ for curves C_1, C_2 that have the same initial and terminal points because C_1 followed by $-C_2$ is a closed curve.)

Corollary. If C is a closed curve, then $\int_C P(z)\,dz = 0$ for every polynomial $P(z)$.

(5) *M-L* **Inequality.** If $|f(z)| \le M$ for every z on C and C has length L, then

$$\left|\int_C f(z)\,dz\right| = \left|\int_a^b f(z(t))z'(t)\,dt\right| \le \int_a^b |f(z(t))||z'(t)|\,dt = \int_C |f(z)||dz| \le M\int_a^b |z'(t)|\,dt = ML.$$

(6) If $f_n(z)$ converges uniformly to $f(z)$ on C, then $\lim_{n\to\infty}\int_C f_n(z)\,dz = \int_C f(z)\,dz$.

(**Proof.** Let L be the length of C. For any $\varepsilon > 0$, because $f_n(z)$ converges uniformly to $f(z)$ on C, there is N such that $n \ge N \Rightarrow |f_n(z) - f(z)| < \varepsilon$ for all z on C. Then for $n \ge N$,

$$\left|\int_C f_n(z)\,dz - \int_C f(z)\,dz\right| = \left|\int_C (f_n(z) - f(z))\,dz\right| \le \varepsilon L.)$$

Exercises

1. Find $\int_C \frac{1}{z^2}\,dz$, where C is a smooth curve from 1 to -1 not passing through the origin.

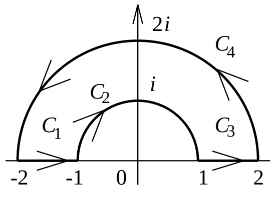
2. Suppose $f(z)$ is holomorphic and $|f(z) - 1| < 1$ in a region G. Show that $\int_C \frac{f'(z)}{f(z)}\,dz = 0$ for every closed curve C in G, assuming f' is continuous.

3. (a) Find $\int_C \frac{z}{\bar{z}}\,dz$, where C is the simple closed curve that goes from -2 to -1 along the real axis, then goes in the clockwise direction on the unit circle to 1, then goes from 1 to 2 along the real axis and finally goes back to -2 in the counterclockwise direction on the circle $|z| = 2$.

   (b) Find $\int_{|z|=1} |z-1|\,|dz|$, where the unit circle $|z| = 1$ is given the counterclockwise orientation.

4. Evaluate the integral $\int_C \frac{z}{\bar{z}}\,dz$, where C is the curve shown below.

   [*Hint*: $\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$]



5. If $\operatorname{Re} a \le 0$ and $\operatorname{Re} b \le 0$, show that $|e^a - e^b| \le |a - b|$.

6. Suppose f is holomorphic on $B(0,r)$ and $\operatorname{Re} f'(z) > 0$ for all $z \in B(0,r)$. Show that f is one-to-one. [*Hint*: Express $f(a) - f(b)$ as an integral.]

7. If f is a continuous real-valued function and $|f(z)| \leq 1$, show that $\left| \int_{|z|=1} f(z)\, dz \right| \leq \int_0^{2\pi} |\sin t|\, dt = 4.$

   [*Hint*: Let I be the value of the integral, then $I = |I|e^{i\theta}$.]

10. Cauchy Theory

Cauchy theory deals with the integration of holomorphic functions and its consequences as developed by Augustine Cauchy.

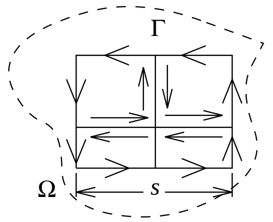
First, we deal with the local Cauchy theory. The word "local" refers to the situation in a neighborhood of a point. We begin with a fundamental property that leads to the whole development.

Definition. (Nonstandard Terminology!) A continuous function f has the *rectangle property in a region* Ω iff $\int_\Gamma f(z)\,dz = 0$ for every rectangle Γ such that Γ and its inside are contained in Ω and its edges are parallel to the real or imaginary axes. Such a rectangle will be referred to as a "*rectangle*" in Ω.

Rectangle Theorem (or Goursat's Theorem). If f is holomorphic on a region Ω, then f has the rectangle property in Ω.

Integral Theorem. If f has the rectangle property in a disk $B(c,R)$ $(0 < R \le \infty)$, then f has an antiderivative there.

Cauchy's Theorem on Disks. If f is holomorphic on a disk $B(c,R)$, then $\int_C f(z)\,dz = 0$ for every closed curve C in $B(c,R)$ (by the rectangle, integral and closed curve theorems.)



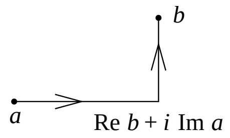
Proof of the Rectangle Theorem. Let Γ be a "rectangle" in Ω. Let s be the length of the larger side of Γ. Suppose $\int_\Gamma f(z)\,dz = I$. Divide the interior of Γ into four congruent subrectangular regions with boundaries Γ_1, Γ_2, Γ_3, Γ_4. Since $\int_\Gamma f(z)\,dz = \sum_{j=1}^{4} \int_{\Gamma_j} f(z)\,dz$, one of the integral must satisfy $\left|\int_{\Gamma_j} f(z)\,dz\right| \ge \frac{|I|}{4}$. Let us define $\Gamma^{(1)} = \Gamma_j$.

Repeat this subdivision to the interior of $\Gamma^{(1)}$ to obtain a $\Gamma^{(2)}$ and so on. We obtain a sequence of "rectangles" $\Gamma^{(1)}, \Gamma^{(2)}, \Gamma^{(3)}, \ldots$ in Ω such that $\left|\int_{\Gamma^{(k)}} f(z)\,dz\right| \ge \frac{|I|}{4^k}$. The rectangular regions bounded by the $\Gamma^{(k)}$'s "shrink" to a point z_0 in Ω.

Since f is holomorphic at z_0, so $g(z) = \frac{f(z)-f(z_0)}{z-z_0} - f'(z_0) \to 0$ as $z \to z_0$. So given $\varepsilon > 0$, there is $\delta > 0$ such that $|z-z_0| \le \delta \Rightarrow |g(z) - 0| \le \varepsilon$. Observe that the sides of $\Gamma^{(k)}$ are $\le \frac{s}{2^k}$, so $|z - z_0| \le \frac{\sqrt{2}s}{2^k}$ for all z in $\Gamma^{(k)}$. Choose k large so that $\frac{\sqrt{2}s}{2^k} < \delta$, then

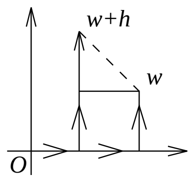
$$\begin{aligned}
\frac{|I|}{4^k} &\le \left|\int_{\Gamma^{(k)}} f(z)\,dz\right| = \left|\int_{\Gamma^{(k)}} (\underbrace{f(z_0) + f'(z_0)(z-z_0)}_{\text{polynomial}} + g(z)(z-z_0))\,dz\right| = \left|\int_{\Gamma^{(k)}} g(z)(z-z_0)\,dz\right|. \\
&\le \left(\frac{\varepsilon\sqrt{2}s}{2^k}\right)\left(4\frac{s}{2^k}\right) = \frac{4\sqrt{2}s^2\varepsilon}{4^k} \Rightarrow |I| \le 4\sqrt{2}s^2\varepsilon.
\end{aligned}$$

Since ε is arbitrary, $\displaystyle\int_\Gamma f(z)\,dz = I = 0$. **QED.**



Proof of the Integral Theorem. Without loss of generality, let $c = 0$. For w in the disk, define $F(w) = \displaystyle\int_{C_{0,w}} f(z)\,dz$, where $C_{a,b}$ denotes the curve from a to $\operatorname{Re} b + i \operatorname{Im} a$ then to b. By the rectangle property, $\displaystyle\int_\Gamma f(z)\,dz = 0$ for the boundary Γ of every rectangle inside the disk.

We will show $F' = f$. Fix w in the disk. For any $\varepsilon > 0$, since f is continuous at w, there is $\delta > 0$ such that $|z - w| \le \delta \Rightarrow |f(z) - f(w)| \le \varepsilon$.



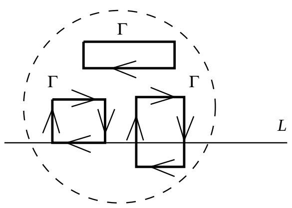
For $|h| \le \delta$, length of $C_{w,w+h} \le 2|h|$. Observe that $f(w) = \frac{1}{h}\int_{C_{w,w+h}} f(w)\,dz$. So,

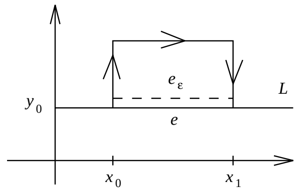
$$\left|\frac{F(w+h) - F(w)}{h} - f(w)\right| = \left|\frac{1}{h}\int_{C_{w,w+h}} (f(z) - f(w))\,dz\right| \le \frac{1}{|h|}\varepsilon \cdot 2|h| = 2\varepsilon.$$

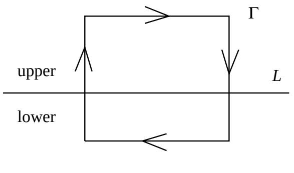
Therefore, $\displaystyle\lim_{h\to 0} \frac{F(w+h) - F(w)}{h} = f(w)$. **QED.**

Occasionally, we come across some continuous functions that are "almost" holomorphic in their domains (with exceptions only on some "thin" subsets), it is remarkable that such functions still have the rectangle property on their domains, as the following theorem asserts.

Extension Theorem. If f is continuous on a region D and holomorphic on $D \setminus L$, where L is a (horizontal) line segment (or in the extreme case, a point), then f has the rectangle property on D.







Proof. Let Γ be a "rectangle" in D. There are three cases to consider.

(1) If Γ and its interior do not contain any point of L, then Γ is a "rectangle" in $D \setminus L$ and $\displaystyle\int_\Gamma f(z)\,dz = 0$ by the holomorphicity of f.

(2) If Γ has a (horizontal) edge e containing points of L, let Γ_ε be the rectangle that has the same edges as Γ except the edge e is replaced by an edge e_ε, which is ε unit from L on the same side as Γ. By (1), $\displaystyle\int_{\Gamma_\varepsilon} f(z)\,dz = 0$. Now as $\varepsilon \to 0$, the (uniform) continuity of f on e implies

$$\int_{e_\varepsilon} f(z)\,dz = \int_{x_1}^{x_0} f(x + i(y_0 + \varepsilon))\,dx \to \int_{x_1}^{x_0} f(x + iy_0)\,dx = \int_e f(z)\,dz.$$

Since $0 = \displaystyle\int_{\Gamma_\varepsilon} f(z)\,dz \to \int_\Gamma f(z)\,dz$ as $\varepsilon \to 0$, we get $\displaystyle\int_\Gamma f(z)\,dz = 0$.

(3) If the interior or a vertical edge of Γ contains a point of L, then $\displaystyle\int_\Gamma f(z)\,dz = \int_{\Gamma_{\text{upper}}} f(z)\,dz + \int_{\Gamma_{\text{lower}}} f(z)\,dz = 0$, where we used the (horizontal) line through L to form two new "rectangles" and applied (2). **QED.**

Now we turn to the (global) Cauchy theory, which deals with the integration of holomorphic functions on (closed) curves in (simply connected) domains. The curves need not be restricted to lie inside disks.

Definition. A continuous function f defined on a region D has the *polygon property* iff $\displaystyle\int_\Gamma f(z)\,dz = 0$ for every right-angled polygon Γ such that Γ and its inside are contained in D.

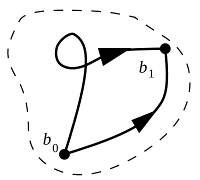
Polygon Theorem. Every holomorphic function on a simply connected region has the polygon property.

Proof. Subdivide the inside of every right-angled polygons into rectangular regions. Then apply the rectangle theorem. **QED.**

Integral Theorem. Every funcion f having the polygon property on a simply connected region D has an antiderivative F on D.

Proof. Fix $z_0 \in D$. Define $F(z) = \int_C f(z)\,dz$, where C is a polygonal line from z_0 to z consisting of horizontal or vertical segments. (If $\tilde{C}$ is another such polygonal line, then C and $\tilde{C}$ form finitely many right-angled polygons. Hence $\int_C f(z)\,dz - \int_{\tilde{C}} f(z)\,dz = 0$ and F is well-defined.) That $F' = f$ follows by the same argument as in the first integral theorem. **QED.**

Cauchy's Theorem. If f is holomorphic on a simply connected region D and C is a closed curve in D, then (by the polygon, integral and closed curve theorems,) $\int_C f(z)\,dz = 0$.



Cauchy's Theorem for Homotopic Curves. If C_1, C_2 are two curves with the same initial and terminal points in a simply connected region D, then $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$ for every holomorphic function f on D. (This follows because C_1 and $-C_2$ form a closed curve, so we may apply Cauchy's theorem above.)

As an appication of the (global) Cauchy theorems, we will present a theorem that deals with a global question. For a holomorphic function without roots in a domain, locally (i.e. near every point) we can take the logarithm of the function. It is of interest to know if there is a continuous logarithm of the function on the whole domain.

Logarithm Theorem. Let f be holomorphic on a simply connected region D without any root in D. Fix $z_0 \in D$ and choose a choice of $\log f(z_0)$, then $G(z) = \int_C \frac{f'(w)}{f(w)}\,dw + \log f(z_0)$ (where C is any curve in D from z_0 to z) defines a holomorphic branch of $\log f(z_0)$ on D (i.e. $e^{G(z)} = f(z)$ for all $z \in D$). In particular, if f is entire without any root, then $f = e^G$ for some entire function G.

Proof. Since $\frac{f'}{f}$ is holomorphic on D, the proof of the integral theorem implies $G' \equiv \frac{f'}{f}$. Consider the function $A(z) = e^{-G(z)}f(z)$. We have $A'(z) = e^{-G(z)}(-G'(z)f(z)+f'(z)) \equiv 0$. Since $A(z_0) = e^{-G(z_0)}f(z_0) = e^{-\log f(z_0)}f(z_0) = 1$ and D is connected, $A \equiv 1$. Therefore $e^G \equiv f$. **QED.**

Remarks. (1) If D is simply connected and not containing 0, then $\log z = \int_{z_0}^{z} \frac{dw}{w} + \log z_0$ is a holomorphic logarithm on D. Again because of simply connectedness, the path of integration can be any curve in D from z_0 to z.

(2) For $\alpha \in \mathbb{C}$, $f(z)^\alpha = e^{\alpha \log f(z)}$ wherever $\log f$ is defined.

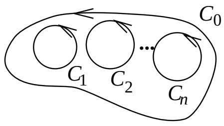
Example. We will show that there are continuous (in fact, holomorphic) branches of $\log\left(\frac{z+1}{z-1}\right)$ and $\sqrt{\frac{z+1}{z-1}}$ on $\mathbb{C} \setminus [-1,1]$. Note that $\mathbb{C} \setminus [-1,1]$ is not a simply connected region.

However, the region $D = \mathbb{C} \setminus ((-\infty,-1] \cup [1,+\infty))$ is simply connected. The function $\frac{1+w}{1-w}$ is holomorphic on D and has no roots there. By the logarithm theorem, there is a holomorphic branch $G(w) = \log\left(\frac{1+w}{1-w}\right)$

on D. Let $R(z) = \dfrac{1}{z}$, then $F(z) = G(R(z)) = \log\left(\dfrac{z+1}{z-1}\right)$ and $e^{\frac{1}{2}F(z)} = \sqrt{\dfrac{z+1}{z-1}}$ are holomorphic on $R^{-1}(D) = \mathbb{C}\setminus[-1,1]$.

Exercises

1. Let $C_1, \dots, C_n$ be simple closed curves inside a simple closed curve C_0, all with the counterclockwise orientation. Suppose no C_i is inside a C_j for positive i, j $(i \neq j)$. Let S be the points inside C_0, but outside all C_j, $j = 1, \dots, n$. If h is holomorphic on $C_0, C_1, \dots, C_n$ and on S, show that $\displaystyle\int_{C_0} h(z)\,dz = \sum_{j=1}^{n} \int_{C_j} h(z)\,dz$. [*Hint*: Draw a sketch and introduce cross-cuts to connect C_0 with C_j.]



2. Find $\displaystyle\int_C \frac{1}{z}\,dz$, where C is a counterclockwise regular n-gon $(n > 2)$ with center at the origin.

11. Power Series Representation

Below we will show that holomorphic functions have power series representations locally. First we introduce the concept of the *winding number* of a closed curve and deduce an integral formula for a holomorphic function. Then the power series represntation follows by applying the formula for summing geometric series to the integrand in the integral formula.

Definition. Suppose C is a closed curve and $a \notin C$. $n(C,a) = \dfrac{1}{2\pi i}\displaystyle\int_C \frac{dz}{z-a}$ is called the *winding number* of C around a.

Example. Let $C\colon z(t) = a + Re^{int}$ $(0 \le t \le 2\pi)$. This is the circle of radius R centered at a that "winds" around a $|n|$ times (counterclockwise if $n > 0$, clockwise if $n < 0$.) We have

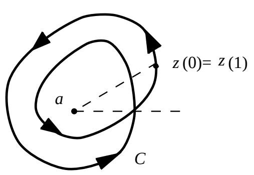
$$n(C,a) = \frac{1}{2\pi i}\int_C \frac{dz}{z-a} = \frac{1}{2\pi i}\int_0^{2\pi} \frac{1}{Re^{int}} Rine^{int}\, dt = n.$$

Theorem. For any closed curve C and $a \notin C, n(C,a)$ is an integer.

Proof. Let C be given by $z\colon [0,1] \to \mathbb{C}$. Since C is closed, $z(0) = z(1)$. For $0 \le s \le 1$, define

$$F(s) = \int_{C|_{[0,s]}} \frac{dz}{z-a} = \int_0^s \frac{z'(t)}{z(t)-a}\, dt.$$

Then $F(0) = 0$ and $F'(s) = \dfrac{z'(s)}{z(s)-a}$. This implies $\dfrac{d}{ds}((z(s)-a)e^{-F(s)}) = (z'(s)-(z(s)-a)F'(s))e^{-F(s)} \equiv 0$. So $(z(s)-a)e^{-F(s)}$ is constant. Then $(z(1)-a)e^{-F(1)} = (z(0)-a)e^{-F(0)} = z(1)-a$, which implies $e^{-F(1)} = 1$. Therefore $F(1) = 2\pi ik$, where k is an integer, and $n(C,a) = \dfrac{1}{2\pi i}F(1) = k$. **QED.**



Remark. Integrating both sides of $F'(s) = \dfrac{z'(s)}{z(s)-a}$, then using $F(0) = 0$, we get

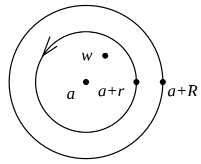
$$F(s) = \log\left(\frac{z(s)-a}{z(0)-a}\right) = \ln\left|\frac{z(s)-a}{z(0)-a}\right| + i\arg\left(\frac{z(s)-a}{z(0)-a}\right).$$

Therefore, if arg is continuous, then $F(1) = F(1) - F(0) = i\Delta_C \arg(z-a) = 2\pi i\times$(number of times C "winds" around a), where $\Delta_C \arg(z-a) = \arg(z(1)-a) - \arg(z(0)-a)$ denotes the change of argument about a as C is traversed.

Cauchy's Integral Formula. If f is holomorphic on $B(c,R)$, $0 < R \le \infty$ and C is a closed curve in $B(c,R) \setminus \{a\}$, then $f(a) = \dfrac{1}{2\pi i\, n(C,a)}\displaystyle\int_C \frac{f(z)}{z-a}\, dz$.

Proof. The function $g(z) = \begin{cases} \dfrac{f(z)-f(a)}{z-a} & \text{if } z \ne a \\ f'(a) & \text{if } z = a \end{cases}$ is holomorphic on $B(c,R) \setminus \{a\}$ and continuous on $B(c,R)$. By the extension, integral and closed curve theorems, $\displaystyle\int_C \frac{f(z)-f(a)}{z-a}\, dz = \int_C g(z)\, dz = 0$. Therefore, $\displaystyle\int_C \frac{f(z)}{z-a}\, dz = \int_C \frac{f(a)}{z-a}\, dz = f(a) 2\pi i\, n(C,a)$. **QED.**

Power Series Representation. If f is holomorphic on the disk $B(a,R), 0 < R \le \infty$, then $f(w) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(w-a)^n$ for all w in $B(a,R)$.



Proof. For w in $B(a,R)$, let r be such that $|w-a| < r < R$. Then the circle $|z-a| = r, z(t) = a + re^{it}$ winds about a and w once. For z on this circle, $\left|\frac{w-a}{z-a}\right| = \frac{|w-a|}{r} < 1$ and $\frac{1}{z-w} = \frac{1}{z-a}\left(\frac{1}{1-\frac{w-a}{z-a}}\right) = \frac{1}{z-a}\sum_{n=0}^{\infty}\left(\frac{w-a}{z-a}\right)^n$. By Cauchy's integral formula,

$$f(w) = \frac{1}{2\pi i}\int_{|z-a|=r}\frac{f(z)}{z-w}dz = \frac{1}{2\pi i}\int_{|z-a|=r}\sum_{n=0}^{\infty}\frac{(w-a)^n f(z)}{(z-a)^{n+1}}dz = \sum_{n=0}^{\infty}\underbrace{\left(\frac{1}{2\pi i}\int_{|z-a|=r}\frac{f(z)}{(z-a)^{n+1}}dz\right)}_{\beta_n}(w-a)^n.$$

(The last equality follows from uniform convergence for all z on the circle by Weierstrass M-test

$$\sum_{n=0}^{\infty}\left|\frac{(w-a)^n}{(z-a)^{n+1}}f(z)\right| \le \sum_{n=0}^{\infty}\frac{|w-a|^n}{r^{n+1}}M = \frac{M}{r-|w-a|} < \infty,$$

where $M = \max_{|z-a|=r}|f(z)|$.) The power series representation $f(w) = \sum_{n=0}^{\infty}\beta_n(w-a)^n$ is valid for all $w \in B(a,r)$. Therefore, by Taylor's theorem for power series, $\beta_n = \frac{f^{(n)}(a)}{n!}$. Finally, we let $r \to R$.

QED.

Examples of Power Series of Common Functions.

$e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!}$ (all z), $\cos z = \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{(2n)!}$ (all z), $\sin z = \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)!}$ (all z),

$\cosh z = \frac{e^z + e^{-z}}{2} = \sum_{n=0}^{\infty}\frac{z^{2n}}{(2n)!}$ (all z), $\sinh z = \frac{e^z - e^{-z}}{2} = \sum_{n=0}^{\infty}\frac{z^{2n+1}}{(2n+1)!}$ (all z),

$\operatorname{Log} z = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}(z-1)^n}{n}$ (for $|z-1| < 1$), $(1+z)^\alpha = 1 + \sum_{n=1}^{\infty}\frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}z^n$ (for any complex α, $|z| < 1$.)

Corollary 1. Holomorphic functions are infinitely differentiable (because power series are).

This is an amazing fact that is certainly not true for differentiable functions of real variables!

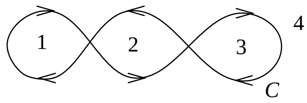
Corollary 2. $g(z) = \begin{cases} \frac{f(z)-f(a)}{z-a} & \text{if } z \ne a \\ f'(a) & \text{if } z = a \end{cases}$ can be represented by $f'(a) + \frac{f''(a)}{2!}(z-a) + \frac{f'''(a)}{3!}(z-a)^2 + \dots$ near a, hence it is holomorphic at a.

Corollary 3. If $f(z)$ is holomorphic on Ω and has roots $z_1, z_2, \dots, z_k$, then there is a holomorphic function $g_k(z)$ on Ω such that $f(z) \equiv (z-z_1)(z-z_2)\dots(z-z_k)g_k(z)$.

Proof. For $n = 1$, $g_1(z) = \begin{cases} \frac{f(z)-f(z_1)}{z-z_1} & \text{if } z \ne z_1 \\ f'(z_1) & \text{if } z = z_1 \end{cases}$ is holomorphic on Ω. The inductive step follows by observing that $g_n(z) = (z-z_{n+1})g_{n+1}(z)$ and getting g_{n+1} as in case $n = 1$. **QED.**

Exercises

1. The following curve C divides the plane into 4 regions. For each region, state the winding number of C around points in that regions. (Give answers by inspection, no computation is needed.)



2. Find $\displaystyle\lim_{z\to 0} \frac{z\,\mathrm{Log}(z^{49}+1)}{(\cos z^{25})-1}$.

3. Find $\displaystyle\int_{|z|=1} \frac{\sin z}{z}\,dz$, $\displaystyle\int_{|z|=1} \frac{\sin z}{z^2}\,dz$ and $\displaystyle\int_{|z|=2} \frac{\sin z}{z^2-1}\,dz$, where the circles are oriented counterclockwisely.

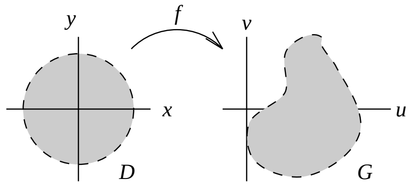
4. If f is holomorphic on the open unit disk D and $|z|+|f(z)|\le 1$ for all $z\in D$, then show that $f\equiv 0$.

5. Suppose f is holomorphic on $D=\{z : |z|<1\}$ and f is an even function (i.e. $f(z)=f(-z)$). Show that there is a holomorphic function g on D such that $g(x)=f(\sqrt{x})$ for all positive real number $x<1$.

6. Prove that $f(z)=\displaystyle\int_0^1 \frac{\sin zt}{t}\,dt$ is an entire function by obtaining a power series expansion for f.

7. Suppose f is holomorphic on the closed unit disk $\{z : |z|\le 1\}$ and $|a|<1$, show that $(1-|a|^2)|f(a)|\le \displaystyle\frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|\,d\theta$. [*Hint*: Consider integrating $f(z)\dfrac{1-\overline{a}z}{z-a}$.]

8. Let $f(z)=u(z)+iv(z)$ (or $f(x,y)=(u(x,y),v(x,y))$) be a one-to-one holomorphic function from the open unit disk $D=\{z\colon |z|<1\}$ onto a domain G with finite area.



   (a) Show that $J_f(x,y) \stackrel{\text{def}}{=} \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = |f'(z)|^2$.

   (b) For distinct nonnegative integers m, n, show $\displaystyle\int_D z^m \overline{z^n}\,dA = 0$ (*orthogonality relation*), where $dA = r\,dr\,d\theta = dx\,dy$ is the area differential.

   (c) Show that if $f(z)=\displaystyle\sum_{n=0}^{\infty} c_n z^n$ is the power series for f in D, then area of $G=\pi\displaystyle\sum_{n=1}^{\infty} n|c_n|^2$.

9. Suppose f is holomorphic on the open unit disk $\{z : |z|<1\}$ such that $\displaystyle\sum_{n=0}^{\infty}\left|\frac{f^{(n)}(0)}{n!}\right|^2<\infty$. Prove that

$$\lim_{r\to 1^-} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2\,d\theta = \sum_{n=0}^{\infty}\left|\frac{f^{(n)}(0)}{n!}\right|^2.$$

10. Suppose $f(z)=\displaystyle\sum_{n=0}^{\infty} a_n z^n$ and $g(z)=\displaystyle\sum_{n=0}^{\infty} b_n z^n$ converges for all $z\in B(0,1)$. Let $|z_0|<r<1$.

   (a) Show that $\displaystyle\sum_{n=0}^{\infty} b_n f(w)\frac{z_0^n}{w^{n+1}}$ converges uniformly on the circle $\{w : |w|=r\}$.

(b) Show that $\dfrac{1}{2\pi i}\displaystyle\int_{|w|=r} f(w)g\left(\frac{z_0}{w}\right)\frac{dw}{w} = \sum_{n=0}^{\infty} a_n b_n z_0^n$.

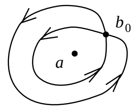
11. Prove $\displaystyle\overline{\lim_{n\to\infty}}\left|\sum_{j=1}^{M} a_j^n\right|^{1/n} = \max_{j=1,\cdots,M}|a_j|$. [*Hint*: Consider the power series with $\displaystyle\sum_{j=1}^{M} a_j^n$ as coefficients.]

12. Consequences of Cauchy Theory

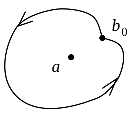
Cauchy Integral Formula for Derivatives. If f is holomorphic on $B(a,R)$, then for $n = 0, 1, 2, \ldots$ and $0 < r < R$, we have $f^{(n)}(a) = \dfrac{n!}{2\pi i} \displaystyle\int\limits_{|z-a|=r} \dfrac{f(z)}{(z-a)^{n+1}}\, dz$ (since both sides equal $n!\beta_n$ as shown in the proof of the power series representation).

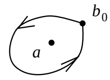
Notice this formula said something amazing, namely the derivative at a point can be given by an integral!

We now give an example which illustrates the fact that the Cauchy integral formula for derivatives is also true for closed curves in a region, provided winding number factors are included. This will be proved later.

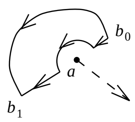
Example. Let f be entire, find $\displaystyle\int_C \frac{f(z)}{(z-a)^{20}}\, dz$, where C is the curve 

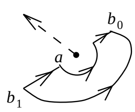
.

Solution. $\displaystyle\int_C = \int_{C_1} + \int_{C_2}$, where C_1 is the curve 

and C_2 is the curve 

.

Since $\dfrac{f(z)}{(z-a)^{20}}$ is holomorphic on the slit planes 



, which are simply connected, by Cauchy's theorem for homotopic curves, $\displaystyle\int_{C_1} = \int\limits_{|z-a|=\varepsilon}$. Similarly, $\displaystyle\int_{C_2} = \int\limits_{|z-a|=\varepsilon}$.

So, $\displaystyle\int_C \frac{f(z)}{(z-a)^{20}}\, dz = 2 \int\limits_{|z-a|=\varepsilon} \frac{f(z)}{(z-a)^{20}}\, dz = 2\pi i\, n(C,a) \frac{f^{(19)}(a)}{19!}$.

When combined with the M-L inequality, the Cauchy integral formula for derivatives gives estimates on the size of derivatives or coefficients of power series.

Liouville's Theorem. A bounded entire function is constant. (More generally, if $f(z)$ is entire and if there are constants A and B and $\alpha \geq 0$ such that $|f(z)| \leq A + B|z|^\alpha$ when $|z|$ is large (i.e. $|z| > r$ for some $r > 0$), then $f(z)$ is a polynomial of degree $k \leq \alpha$.)

Proof. Since $f(z)$ is holomorphic on $B(0,\infty)$, $f(z) = \displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$ for all z (in $B(0,\infty)$). We will show that $f^{(n)}(0) = 0$ for $n > \alpha$. This implies $f(z)$ is a polynomial of degree at most $k \leq \alpha$.

Let $n > \alpha$ and $R > r$. By the Cauchy integral formula for derivatives and M-L inequality, as $R \to \infty$,

$$\left|f^{(n)}(0)\right| = \left|\frac{n!}{2\pi i} \int\limits_{|z|=R} \frac{f(z)}{z^{n+1}}\, dz\right| \leq \frac{n!}{2\pi}\left(\frac{A + BR^\alpha}{R^{n+1}} 2\pi R\right) = \frac{n!(A + BR^\alpha)}{R^n} \to 0.$$

(We can take limit as $R \to \infty$ because $f(z)$ is defined in the whole plane, hence the circle $|z| = R$ would always be in the domain of $f(z)$, so Cauchy integral formula for derivatives can be applied!) Therefore, $f^{(n)} = 0$ and $f(z) = f(0) + f'(0)z + \ldots + f^{(k)}(0)z^k$ with $k \leq \alpha$. **QED.**

In an advanced course, a theorem called the *Little Picard Theorem* is usually proved, which asserts that an entire function whose range misses (i.e. not includes) two complex numbers must be constant. This is a big improvement over Liouville's theorem, since bounded functions are functions whose ranges are bounded sets, hence miss many complex numbers. An example of an entire function whose range misses only one value is e^z. Its range is $\mathbb{C} \setminus \{0\}$, hence misses only 0.

L' Hôpital's Rule. If $f(z)$ and $g(z)$ are holomorphic in Ω and for some $a \in \Omega$, $f(a) = f'(a) = \ldots = f^{(n-1)}(a) = 0, g(a) = g'(a) = \ldots = g^{(n-1)}(a) = 0$ and either $f^{(n)}(a) \neq 0$ or $g^{(n)}(a) \neq 0$ for some $a \in \Omega$, then $\displaystyle\lim_{z \to a} \frac{f(z)}{g(z)} = \frac{f^{(n)}(a)}{g^{(n)}(a)}$.

Proof. Near a, the power series of $f(z)$ is $\displaystyle\sum_{k=n+1}^{\infty} \frac{f^{(k)}(a)}{k!}(z-a)^k = \frac{f^{(n)}(a)}{n!}(z-a)^n + F(z)(z-a)^{n+1}$. Similarly, the power series of $g(z)$ near a has a similar form. Then

$$\lim_{z \to a} \frac{f(z)}{g(z)} = \lim_{z \to a} \frac{f^{(n)}(a) + n!F(z)(z-a)}{g^{(n)}(a) + n!G(z)(z-a)} = \frac{f^{(n)}(a)}{g^{(n)}(a)}.$$

QED.

Definition. If a holomorphic function $f(z)$ has a root a such that $f(a) = f'(a) = \ldots = f^{(n-1)}(a) = 0$ but $f^{(n)}(a) \neq 0$, it is a *root of order (or multiplicity)* n. (Equivalently, this means that $f(z) = \dfrac{f^{(n)}(a)}{n!}(z-a)^n + \dfrac{f^{(n+1)}(a)}{(n+1)!}(z-a)^{n+1} + \cdots$ near a with $f^{(n)}(a) \neq 0$ or $f(z) = (z-a)^n g(z)$ with $g(a) \neq 0$.) A root of order one, two or three is called a *simple, double or triple root*, respectively.

Example. Find $\displaystyle\lim_{z \to 0} \frac{(\sin z^2)(\cos z - 1)}{e^{z^4} - 1}$.

When w is near 0, $\sin w = w - \dfrac{w^3}{3!} + \ldots$, $\cos w - 1 = -\dfrac{w^2}{2!} + \dfrac{w^4}{4!} - \ldots$ and $e^w - 1 = w + \dfrac{w^2}{2!} + \ldots$.

So $\displaystyle\lim_{z \to 0} \frac{(\sin z^2)(\cos z - 1)}{e^{z^4} - 1} = \lim_{z \to 0} \frac{(z^2 - \frac{z^6}{3!} + \ldots)(-\frac{z^2}{2!} + \frac{z^4}{4!} - \ldots)}{z^4 + \frac{z^8}{2!} + \ldots} = \lim_{z \to 0} \frac{-\frac{z^4}{2} + \ldots}{z^4 + \ldots} = -\frac{1}{2}$. (Incidentally, from the power series, we can also see that 0 is a double root for $\sin z^2, \cos z - 1$ and a root of order 4 for $e^{z^4} - 1$.)

Fundamental Theorem of Algebra. Every nonconstant polynomial $P(z)$ with complex coefficients has a root in $\mathbb{C}$.

Proof. Suppose $P(z)$ has no root, then $f(z) = \dfrac{1}{P(z)}$ is entire. Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$ with $n \geq 1, a_n \neq 0$. As $z \to \infty$, eventually $|z| > 1$, then

$$\begin{aligned} |P(z)| &\geq |a_n z^n| - |a_{n-1}z^{n-1}| - \cdots - |a_0| = |z|^{n-1}\left(|a_n z| - |a_{n-1}| - \cdots - \left|\frac{a_0}{z^{n-1}}\right|\right) \\ &\geq |z|^{n-1}(|a_n z| - |a_{n-1}| - \cdots - |a_0|) \to \infty. \end{aligned}$$

So, $\displaystyle\lim_{z \to \infty} P(z) = \infty$ and $\displaystyle\lim_{z \to \infty} \frac{1}{P(z)} = 0$. Then $|f(z)| \leq 1$ when $|z|$ is large. By Liouville's theorem, $f(z)$ is a polynomial of degree 0, i.e. constant. Then $P(z) = \dfrac{1}{f(z)}$ is constant, a contradiction. **QED.**

Identity Theorem. Suppose f is holomorphic on a region D and the roots of f has a limit point in D (i.e. there are $a_n \in D$ such that $f(a_n) = 0$ and $a_n \to a \in D$), then $f \equiv 0$.

Uniqueness Theorem. Suppose g and h are holomorphic on a region D and $\{z\colon g(z) = h(z)\}$ has a limit point in D, then $g \equiv h$ (by considering $f = g - h$).

Proof of the Identity Theorem. Let $A = \{z \in D\colon z \text{ is a limit of roots of } f(z)\}$ and $B = D \setminus A$. Then $A \cup B = D, A \cap B = \emptyset$ and $A \neq \emptyset$ by hypothesis.

If $a \in A$, then $a = \lim_{n\to\infty} a_n$ with $f(a_n) = 0$. The power series of $f(z)$ at a, namely $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(z-a)^k$, has roots a_n with limit at the center a, so $f(z) = 0$ in a disk around a. Then every point in this disk belongs to A. Hence, this disk is a neighborhood of a in A. So A must be open.

If $b \in B$, then there is a disk around b such that $f(z) \neq 0$ in the disk (except possibly at b only). Then all points of this disk cannot be in A. Hence they must be in B. Then this disk is a neighborhood of b in B. So B must be open.

Since D is connected, $B = \emptyset$ and $A = D$. Therefore $f \equiv 0$. **QED.**

Examples. (1) The functions $g(z) = \sin 2z$ and $h(z) = 2\sin z\cos z$ are entire. Since $g(x) = h(x)$ for every real number x and the real numbers have limit points (*e.g.* $\frac{1}{n} \to 0$ as $n \to \infty$), by the uniqueness theorem, $g \equiv h$. Therefore $\sin 2z = 2\sin z\cos z$ for all complex number z.

(2) The function $f(z) = \sin(\frac{1}{z})$ is holomorphic on $D = \mathbb{C}\setminus\{0\}$. Its roots are $\frac{1}{n\pi}$ ($n \in \mathbb{Z}$). As $n \to \infty$, $\frac{1}{n\pi} \to 0$, but we cannot conclude $f(z) \equiv 0$ because 0 is **not** in D.

Mean Value Theorem. If f is holomorphic on a region D containing $\overline{B(a,r)}$, then (by the Cauchy integral formula) $f(a) = \frac{1}{2\pi i}\int\limits_{|z-a|=r} \frac{f(z)}{z-a}\,dz = \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{i\theta})\,d\theta$, which is the average of f on the circle $C(a,r)$.

Maximum Modulus Theorem. If f is a nonconstant holomorphic function on a region D and $B(a,r)$ is contained in D, then $|f(a)| < |f(w)|$ for some w in $B(a,r)$. (In addition, if D is bounded and f is also continuous on the boundary ∂D, then the maximum of $|f(z)|$ can only (and must) occur on ∂D.)

Proof. Let $0 < r' < r$ and $M = \max_{0\le\theta\le 2\pi} |f(a+r'e^{i\theta})|$. Since $|f(a+r'e^{i\theta})|$ is a continuous function on $[0,2\pi]$, $M = |f(a+r'e^{i\theta_0})|, 0 \le \theta_0 \le 2\pi$. Observe that

$$|f(a)| = \frac{1}{2\pi}\left|\int_0^{2\pi} f(a+r'e^{i\theta})\,d\theta\right| \le \frac{1}{2\pi}\int_0^{2\pi} |f(a+r'e^{i\theta})|\,d\theta \le \frac{1}{2\pi}\int_0^{2\pi} M\,d\theta = M.$$

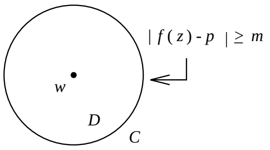
If $|f(a)| < M$, then $|f(a)| < |f(w)|$ for $w = a+r'e^{i\theta_0}$ and we are done. If $|f(a)| = M$, then $|f(a+r'e^{i\theta})| \equiv M$ because $|f(a+r'e^{i\theta})|$ is continuous in θ. If this is true for all $r' < r$, then $f \equiv f(a)$ on $B(a,r)$ by the Cauchy-Riemann equations and hence on D by the uniqueness theorem, contradicting the hypothesis. Otherwise $|f(a+r'e^{i\theta})| \not\equiv |f(a)|$ for some $r' < r$, then some $w = a + r'e^{i\theta_0}$ will satisfy $|f(a)| < |f(w)|$. **QED.**

Minimum Modulus Theorem. If f is a nonconstant holomorphic function on a region D and $B(a,r)$ is contained in D, then $|f(a)| > |f(w)|$ for some w in $B(a,r)$ unless $f(a) = 0$. (In addition, if D is bounded and f is also continuous on the boundary ∂D, then either f has a root in D or the minimum of $|f(z)|$ can only occur on ∂D.)

Proof. If $f(a) \neq 0$, then $f(z) \neq 0$ in some subdisk of $B(a,r)$. Apply the maximum modulus theorem to $\frac{1}{f(z)}$ on the subdisk. **QED.**

Open Mapping Theorem. If f is a nonconstant holomorphic function on an open set U, then $f(U)$ is an open set. (In particular, $f(U)$ has no boundary points.)

Proof. Let $w \in U$. (If each of the circles $C_k = C(w, \frac{1}{k})$ has a point w_k such that $f(w_k) = f(w)$, then since $w_k \to w \in U$, the uniqueness theorem will imply $f(z) \equiv f(w)$.) Now, since f is nonconstant, so there must be a circle $C = C(w, \frac{1}{k})$ such that $f(z) \neq f(w)$ for all z on C. Let $m = \frac{1}{2}\min_{z\in C}|f(z) - f(w)|$. We will show $B(f(w), m) \subset f(U)$. (This will imply every point in $f(U)$ has a neighborhood in $f(U)$, hence $f(U)$ is open.)



Let $p \in B(f(w), m)$ and consider the minimum modulus of $f(z) - p$ on $\overline{B(w,r)}$. For $z \in C, |f(z) - p| \geq |f(z) - f(w)| - |f(w) - p| \geq 2m - m = m$. However, $|f(w) - p| < m$. So the minimum modulus of $f(z) - p$ is not in C, hence $f(z) - p$ has a root inside C. Then $p \in f(U)$. So $B(f(w), m) \subset f(U)$. **QED.**

In particular, the open mapping theorem implies the range of a holomorphic function cannot be an arc or a curve or any closed set. We saw this before by using the Cauchy-Riemann equations.

Schwarz's Lemma. Let D be the open unit disk. If $f\colon D \to D$ is holomorphic and $f(0) = 0$, then $|f(z)| \leq |z|$ for any $z \in D \setminus \{0\}$ and $|f'(0)| \leq 1$. Equality holds in either case only for $f(z) \equiv e^{i\theta}z$, θ real.

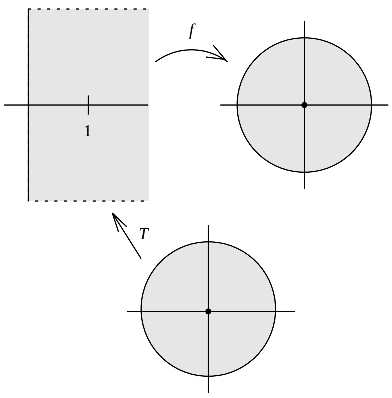
Proof. Define $g(z) = \begin{cases} \dfrac{f(z)}{z} & \text{if } 0 < |z| < 1 \\ f'(0) & \text{if } z = 0 \end{cases}$, then g is holomorphic on D. On the circle $|z| = r < 1$, $|g(z)| = \dfrac{|f(z)|}{|z|} \leq \dfrac{1}{r}$. Maximum modulus theorem implies $|g(z)| \leq \dfrac{1}{r}$ for $|z| \leq r$. Let $r \to 1^-$, we get $|g(z)| \leq 1$ for $|z| < 1$, which implies $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$.

If $|f(z)| = |z|$ for some $z \in D$ or $|f'(0)| = 1$, then $|g(z)| = 1$ for some $z \in D$. The maximum modulus theorem implies $g \equiv$ constant ($\equiv e^{i\theta}$), then $f(z) \equiv e^{i\theta}z$. **QED.**

Remarks. If we recall that every proper simply connected domain is conformally equivalent to the open unit disk D by the Riemann mapping theorem, then we see that Schwarz's lemma is not just about holomorphic mappings from D to D, but it can be viewed as a statement about mappings from simply connected domains to simply connected domains.

This is also true of many other theorems in complex analysis with settings on D. For theorems that are true for all simply connected domains, often all we need to prove is the case for the open unit disk, then the general case usually follows by the Riemann mapping theorem and conformal mappings.

Example. If f is holomorphic on the right half plane such that $|f(z)| \leq 1$ and $f(1) = 0$, what is the maximum of $|f(2)|$ as all such functions f are considered? Which ones attain the maximum?



Solution. Let $T(z) = \dfrac{1-z}{1+z}$, which maps the unit disk D onto the right half plane and let $g(z) = f(T(z))$, then g is holomorphic on D, $|g(z)| \leq 1$ and $g(0) = 0$. By Schwarz's lemma, $|f(T(z))| = |g(z)| \leq |z|$. (We want $T(z) = 2$, so take $z = -\frac{1}{3}$.) Then $|f(2)| = |f(T(-\frac{1}{3}))| \leq \frac{1}{3}$. So $\max|f(2)| = \frac{1}{3}$. For this to happen, we must have $|g(-\frac{1}{3})| = \frac{1}{3}$, then $f(T(z)) = g(z) = e^{i\theta}z$. Therefore, $f(w) = e^{i\theta}T^{-1}(w) = e^{i\theta}\dfrac{1-w}{1+w}$.

Exercises

1. If the range of an entire function lies in the right half plane $\mathrm{Re}\, w > 0$, show that the function is a constant function. [*Hint*: Compose with a Möbius mapping.]

2. Suppose a polynomial is bounded by 1 on the open unit disk. Show that all of its coefficients are bounded by 1.

3. Show that if f is holomorphic on $\{z : |z| \leq 1\}$, then there must be some positive integer k such that $f\left(\frac{1}{k}\right) \neq \frac{1}{k+1}$.

4. Suppose f is holomorphic on the annulus $\{z : 1 \leq |z| \leq 2\}, |f(z)| \leq 1$ for $|z| = 1$ and $|f(z)| \leq 4$ for $|z| = 2$. Prove that $|f(z)| \leq |z|^2$ throughout the annulus.

5. Let D be the open unit disk. If $f : D \to D$ is holomorphic with at least two fixed points (i.e. points w such that $f(w) = w$), show that $f(z) \equiv z$. [*Hint*: By composing with a suitable Möbius mapping, one of the fixed points may be moved to the origin.]

6. Let f be an entire function which is real on the real axis and imaginary on the imaginary axis, show that f is an odd function, i.e. $f(z) = -f(-z)$.

7. Suppose f is a nonconstant holomorphic function on the closed annulus $A = \{z : 1 \leq |z| \leq 2\}$. If f sends the boundary circles of A into the unit circle, show that f must have a root in A.

8. Use the uniqueness theorem to prove the identity $\sin(w + z) = \sin w \cos z + \cos w \sin z$ for all complex numbers w and z.

9. Find $\displaystyle\int_{|z|=1} \frac{dz}{z^4 + 4z^2}$, where the unit circle is given the counterclockwise orientation.

10. Find $\displaystyle\int_0^{2\pi} \cos(\cos\theta)\cosh(\sin\theta)\, d\theta$. [*Hint*: Consider $\cos e^{i\theta}$.]

11. Suppose f and g are holomorphic on the closed unit disk. Show that $|f(z)| + |g(z)|$ takes its maximum on the boundary. [*Hint*: Consider $f(z)e^{i\alpha} + g(z)e^{i\beta}$ for appropriate α and β.]

12. Suppose f is entire and $|f'(z)| \leq |z|$ for all z . Show that $f(z) = a + bz^2$ with $|b| \leq \frac{1}{2}$.

13. Find the maximum of $\left|f\left(\frac{1}{2}\right)\right|$, where f is holomorphic on $D = \{z : |z| < 2\}, f(1) = 0$ and $|f(z)| \leq 10$ for $z \in D$.

14. Find all holomorphic function(s) f defined on the open unit disk D satisfying $f\left(\frac{1}{2}\right) = \frac{2}{3}$ and $f(z) = (2 - f(z))f(2z)$ for all $z \in D$.

15. Let $H = \{z : \mathrm{Re}\, z > 0\}$. Suppose $f : H \to H$ is holomorphic and $f(1) = 1$. Show that $|1 - f(2)| \leq \frac{1}{3}|1 + f(2)|$.

16. If f is entire and $\mathrm{Re}\, f'(z) > 0$ for all complex numbers z, prove that f is a polynomial of degree 1.

17. Given the polynomial $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$. Prove that $\max_{|z|=1} |P(z)| \geq 1$.

18. If f is an entire function mapping the unit circle into the unit circle (i.e. $|f(z)| = 1$ for $|z| = 1$), then $f(z) = e^{i\theta}z^n$ for some real θ and some positive integer n. [*Hint*: In the unit disk, f has finitely many roots $\alpha_1, \ldots, \alpha_n$, repeated according to multiplicities. Recall $\left|\frac{z - \alpha_j}{1 - \overline{\alpha_j}z}\right| = 1$ for $|z| = 1$. Use the modulus theorems to show $f(z) = e^{i\theta}\prod_{j=1}^{n}\frac{z - \alpha_j}{1 - \overline{\alpha_j}z}$ first.]

19. Let f and g be holomorphic on a domain U. If $f\overline{g}$ is holomorphic on U, show that either $f \equiv 0$ or g is a constant function.

20. Let f be holomorphic on the open unit disk D. Show that there is a sequence $\{z_n\}$ in D with $|z_n| \to 1$ such that $\{f(z_n)\}$ is a bounded sequence. [*Hint*: Consider the roots of f.]

21. Let w and z be in the open unit disk D. If $f : D \to D$ is holomorphic and $f(w) = z$, prove that $|f'(w)| \leq \dfrac{1 - |z|^2}{1 - |w|^2}$.

22. (*Study's Theorem*) Let G be a convex domain and f be a one-to-one holomorphic function from the open unit disk onto G. Prove that $f(B(0, r))$ is a convex domain for $0 < r < 1$. (Recall a set S is convex if $w, z \in S$ implies $tw + (1 - t)z \in S$ for all$t \in [0, 1]$.)

13. Harmonic Functions and Conjugates

Definitions. A real-valued function $u(x,y)$ having continuous second order partial derivatives on a region D is *harmonic* on D iff it satisfies the Laplace equation $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}\equiv 0$ on D. If v is harmonic on D and $u+iv$ is holomorphic on D, then we say v is a *harmonic conjugate* of u on D.

Theorem. If $f=u+iv$ is holomorphic on D, then u (and similarly, v) is harmonic on D.

Proof. By the Cauchy-Riemann equations, we get $\dfrac{\partial^2 u}{\partial x^2}=\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right)=\dfrac{\partial}{\partial x}\left(\dfrac{\partial v}{\partial y}\right)=\dfrac{\partial}{\partial y}\left(\dfrac{\partial v}{\partial x}\right)=-\dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial y}\right)=-\dfrac{\partial^2 u}{\partial y^2}$. Similarly, $\dfrac{\partial^2 v}{\partial x^2}=-\dfrac{\partial^2 v}{\partial y^2}$. **QED.**

Remark. In general, the converse of the theorem is false, which can be seen from the example $u(x,y)=\ln\sqrt{x^2+y^2}$ on $\mathbb{C}\setminus\{0\}$. If $u=\operatorname{Re} f$ there, then locally, the Cauchy-Riemann equations imply $(f(z)-\log z)'=\dfrac{\partial}{\partial x}(u(z)-\ln|z|)-i\dfrac{\partial}{\partial y}(u(z)-\ln|z|)=0$. So the only possibility is $f(z)=\log z+\text{constant}$, which is not continuous on $\mathbb{C}\setminus\{0\}$. However, for simply connected domains, the converse of the theorem is true.

Theorem. If u is harmonic on a simply connected region D, then u is the real part of a holomorphic function f on D.

Proof. Define $g(z)=\dfrac{\partial u}{\partial x}(z)-i\dfrac{\partial u}{\partial y}(z)$, then g is holomorphic on D because u has continuous second order partial derivatives and the Cauchy-Riemann equations $\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right)=\dfrac{\partial}{\partial y}\left(-\dfrac{\partial u}{\partial y}\right)$, $\dfrac{\partial}{\partial x}\left(-\dfrac{\partial u}{\partial y}\right)=-\dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right)$ are satisfied on D. By the integral theorem, g has an antiderivative f, i.e. $f'(z)=g(z)$. Now

$$\frac{\partial}{\partial x}(\operatorname{Re} f-u)=\operatorname{Re} f_x-\frac{\partial u}{\partial x}=\operatorname{Re} f'-\frac{\partial u}{\partial x}=0 \text{ and } \frac{\partial}{\partial y}(\operatorname{Re} f-u)=\operatorname{Re} f_y-\frac{\partial u}{\partial y}=\operatorname{Re}(if')-\frac{\partial u}{\partial y}=0.$$

So $\operatorname{Re} f-u\equiv\text{constant}$, i.e. $u\equiv\operatorname{Re}(f+\text{constant})$ on D. **QED.**

Below we shall give some rules for recovering a holomorphic function $f=u+iv$ from u (and v). Since no proofs of these rules will be offered, proper checking should be made to ensure the functions obtained have the correct real and imaginary parts.

Rule A. If u is harmonic on D containing an interval of the real axis, then $f'(z)=\dfrac{\partial u}{\partial x}(z,0)-i\dfrac{\partial u}{\partial y}(z,0)$ and f is obtained by integrating f'.

Rule B. If u is harmonic on $B(z_0,r)$, $z_0=x_0+iy_0$, then take $f(z)=2u\left(\dfrac{z+\overline{z}_0}{2},\dfrac{z-\overline{z}_0}{2i}\right)-u(x_0,y_0)$.

Rule C. If u, v are harmonic conjugate on $B(0,r)$, then take $f(z)=u(z,0)+iv(z,0)$.

Examples. (1) $u(x,y)=x^2-y^2$, $\dfrac{\partial u}{\partial x}(x,y)=2x=-\dfrac{\partial v}{\partial y}(x,y)$, $\dfrac{\partial u}{\partial y}(x,y)=-2y=\dfrac{\partial v}{\partial x}(x,y)$, which imply $v(x,y)=2xy+\text{constant}$.

Rule A: $f'(z)=2z-i0\Rightarrow f(z)=z^2$.

Rule B: Take $z_0 = 0$, $f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) = 2\left(\frac{z^2}{4} + \frac{z^2}{4}\right) = z^2$.
Rule C: Take $v(x, y) = 2xy$. Then $f(z) = (z^2 - 0) + i0 = z^2$.

(2) $u(x, y) = e^{-y}\sin x$, $\frac{\partial u}{\partial x}(x, y) = e^{-y}\cos x = -\frac{\partial v}{\partial y}(x, y)$, $\frac{\partial u}{\partial y}(x, y) = -e^{-y}\sin x = \frac{\partial v}{\partial x}(x, y)$, which imply $v(x, y) = -e^{-y}\cos x + \text{constant}$.

Rule A: $f'(z) = \cos z + i\sin z = e^{iz} \Rightarrow f(z) = -ie^{iz}$.
Rule B: Take $z_0 = 0, f(z) = 2e^{-z/2i}\sin\frac{z}{2} = e^{-z/2i}\frac{e^{iz/2} - e^{-iz/2}}{i} = -ie^{iz} + i$.
Rule C: Take $v(x, y) = -e^{-y}\cos x$. Then $f(z) = \sin z + i(-\cos z) = -ie^{iz}$.

(3) $u(x, y) = \frac{x}{x^2 + y^2}$, $\frac{\partial u}{\partial x}(x, y) = \frac{y^2 - x^2}{(x^2 + y^2)^2} = -\frac{\partial v}{\partial y}(x, y)$, $\frac{\partial u}{\partial y}(x, y) = \frac{-2xy}{(x^2 + y^2)^2} = \frac{\partial v}{\partial x}(x, y)$, which imply $v(x, y) = -\frac{y}{x^2 + y^2} + \text{constant}$.

Rule A: $f'(z) = -\frac{z^2}{z^4} = -\frac{1}{z^2} \Rightarrow f(z) = \frac{1}{z}$.
Rule B: Take $z_0 = 1$, $f(z) = 2\frac{(z + 1)/2}{(z + 1)^2/4 + (z - 1)^2/-4} - 1 = \frac{1}{z}$.
Rule C: Although $u(0, 0)$ is undefined, we will try and see. Take $v(x, y) = -\frac{y}{x^2 + y^2}$. Then $f(z) = \frac{1}{z} + 0 = \frac{1}{z}$.

Mean-Value Theorem for Harmonic Functions. If u is harmonic on $B(z_0, R)$, then for $0 \le r < R$, $u(z_0) = \operatorname{Re} f(z_0) = \operatorname{Re}\left(\frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta\right) = \frac{1}{2\pi}\int_0^{2\pi}\operatorname{Re} f(z_0 + re^{i\theta})d\theta = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta$.

Maximum/Minimum Principle for Harmonic Functions. If u is a nonconstant harmonic function on an open set D, then u has no maximum or minimum on D. (If D is bounded and u is also continuous on D and its boundary ∂D, then u attains maximum and minimum values on ∂D only.)

Proof. Suppose u has a maximum or minimum at $p \in D$. Let $D_0 = B(p, \varepsilon)$ be a disk in D. Then u has a maximum or minimum at p on D_0. Now $u = \operatorname{Re} f$ on D_0 for some nonconstant holomorphic function f on D_0. By the open mapping theorem, $f(D_0)$ is open in $\mathbb{C}$. Hence $\operatorname{Re} f(D_0) = \{\operatorname{Re} f(z)\colon z \in D_0\} = \{u(z)\colon z \in D_0\}$ is open in $\mathbb{R}$. Then u cannot have a maximum or minimum on $D_0 \subset D$, a contradiction. **QED.**

Corollary. If u_1, u_2 are harmonic on a bounded open set D, continuous on D and its boundary ∂D, $u_1 = u_2$ on ∂D, then (considering $u = u_1 - u_2$, we get) $u_1 \equiv u_2$ on D. So, the function $u \mapsto u|_{\partial D}$ is one-to-one.

Poisson Integral Formula. If u is harmonic on $B(0, 1)$ and continuous on $\overline{B(0, 1)}$, then for $z = re^{i\theta} \in B(0, 1)$, $u(z) = \frac{1}{2\pi}\int_0^{2\pi}\frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2}u(e^{it})\,dt$. This formula gives an inverse to the function $u \mapsto u|_{\partial B(0,1)}$.

Proof. Since $B(0, 1)$ is simply connected, $u = \operatorname{Re} f$ for some holomorphic function f on $B(0, 1)$. For $r < R < 1$, the Cauchy integral formula gives

$$f(z) = \frac{1}{2\pi i}\int\limits_{|w|=R}\frac{f(w)}{w - z}\,dw - \underbrace{\frac{1}{2\pi i}\int\limits_{|w|=R}\frac{f(w)}{w - \frac{R^2}{\overline{z}}}\,dw}_{=0} = \frac{1}{2\pi i}\int\limits_{|w|=R}\left(\frac{1}{w - z} - \frac{1}{w - \frac{R^2}{\overline{z}}}\right)f(w)\,dw.$$

For $w = Re^{it}$ and $z = re^{i\theta}$, we have $dw = iRe^{it}\,dt$ and

$$\begin{aligned}\frac{1}{w-z} - \frac{1}{w - \frac{R^2}{\overline{z}}} &= \frac{1}{Re^{it} - re^{i\theta}} - \frac{1}{Re^{it} - \frac{R^2}{r}e^{i\theta}} = \frac{(r - \frac{R^2}{r})e^{i\theta}}{(Re^{it} - re^{i\theta})(Re^{it} - \frac{R^2}{r}e^{i\theta})} \\ &= \frac{(R^2 - r^2)}{(Re^{i(t-\theta)} - r)(Re^{i(\theta-t)} - r)Re^{it}} = \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - t) + r^2}\frac{1}{Re^{it}}.\end{aligned}$$

So, $f(z) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - t) + r^2} f(Re^{it})\,dt$. Taking the real part of both sides and letting $R \to 1^-$, we get $u(z) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2} u(e^{it})\,dt$. **QED.**

Remark. In an advanced course, it can be shown that if u is continuous on $C(0,1)$, then $u(z)$, defined by the Poisson integral above, is harmonic on $B(0,1)$ and continuous on $\overline{B(0,1)}$.

Exercises

1. If $f = u + iv$ is holomorphic, prove that $u + v$ and uv are harmonic.

2. If u is harmonic, prove that all the partial derivatives of u are harmonic, but u^2 is not harmonic, unless u is constant.

3. Find holomorphic functions whose real parts are

   (a) $x^3y - xy^3$;

   (b) $e^{x^2-y^2}\cos(2xy)$;

   (c) $\arctan\left(\dfrac{y}{x}\right)$;

   (d) $\dfrac{1 - x^2 - y^2}{(1-x)^2 + y^2}$.

4. Is the parenthetical part of the maximum/minimum principle true if D is unbounded? [*Hint*: Consider $u(x, y) = \pm y$ on $\{z : \operatorname{Im} z \geq 0\}$.]

5. Using the Poisson integral formula, find $\displaystyle\int_0^{2\pi} \frac{dt}{5 - 4\cos t}$ and $\displaystyle\int_0^{2\pi} \frac{3e^{\cos t}\cos(\sin t)}{5 - 4\cos(t - \sqrt{2})}\,dt$.

6. Show that in polar coordinates, Laplace's equation is $\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0$.

7. (a) Suppose g is holomorphic on the closed unit disk, $g(0) = 1$ and $\operatorname{Re} g(z) > 0$ for $|z| < 1$. Use the Poisson integral formula to show that $\operatorname{Re} g(z) \leq \dfrac{1 + |z|}{1 - |z|}$ for $|z| < 1$.

   (b) Suppose $f : \{z : |z| \leq 1\} \to \{z : 0 < |z| < 1\}$ is holomorphic. Show that $|f(z)| \leq |f(0)|^{\frac{1+|z|}{1-|z|}}$ for $|z| < 1$.

14. Morera's Theorem

The rectangle theorem asserts that if f is holomorphic, then f has the rectangle property. Below we shall prove the converse for continuous functions. Thus, holomorphicity is equivalent to the rectangle property.

Morera's Theorem. If a continuous function f has the rectangle property on an open set U, then f is holomorphic on U.

Proof. For $a \in U$, take a disk $B(a,r)$ in U. Since f has the rectangle property, the integral theorem implies f has an antiderivative F on $B(a,r)$. That is, F is differentiable on $B(a,r)$ and $F' = f$. Then F is infinitely differentiable, so f is differentiable on $B(a,r)$. Since a is arbitrary, f is holomorphic on U. **QED.**

Example. Define $F(z) = \displaystyle\int_0^\infty \frac{e^{zt}}{t+1}\,dt$ for $U = \{z\colon \operatorname{Re} z < 0\}$. (Observe that $\displaystyle\int_0^\infty \left|\frac{e^{zt}}{t+1}\right| dt \le \int_0^\infty e^{xt}\,dt$ $= -\dfrac{1}{x} = \dfrac{1}{|x|}$ for $x = \operatorname{Re} z < 0$. So, F is defined on U.) Now, let Γ be a "rectangle" in U. We have

$$\int_\Gamma \int_0^\infty \frac{e^{zt}}{t+1}\,dt\,dz = \int_0^\infty \int_\Gamma \frac{e^{zt}}{t+1}\,dz\,dt = 0$$

because for a fixed positive t, e^{zt} is holomorphic on U. (The interchange of integrals is valid because

$$\int_\Gamma \int_0^\infty \left|\frac{e^{zt}}{t+1}\right| dt\,dz \le \int_\Gamma \frac{1}{|x|}\,dz < \infty,$$

since $x = \operatorname{Re} z$ is bounded away from 0 on Γ.) Therefore, F is holomorphic on U by Morera's theorem.

As an application of Morera's theorem, we will consider limits of holomorphic functions. In general, the limit of a sequence of continuous functions may not be continuous (*e.g.* $f_n(x) = x^n$, $0 \le x \le 1$ has the limit $f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \end{cases}$, which is **not** continuous). So the limit of a sequence of holomorphic functions may not be holomorphic. However, if the convergence to the limit is uniform, then it is holomorphic. In that case, the derivatives will also converge uniformly to the derivative of the limit.

Weierstrass' Theorem. If f_n is a sequence of holomorphic functions on a region D and f_n converges uniformly to f on every closed disk $\overline{B(a,r)} = \{z\colon |z-a| \le r\}$ in D, then f is holomorphic on D. (Furthermore, for any positive integer k, $f_n^{(k)}$ will also converge uniformly to $f^{(k)}$ on every closed disk in D, i.e. $\displaystyle\lim_{n\to\infty} \frac{d^k}{dz^k} f_n = \frac{d^k}{dz^k} \lim_{n\to\infty} f_n$ uniformly on D.)

Proof. For every a in D, take a closed disk $\overline{B(a,r)}$ in D. Since the f_n's are continuous and the convergence is uniform, f is continuous on $\overline{B(a,r)}$. For a "rectangle" Γ in $B(a,r)$, $\displaystyle\int_\Gamma f(z)\,dz = \int_\Gamma \lim_{n\to\infty} f_n(z)\,dz =$ $\displaystyle\lim_{n\to\infty} \int_\Gamma f_n(z)\,dz = 0$. By Morera's theorem, f is holomorphic at a for any $a \in D$.

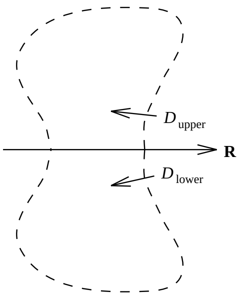
(For the parenthetical statement, let k be a positive integer, $\overline{B(a,r)} \subset D$ and $\varepsilon > 0$ be given. Then there is $R > r$ such that $\overline{B(a,r)} \subset \overline{B(a,R)} \subset D$. Fix a ρ such that $0 < \rho < R - r$. Since f_n converges uniformly to f on $\overline{B(a,R)}$, there is N such that $n \ge N \Rightarrow |f_n(w) - f(w)| < \varepsilon$ for all $w \in \overline{B(a,R)}$. Now for any arbitrary $z \in \overline{B(a,r)}$, $C(z,\rho) \subset \overline{B(a,R)}$. By the Cauchy integral formula for derivatives and the M-L inequality,

$$\left|f_n^{(k)}(z) - f^{(k)}(z)\right| = \left|\frac{k!}{2\pi i}\int_{|w-z|=\rho} \frac{f_n(w)-f(w)}{(w-z)^{k+1}}\,dw\right| \le \frac{k!}{2\pi}\left(\frac{1}{\rho^{k+1}}\varepsilon\right)2\pi\rho = \frac{k!\varepsilon}{\rho^k}.$$

Since z is arbitrary in $\overline{B(a,r)}$, uniform convergence is proved.) **QED.**

The next type of applications of Morera's theorem concerns extending holomorphic functions across line segments.

Schwarz Reflection Principle. Let D be a symmetric region with respect to the real axis, $D_{\text{upper}} = \{z \in D : \operatorname{Im} z > 0\}$ and $D_{\text{lower}} = \{z \in D : \operatorname{Im} z < 0\}$. If f is continuous on $D_{\text{upper}} \cup (D \cap \mathbb{R})$, holomorphic on D_{upper} and real-valued on $D \cap \mathbb{R}$, then f can be extended to a holomorphic function on D by defining $f(z) = \overline{f(\overline{z})}$ for $z \in D_{\text{lower}}$.



Proof. The hypothesis that f is real-valued on $D \cap \mathbb{R}$ is equivalent to $f(x) = \overline{f(x)}$ for all $x \in D \cap \mathbb{R}$. This makes the extension continuous on all of D. On D_{lower}, we can apply the definition of derivative to get

$$f'(z) = \lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = \lim_{h\to 0}\overline{\frac{f(\overline{z}+\overline{h})-f(\overline{z})}{\overline{h}}} = \overline{f'(\overline{z})}$$

for all $z \in D_{\text{lower}}$. So, f is holomorphic on $D \setminus \mathbb{R}$. Then the extension theorem and Morera's theorem imply f is holomorphic on D.

QED.

Exercises

1. Use Morera's Theorem to show that $f(z) = \displaystyle\int_0^1 \frac{\sin zt}{t}\,dt$ is an entire function.

2. Prove that if f is continuous on the closed unit disk $\{z : |z| \le 1\}$, holomorphic on the open unit disk $\{z : |z| < 1\}$ and real-valued on the unit circle $\{z : |z| = 1\}$, then f is a constant function.

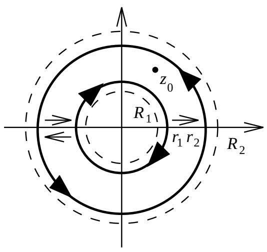
3. Let f be an entire function which is real on the real axis.
   (a) Use Schwarz reflection principle to prove that $f(z) = \overline{f(\overline{z})}$ for every complex number z.
   (b) If f is also imaginary on the imaginary axis, prove that f is an odd function by considering the power series of f at the origin.

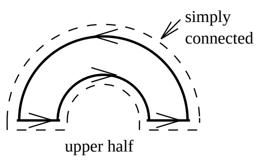
15. Isolated Singularities

In this section, we will consider a two-sided infinite series representation near a singular point in the domain of a holomorphic function.

Definition. We say $\sum_{k=-\infty}^{\infty} w_k$ *converges* to L iff $\sum_{k=-\infty}^{-1} w_k$, $\sum_{k=0}^{\infty} w_k$ converge and $\sum_{k=-\infty}^{-1} w_k + \sum_{k=0}^{\infty} w_k = L$.

Laurent Series Representation. If f is holomorphic on the annulus $A = \{z\colon 0 \le R_1 < |z-a| < R_2 \le \infty\}$, then for all $z \in A$, $f(z) = \sum_{k=-\infty}^{\infty} a_k (z-a)^k$, where $a_k = \frac{1}{2\pi i} \int\limits_{|w-a|=r} \frac{f(w)}{(w-a)^{k+1}}\, dw$ for any r between R_1 and R_2. (Also the convergence is absolute on A, uniform on any smaller annulus $\{z\colon R_1 < R_1' \le |z-a| \le R_2' < R_2\}$ and the coefficients a_k are <u>unique for A</u> .) The series is called the *Laurent series* of $f(z)$ on A.





Proof. By a change of variable, we may assume $a = 0$. Let $R_1 < r_1 < r_2 < R_2$. Also, let C_1, C_2 be the counterclockwise circles $|z| = r_1, |z| = r_2$. For $r_1 < |z_0| < r_2$, the function $g(z) = \begin{cases} \dfrac{f(z)-f(z_0)}{z-z_0} & \text{if } z \ne z_0, z \in A \\ f'(z_0) & z = z_0 \end{cases}$ is holomorphic on A.

Taking the line segments $[r_1, r_2]$ and $[-r_2, -r_1]$ as cross-cuts, we get by Cauchy's theorem, $\int\limits_{C_2 - C_1} g(w)\, dw = 0$. Using the definition of $g(z)$ and grouping terms, we get

$$\int_{C_2} \frac{f(w)}{w-z_0} dw - \int_{C_1} \frac{f(w)}{w-w_0} dw = f(z_0) \left(\int_{C_2} \frac{dw}{w-z_0} - \int_{C_1} \frac{dw}{w-z_0} \right) = 2\pi i f(z_0).$$

For $w \in C_2, |z_0| < |w| = r_2, \dfrac{1}{w-z_0} = \dfrac{1}{w(1-\frac{z_0}{w})} = \dfrac{1}{w}\left(1 + \dfrac{z_0}{w} + \left(\dfrac{z_0}{w}\right)^2 + \dots\right) = \sum_{k=0}^{\infty} \dfrac{z_0^k}{w^{k+1}}$.

For $w \in C_1, |w| = r_1 < |z_0|, \dfrac{1}{w-z_0} = \dfrac{-1}{z_0(1-\frac{w}{z_0})} = -\dfrac{1}{z_0}\left(1 + \dfrac{w}{z_0} + \left(\dfrac{w}{z_0}\right)^2 + \dots\right) = -\sum_{k=-\infty}^{-1} \dfrac{z_0^k}{w^{k+1}}$.

Both series converge uniformly in w because $\left|\dfrac{z_0}{w}\right| < 1$ on C_2 and $\left|\dfrac{w}{z_0}\right| < 1$ on C_1. So, for $R_1 < r < R_2$,

$$2\pi i f(z_0) = \underbrace{\sum_{k=0}^{\infty} \int_{C_2} \frac{f(w) z_0^k}{w^{k+1}}\, dw}_{I} + \underbrace{\sum_{k=-\infty}^{-1} \int_{C_1} \frac{f(w) z_0^k}{w^{k+1}}\, dw}_{II} = \sum_{k=-\infty}^{\infty} \bigg(\int\limits_{|w|=r} \frac{f(w)}{w^{k+1}}\, dw \bigg) z_0^k.$$

(Observe that $\int_{C_2} = \int\limits_{|w|=r} = \int_{C_1}$ by using cross-cuts as above.) Letting $r_1 \to R_1$ and $r_2 \to R_2$, we get the result. (For the parenthetical statement, we observe that the series I and II are power series in z_0 and z_0^{-1}, respectively. Hence, the absolute and uniform convergence properties and the uniqueness of coefficients follow from the corresponding properties for power series.) **QED.**

An important point of the theorem is that the Laurent series coefficients for a holomorphic function on an annulus are unique. (To elaborate on this, suppose $f(z) = \sum_{k=-\infty}^{\infty} b_k(z-a)^k$ uniformly on the same annulus, then $\frac{1}{2\pi i}\int_{|w-a|=r} \frac{f(w)}{(w-a)^{n+1}}dw = \sum_{k=-\infty}^{\infty} b_k\left(\frac{1}{2\pi i}\int_{|w-a|=r}(w-a)^{k-n-1}\,dw\right) = b_n$. Therefore, the coefficients b_k's are the same as the coefficients a_k's in the Laurent series representation above.) So in computing Laurent series, we do not have to compute the coefficients by integrals. Instead, we can apply other methods (such as the formula for summing geometric series or power series of common functions,) depending on the given functions and annuli.

Example. Find the Laurent series of $f(z) = \frac{1}{z^2(1-z)}$ for $A_1 = \{z\colon 0 < |z| < 1\}, A_2 = \{z\colon 1 < |z| < \infty\}$ and $A_3 = \{z\colon 0 < |z-1| < 1\}$.

Solution. On A_1, $\frac{1}{z^2(1-z)} = \frac{1}{z^2}(1+z+z^2+\ldots) = \frac{1}{z^2}+\frac{1}{z}+1+z+z^2+\ldots$.

On A_2, $\left|\frac{1}{z}\right| < 1$, $\frac{1}{z^2(1-z)} = -\frac{1}{z^3}\left(\frac{1}{1-\frac{1}{z}}\right) = -\frac{1}{z^3}(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\ldots) = \ldots - \frac{1}{z^6}-\frac{1}{z^5}-\frac{1}{z^4}-\frac{1}{z^3}$.

On A_3, the series is of the form $\sum_{k=-\infty}^{\infty} a_k(z-1)^k$. We make use of the fact $|z-1|<1$ and write $z = 1+(z-1)$, so $\frac{1}{z^2(1-z)} = \frac{1}{(1+(z-1))^2}\cdot\frac{1}{(1-z)}$. For $w = z-1$, $|w|<1$, $\frac{1}{(1+w)^2} = -\frac{d}{dw}\left(\frac{1}{1+w}\right) = -\sum_{n=0}^{\infty}\frac{d}{dw}(-w)^n = \sum_{n=1}^{\infty}(-1)^{n+1}nw^{n-1}$, so

$$\frac{1}{z^2(1-z)} = -\frac{1}{(z-1)}[1-2(z-1)+3(z-1)^2-4(z-1)^3+\ldots] = -\frac{1}{z-1}+2-3(z-1)+4(z-1)^2-\ldots.$$

Definitions. If $f(z)$ is holomorphic in a deleted neighborhood of z_0 (i.e. on some $B(z_0,\varepsilon)\setminus\{z_0\}$), then $f(z)$ is said to have an *isolated singularity* at z_0. The *Laurent expansion of* $f(z)$ *at* z_0 is the Laurent series of $f(z)$ on $A = \{z\colon 0 < |z-z_0| < \varepsilon\}$. If it is of the form

(i) $a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ \ldots$, we say z_0 is a *removable singularity* of $f(z)$ (in this case, we can define $f(z_0) = a_0$ and $f(z)$ becomes holomorphic at z_0 because the Laurent series is the same as the power series of $f(z)$ in $B(z_0,\varepsilon)$);

(**Example:** $f(z) = \frac{\sin z}{z}$ has a removable singularity at 0. For $0 < |z| < \varepsilon$, $\frac{\sin z}{z} = \frac{z - \frac{z^3}{3!}+\frac{z^5}{5!}-\ldots}{z}$ $= 1 + 0z - \frac{z^2}{3!} + 0z^3 + \frac{z^4}{5!} - \ldots$. So defining $f(0) = 1$ will make $f(z)$ holomorphic at 0.)

(ii) $\frac{a_{-k}}{(z-z_0)^k} + \frac{a_{-k+1}}{(z-z_0)^{k-1}} + \ \ldots + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \ \ldots$ with $a_{-k} \neq 0$, we say z_0 is a *pole of order* k for $f(z)$ (in this case, $F(z) = (z-z_0)^k f(z)$ is holomorphic at z_0, $F(z_0) = a_{-k} \neq 0$ and $\lim_{z\to z_0} f(z) = \lim_{z\to z_0}\frac{F(z)}{(z-z_0)^k} = \infty$);

(**Example:** $f(z) = \frac{1}{\sin z}$ has a pole (of order 1) at 0. For $0 < |z| < \varepsilon$,

$$\frac{1}{\sin z} = \frac{1}{z(1-\frac{z^2}{3!}+\frac{z^4}{5!}-\ldots)} = \frac{1}{z}(1+\frac{z^2}{6}+\frac{7}{360}z^4+\ldots) = \frac{1}{z}+0+\frac{1}{6}z+0z^2+\frac{7}{360}z^3+\ldots.)$$

(iii) $\ldots + \dfrac{a_{-k}}{(z-z_0)^k} + \dfrac{a_{-k+1}}{(z-z_0)^{k-1}} + \ldots + \dfrac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \ldots$, we say z_0 is an *essential singularity* of $f(z)$.

(**Example:** $f(z) = \sin\dfrac{1}{z}$ has an essential singularity at 0. For $0 < |z| < \varepsilon$,

$$\sin\frac{1}{z} = \frac{1}{z} - \frac{1}{3!}\frac{1}{z^3} + \frac{1}{5!}\frac{1}{z^5} - \ldots = \ldots + \frac{1}{120z^5} + \frac{0}{z^4} - \frac{1}{6z^3} - \frac{0}{z^2} + \frac{1}{z} + 0 + 0z + \ldots.)$$

Definition. If the Laurent series of $f(z)$ on $0 < |z-z_0| < \varepsilon$ is $\displaystyle\sum_{k=-\infty}^{\infty} a_k(z-z_0)^k$, then $\displaystyle\sum_{k=-\infty}^{-1} a_k(z-z_0)^k$ is the *principal part* of f at z_0 and $\displaystyle\sum_{k=0}^{\infty} a_k(z-z_0)^k$ is the *holomorphic part* of f at z_0.

Theorem. Let z_0 be an isolated singularity of $f(z)$.
(i) **(Riemann's principle.)** If $\lim_{z\to z_0}(z-z_0)f(z) = 0$ (*e.g.* $|f(z)| \le M$ near z_0), then z_0 is a removable singularity.
(ii) If there is a positive integer k such that $\lim_{z\to z_0}(z-z_0)^k f(z) \ne 0$, but $\lim_{z\to z_0}(z-z_0)^{k+1} f(z) = 0$, then z_0 is a pole of order k.

Proof. (i) The function $g(z) = \begin{cases} (z-z_0)f(z) & \text{if } z \ne z_0, \\ 0 & \text{if } z = z_0, \end{cases}$ is holomorphic near z_0 and continuous at z_0. By the extension theorem, g is holomorphic at z_0. So, near z_0, $f(z) = \dfrac{g(z)}{z-z_0} = \displaystyle\sum_{k=0}^{\infty} \frac{g^{(k+1)}(z_0)}{(k+1)!}(z-z_0)^k$. Therefore, z_0 is a removable singularity of f.

(ii) By (i), $\lim_{z\to z_0}(z-z_0)[(z-z_0)^k f(z)] = 0$ implies $(z-z_0)^k f(z)$ has a removable singularity at z_0. So, near z_0, $(z-z_0)^k f(z) = \displaystyle\sum_{n=0}^{\infty} c_n(z-z_0)^n$. Now $c_0 = \lim_{z\to z_0}(z-z_0)^k f(z) \ne 0$. Therefore, near z_0, $f(z) = \dfrac{c_0}{(z-z_0)^k} + \dfrac{c_1}{(z-z_0)^{k-1}} + \cdots$, i.e. f has a pole of order k at z_0. **QED.**

Casorati-Weierstrass Theorem. If z_0 is an essential singularity of $f(z)$, then the range set $f(D_*) = \{f(z)\colon 0 < |z-z_0| < \varepsilon\}$ is dense in $\mathbb{C}$ for every $\varepsilon > 0$. (Here "dense" means in every open disk in $\mathbb{C}$, regardless of size and location, we can find a point of $f(D_*)$.)

Proof. Suppose $f(D_*)$ is not dense in $\mathbb{C}$ for some $\varepsilon > 0$. Then $f(D_*)$ misses some disk $B(a,r)$, i.e. $|f(z) - a| \ge r$ for $0 < |z-z_0| < \varepsilon$. It follows that $g(z) = \dfrac{1}{f(z)-a}$ is bounded (by $\dfrac{1}{r}$) near z_0. By Riemann's principle, z_0 is a removable singularity of $g(z)$, i.e. g can be defined at z_0 so as to be holomorphic there. Now $f(z) = a + \dfrac{1}{g(z)}$. If $g(z_0) \ne 0$, then $\lim_{z\to z_0}(z-z_0)f(z) = 0$, which forces f to have a removable singularity at z_0, a contradiction. If $g(z_0) = 0$, then $g(z) = (z-z_0)^k h(z)$ for some positive integer k and some holomorphic function $h(z)$ near z_0 with $h(z_0) \ne 0$. It follows that $\lim_{z\to z_0}(z-z_0)^k f(z) = \dfrac{1}{h(z_0)} \ne 0$ and $\lim_{z\to z_0}(z-z_0)^{k+1} f(z) = 0$. Then f has a pole of order k at z_0, a contradiction. **QED.**

In an advanced course, a theorem called the *Great Picard Theorem* is usually proved, which asserts that $f(D_*)$ can miss at most one complex number and in fact, the equation $f(z) = c$ has infinitely many solutions in D_*, except perhaps for one complex value c. (For example, 0 is an essential singularity of $e^{1/z}$ and in every neighborhood of 0, the range of $e^{1/z}$ is $\mathbb{C}\setminus\{0\}$.) This is stronger than the Casorati-Weierstrass theorem.

Summary. Let z_0 be an isolated singularity of f. Then

(i) z_0 is a removable singularity of $f \iff \lim_{z\to z_0} f(z) \in \mathbb{C} \iff f(z)$ is bounded near z_0 (i.e. $|f(z)| \le M$ near z_0 for some M) $\iff \lim_{z\to z_0} (z-z_0)f(z) = 0$;

(ii) z_0 is a pole of $f \iff \lim_{z\to z_0} f(z) = \infty$; (also, z_0 is a pole of order $k \iff \lim_{z\to z_0} (z-z_0)^k f(z) \neq 0$ and $\lim_{z\to z_0} (z-z_0)^{k+1} f(z) = 0$;)

(iii) z_0 is an essential singularity of $f \iff f(D_*) = \{f(z)\colon 0 < |z-z_0| < \varepsilon\}$ is dense in $\mathbb{C}$ for any $\varepsilon > 0 \iff \lim_{z\to z_0} f(z)$ doesn't exist.

(**Reasons.**

(i) If z_0 is a removable singularity, then near z_0, $f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k$, so $\lim_{z\to z_0} f(z) = a_0 \in \mathbb{C}$.
If $\lim_{z\to z_0} f(z) = A \in \mathbb{C}$, then for each $\varepsilon > 0$, there is $\delta > 0$ such that $0 < |z-z_0| < \varepsilon \Rightarrow |f(z) - A| < \delta$. So $|f(z)| < A + \delta$ in $0 < |z-z_0| < \varepsilon$.
If $|f(z)| \le M$ near z_0, then $|(z-z_0)f(z)| \le M|z-z_0| \to 0$ as $z \to z_0$.
If $\lim_{z\to z_0} (z-z_0)f(z) = 0$, then z_0 is a removable singularity by Riemann's Principle.

(ii) If z_0 is a pole (of order k), then from the definition, we have $\lim_{z\to z_0} f(z) = \infty$ (and $\lim_{z\to z_0} (z-z_0)^k f(z) \neq 0$, $\lim_{z\to z_0} (z-z_0)^{k+1} f(z) = 0$).
If $\lim_{z\to z_0} f(z) = \infty$, then z_0 is not a removable singularity by (i) above and it is not an essential singularity by Casorati-Weierstrass Theorem (because the range of f misses small values near z_0). (The converse part of the parenthetical statement was proved in a theorem earlier.)

(iii) If z_0 is an essential singularity, then $f(D_*)$ is dense in $\mathbb{C}$ for any $\varepsilon > 0$ by Casorati-Weierstrass Theorem.
If $f(D_*)$ is dense in $\mathbb{C}$, then $\lim_{z\to z_0} f(z)$ cannot exist.
If $\lim_{z\to z_0} f(z)$ doesn't exist, then by (i) and (ii), z_0 is not a removable singularity or a pole, hence it must be an essential singularity.)

Exercises

1. Identify the isolated singularities of the following functions and classify each as a removable singularity, a pole (and its order) or an essential singularity:

   (a) $\dfrac{1}{z^4 + z^2}$;   (b) $\cot z$;

   (c) $\dfrac{e^{1/z^2}}{z-1}$;   (d) $\dfrac{z^2 - 1}{\sin \pi z}$.

2. Find the Laurent series of $\dfrac{1}{z^2 - 4}$ on

   (a) $0 < |z-2| < 4$;   (b) $2 < |z| < \infty$.

3. Find $\displaystyle\int_{|z|=r} \sin\frac{1}{z}\, dz$ for positive $r \neq \dfrac{1}{n\pi}$. As usual, the circle $|z| = r$ is given the counterclockwise orientation.

4. Suppose f is holomorphic on $\mathbb{C}\setminus\{0\}$ and satisfies $|f(z)| \le \sqrt{|z|} + \dfrac{1}{\sqrt{|z|}}$. Prove that f is a constant function.

5. Find the roots of $\sinh z$. What is the largest r for which there exist $c_0,\ c_1,\ c_2,\ \ldots \in \mathbb{C}$ such that $\dfrac{(z^2+\pi^2)(e^z-1)}{\sinh z} = \displaystyle\sum_{n=0}^{\infty} c_n z^n$ for $|z| < r$?

6. Let G be a region and $f : G \to \mathbb{C}$ be continuous. If f^2 is holomorphic on G, show that f is holomorphic on G. [*Hint*: First show f is holomorphic at z such that $f(z) \neq 0$. Then consider the singularity type of the roots of f.]

7. If f has a pole at 0, show that e^f cannot have a pole at 0.

8. Let f be holomorphic on $\{z : R < |z| < \infty\}$. We say ∞ is a removable singularity, a pole of order k or an essential singularity of $f(z)$ if and only if 0 is a removable singularity, a pole of order k or essential singularity of $f\left(\dfrac{1}{z}\right)$, respectively.
   (a) Prove that an entire function with a pole at ∞ is a polynomial.
   (b) Prove that a holomorphic function on $\mathbb{C} \cup \{\infty\}$ except for isolated poles must be a rational function.

9. Prove that the image of the plane under a nonconstant entire mapping f is dense in the plane. [*Hint*: If f is not a polynomial, then consider $f\left(\dfrac{1}{z}\right)$.]

10. Can the positive integers $\{1, 2, 3, \ldots\}$ be partitioned into a finite number of sets $S_1, S_2, \ldots, S_k$, each of which is an arithmetic progression, i.e.

$$S_1 = \{a_1, a_1 + d_1, a_1 + 2d_1, \ldots\}$$

$$S_2 = \{a_2, a_2 + d_2, a_2 + 2d_2, \ldots\}$$

$$\cdots$$

$$S_k = \{a_k, a_k + d_k, a_k + 2d_k, \ldots\},$$

and such that there are no equal common differences (i.e. $d_i \neq d_j$ for $i \neq j$)? [*Hint*: Consider the series $\displaystyle\sum_{n \in S_j} z^n$ for $j = 1, 2, \ldots, k$.] (This exercise is due to D. J. Newman.)

16. Residues and Roots

Definition. Let $\sum_{k=-\infty}^{\infty} a_k(z-z_0)^k$ be the Laurent series of $f(z)$ on $0<|z-z_0|<\varepsilon$, then the *residue* of $f(z)$ at z_0 is $\operatorname*{Res}_{z=z_0} f(z) = \operatorname{Res}(f, z_0) = a_{-1}$.

Observe that if $f(z) = \sum_{k=-\infty}^{\infty} a_k(z-z_0)^k$ for $0<|z-z_0|<\varepsilon$, then for $0<r<\varepsilon$, $\int\limits_{|z-z_0|=r} f(z)\,dz$ $= \sum_{k=-\infty}^{\infty} a_k \int\limits_{|z-z_0|=r} (z-z_0)^k\,dz = 2\pi i a_{-1}$ because $(z-z_0)^k$ has antiderivative $\dfrac{(z-z_0)^{k+1}}{k+1}$ for $k\neq -1$. This is the reason the word "residue" is used (because a_{-1} is the only coefficient that remains).

Definition. A function f is *meromorphic* on a region D iff f is holomorphic on D except for (isolated) poles.

Residue Theorem. Suppose f is meromorphic on a simply connected region D with poles at $z_1,\ldots,z_n$. Let Γ be a closed curve on D <u>not</u> passing through $z_1,\ldots,z_n$, then

$$\int_\Gamma f(z)\,dz = 2\pi i \sum_{j=1}^{n} n(\Gamma, z_j) \operatorname*{Res}_{z=z_j} f(z).$$

Proof. Let $P_j\left(\dfrac{1}{z-z_j}\right) = \sum_{k=-m_j}^{-1} a_{k,j}(z-z_j)^k$ be the principal part of f at z_j, where m_j is the order of the pole at z_j. Then the function $g(z) = f(z) - \sum_{j=1}^{n} P_j\left(\dfrac{1}{z-z_j}\right)$ is holomorphic on D. (At z_j, the singularity is cancelled.) By Cauchy's theorem,

$$0 = \int_\Gamma g(z)\,dz = \int_\Gamma f(z)\,dz - \sum_{j=1}^{n}\int_\Gamma \sum_{k=-m_j}^{-1} a_{k,j}(z-z_j)^k\,dz = \int_\Gamma f(z)\,dz - 2\pi i \sum_{j=1}^{n} a_{-1,j}\, n(\Gamma,\ z_j).$$

QED.

Remark. The residue theorem is true even if there are essential singularities. The proof for that amounts to replacing Γ by a sequence of small circles about the singularities.

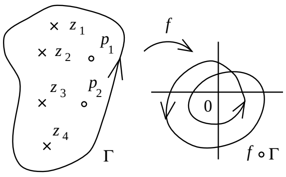
Cauchy Integral Formula for Derivatives. If f is holomorphic on a simply connected region D and a is a point <u>not</u> on a closed curve Γ in D, then $f^{(n)}(a) = \dfrac{n!}{2\pi i\, n(\Gamma,a)} \displaystyle\int_\Gamma \frac{f(w)}{(w-a)^{n+1}}\,dw.$

Proof. Near a, $f(z) = f(a) + f'(a)(z-a) + \ldots + \dfrac{f^{(n)}(a)}{n!}(z-a)^n + \ldots$. By the residue theorem,

$$\int_\Gamma \frac{f(w)}{(w-a)^{n+1}}\,dw = 2\pi i\, n(\Gamma,a) \operatorname*{Res}_{z=a} \frac{f(z)}{(z-a)^{n+1}} = 2\pi i\, n(\Gamma,a)\frac{f^{(n)}(a)}{n!}.$$

QED.

Argument Principle. If f is meromorphic on a simply connected region D and Γ is a simple closed curve in D <u>not</u> passing through the roots z_j nor the poles p_k of f, then $\dfrac{1}{2\pi i}\displaystyle\int_\Gamma \frac{f'(z)}{f(z)}\,dz = n - m$, where n and m are the number of roots and poles (counting multiplicities) of f inside Γ, respectively.



[Note: If we let $w = f(z)$, then $n - m = \dfrac{1}{2\pi i}\displaystyle\int_\Gamma \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_{f\circ\Gamma} \frac{dw}{w}$ $= n(f\circ\Gamma, 0) = \dfrac{\Delta_\Gamma \arg f(z)}{2\pi}$, where $\Delta_\Gamma \arg f(z)$ is the change of argument in $f(z)$ along Γ. This is the reason for the name *Argument Principle.*]

Proof. The isolated singularities of $\dfrac{f'}{f}$ are at the z_j's and p_k's. If the order of z_j as a root of f is n_j, then $f(z) = a_{n_j}(z - z_j)^{n_j} + \ldots$ near z_j and $\dfrac{f'(z)}{f(z)} = \dfrac{n_j}{z - z_j} + \ldots$ near z_j. If the order of p_k as a pole of f is m_k, then $f(z) = \dfrac{a_{-m_k}}{(z-p_k)^{m_k}} + \ldots$ near p_k and $\dfrac{f'(z)}{f(z)} = \dfrac{-m_k}{z - p_k} + \ldots$. By the residue theorem,

$$\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)}\,dz = \sum_j \operatorname*{Res}_{z=z_j} \frac{f'(z)}{f(z)} + \sum_k \operatorname*{Res}_{z=p_k} \frac{f'(z)}{f(z)} = \sum_j n_j + \sum_k (-m_k) = n - m.$$

QED.

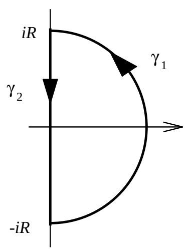
Rouché's Theorem (Estermann-Glicksberg Form). If f and g are holomorphic inside and on a simple closed curve Γ and $|f(z) + g(z)| < |f(z)| + |g(z)|$ for all $z \in \Gamma$, then f and g have the same number of roots inside Γ. (Note the inequality on Γ implies f and g have <u>no</u> roots on Γ.)

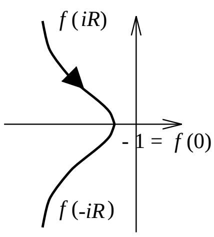
Proof. For $z \in \Gamma, |f(z) + g(z)| < |f(z)| + |g(z)|$ implies $\dfrac{f(z)}{g(z)}$ is not zero or a positive real number (otherwise $|f(z) + g(z)| = |f(z)| + |g(z)|$). Then $\log \dfrac{f(z)}{g(z)}$ can be defined with $0 < \arg \dfrac{f(z)}{g(z)} < 2\pi$. Therefore $0 = \displaystyle\int_\Gamma \left(\log \frac{f(z)}{g(z)}\right)' dz = \int_\Gamma \frac{f'(z)}{f(z)}\,dz - \int_\Gamma \frac{g'(z)}{g(z)}\,dz$ and the result follows from the argument principle.

QED.

Remark. To show f and g have the same number of roots inside Γ, it is sufficient to show $|f(z)+g(z)| < |f(z)|$ or $|f(z) + g(z)| < |g(z)|$.

Examples. (1) Find the number of roots of $f(z) = 17z^5 + z^2 - 1$ in each quadrant and on the real or imaginary axes.





Solution. Because $f'(z) = 85z^4 + 2z, f(0) = -1, f(-\sqrt[3]{\frac{2}{85}}) < 0$, there is one positive real root. Because $f(iy) = -y^2 - 1 + 17y^5 i \neq 0$ for y real, there is no imaginary root.

We now apply the argument principle to the right half plane. Consider Γ to be the boundary of the half disk of radius R (R huge) in the right half plane. On γ_1, $z = Re^{i\theta}$, $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$, because R is huge, $f(z) \approx 17z^5 = 17R^5 e^{i5\theta}$. This implies $\Delta_{\gamma_1} \arg f(z) \approx 5(\dfrac{\pi}{2} - (-\dfrac{\pi}{2})) = 5\pi$.

On $\gamma_2, z = iy$, where y decreases from R to $-R$, $f(iy) = \underbrace{-y^2-1}_{\text{negative}} + 17y^5 i$, $f(iR) = \underbrace{-R^2-1}_{\text{negative}} + \underbrace{17R^5}_{\text{positive}}\, i$, $f(-iR) = \underbrace{-R^2-1}_{\text{negative}}\underbrace{-17R^5}_{\text{negative}} i$. Since $\displaystyle\lim_{R\to+\infty} \frac{17R^5}{R^2+1} = +\infty$, these imply $\Delta_{\gamma_2} \arg f(z) = \arg f(-iR) - \arg f(iR) \approx \dfrac{3\pi}{2} - \dfrac{\pi}{2} = \pi$.

So $\Delta_\Gamma \arg f(z) = 5\pi + \pi = 6\pi$ and there are $\dfrac{6\pi}{2\pi} = 3$ roots inside Γ when R is huge (one of which is real as shown above). Therefore there is one positive real root and two pairs of complex roots in conjugate (one complex root in each quadrant).

(2) Show that all the roots of $f(z) = z^5 + 3z + 1$ are in the disk $|z| < 2$.

Solution. Let $g(z) = -z^5$, which has 5 roots (counting multiplicities) in the disk $|z| < 2$. Let Γ be the circle $|z| = 2$, then for $z \in \Gamma$, $|f(z) + g(z)| = |3z + 1| \le 3|z| + 1 = 7 < |g(z)| = |z|^5 = 2^5 = 32$. So by Rouché's theorem, f has 5 roots inside $|z| = 2$.

(3) What is the smallest positive integer r such that $f(z) = z^5 + 48z + 64$ has a root in the disk $|z| < r$?

Solution. For $|z| = 1$, let $g(z) = -64$, then $|f(z) + g(z)| = |z^5 + 48z| \le |z|^5 + 48|z| = 49 < 64 = |g(z)|$, so by Rouché's theorem, f has no roots inside $|z| = 1$.

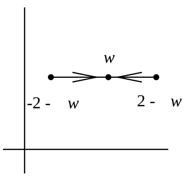
For $|z| = 2$, let $g(z) = -48z$, then $|f(z) + g(z)| = |z^5 + 64| \le |z|^5 + 64 = 96 = 48|z| = |g(z)| \le |f(z)| + |g(z)|$. For equality to hold throughout, we must have $z^5 = 32$ and $f(z) = 0$. Then $32 + 48z + 64 = 0$ implies $z = -2$, which contradicts $z^5 = 32$. So one of the inequality is strict, i.e. on $|z| = 2$, $|f(z) + g(z)| < |f(z)| + |g(z)|$. By Rouché's theorem, f has one root inside $|z| = 2$. Therefore $r = 2$.

(4) If $a_1, \ldots, a_n$, $b \in D = \{z \colon |z| < 1\}$ and $f(z) = \displaystyle\prod_{j=1}^{n} \frac{z - a_j}{1 - \overline{a_j} z}$, then the equation $f(z) = b$ has exactly n solutions (counting multiplicities) in D.

Solution. Let $g(z) = b - f(z)$, then for $|z| = 1$, $|f(z) + g(z)| = |b| < 1 = \displaystyle\prod_{j=1}^{n} \left| \frac{z - a_j}{1 - \overline{a_j} z} \right| = |f(z)|$. By Rouché's theorem, g has n roots in D, since f has n roots $a_1 \ldots, a_n$ in D. So $f(z) = b$ has n solutions in D.

(5) Let $w \in \mathbb{C} \setminus [-2, 2]$ and k a positive integer. Define $f(z) = z^k + \dfrac{1}{z^k}$. Show that f takes the value w exactly k times in $D = \{z \colon |z| < 1\}$ (i.e. $f(z) = w$ has exactly k solutions in D).

Solution. Let $g(z) = f(z) - w$, we want to show g has exactly k roots in D. Let Γ be the unit circle $z(\theta) = e^{i\theta}$, $0 \le \theta \le 2\pi$, $g(e^{i\theta}) = e^{ik\theta} + e^{-ik\theta} - w = 2\cos k\theta - w$.



Then $g \circ \Gamma$ is the line segment joining $-2 - w$ to $2 - w$. Since $w \notin [-2, 2]$, $g \circ \Gamma$ does not wind around 0, so by the argument principle,

$$0 = n(g \circ \Gamma, 0) = \begin{pmatrix} \text{number of roots} \\ \text{of } g(z) \text{ in } D \end{pmatrix} - \begin{pmatrix} \text{number of poles} \\ \text{of } g(z) \text{ in } D \end{pmatrix}.$$

Since 0 is the only pole of $f(z)$ (hence also of $g(z)$) of order k, we have the number of roots of $g(z)$ in $D =$ number of poles of $g(z)$ in $D = k$.

Local Mapping Theorem. Let f be a nonconstant holomorphic function on Ω and z_0 be a root of f of order m on Ω. Then for $\varepsilon > 0$ small, there is a $\delta > 0$ such that the equation $f(z) = w$ with $0 < |w| < \delta$ has exactly m distinct roots (each of order 1) in $B(z_0, \varepsilon)$, i.e. near z_0, f is locally a m-to-1 mapping.

Proof. Since f is nonconstant, by the identity theorem, there cannot be any sequence of roots of f (or f') converging to z_0. So we may pick $\varepsilon > 0$ small so that $f(z) \ne 0$ and $f'(z) \ne 0$ for $z \in \overline{B(z_0, \varepsilon)} \setminus \{z_0\} \subset \Omega$. Let $\delta = \displaystyle\min_{|z - z_0| = \varepsilon} |f(z)|$, then $\delta > 0$. Suppose $0 < |w| < \delta$, then for $|z - z_0| = \varepsilon$, we have $|f(z) + (w - f(z))\,| = |w| < \delta \le |f(z)| + |w - f(z)|$. By Rouché's theorem, f and $w - f$ must have the same number of roots in $B(z_0, \varepsilon)$, namely m. Since $f'(z) \ne 0$ for $z \in B(z_0, \varepsilon)$, the roots of $f(z) = w$ are distinct and of order 1.

QED.

Inverse Mapping Theorem. If f is univalent (i.e. one-to-one and holomorphic) on an open set U, then $f^{-1}\colon f(U) \to U$ is holomorphic.

Proof. (Since f is one-to-one, f is nonconstant. Hence by the open mapping theorem, $f(U)$ is open.) For an arbitrary $a \in U$, we will show that f^{-1} is continuous at $f(a)$. Given $\varepsilon > 0$, by the open mapping theorem, $f(B(a,\varepsilon))$ is open. Since $f(a) \in f(B(a,\varepsilon))$, there is an open disk $B(f(a),\delta) \subset f(B(a,\varepsilon))$ for some $\delta > 0$. Now $w = f(f^{-1}(w))$. So,

$$|w - f(a)| < \delta \Rightarrow w \in B(f(a),\delta) \subset f(B(a,\varepsilon)) \Rightarrow f^{-1}(w) \in B(a,\varepsilon) \Rightarrow |f^{-1}(w) - a| < \varepsilon.$$

Next we will show $f'(z) \neq 0$ for all $z \in U$. Otherwise, suppose $f'(c) = 0$. Then $g(z) = f(z) - f(c)$ has a root of order $m \geq 2$ at c. By the local mapping theorem, there are distinct a and b near c such that $g(a) = g(b)$. Then $f(a) = f(b)$, contradicting the fact f is one-to-one.

Therefore, by the inverse rule, f^{-1} is holomorphic on $f(U)$. **QED.**

Earlier we proved that for $|a| < 1, \theta$ real, $f(z) = e^{i\theta}\dfrac{z-a}{1-\overline{a}z}$ is a one-to-one function from the open unit disk D onto D. Here we will prove the converse.

Theorem. If f is a one-to-one holomorphic function from D onto D, then $f(z) = e^{i\theta}\dfrac{z-a}{1-\overline{a}z}$ for some real θ and $a \in D$.

Proof. Let $a \in D$ such that $f(a) = 0$. Let $T(z) = \dfrac{z+a}{1+\overline{a}z}$, which maps D onto D, the unit circle onto the unit circle and $T(0) = a$. The function $g = f \circ T$ and g^{-1} are one-to-one holomorphic functions of D onto D such that $g(0) = 0 = g^{-1}(0)$. By Schwarz's lemma, $|g(z)| \leq |z|$ and $|g^{-1}(z)| \leq |z|$ ($\iff$ $|z| \leq |g(z)|$). So the equality $|g(z)| = |z|$ holds for all $z \in D$. Then $f \circ T(z) = g(z) \equiv e^{i\theta}z$. Setting $w = T(z)$, we get $f(w) = e^{i\theta}T^{-1}(w) = e^{i\theta}\dfrac{w-a}{1-\overline{a}z}$. **QED.**

Hurwitz's Theorem. Let $\{f_n\}$ be a sequence of holomorphic functions on Ω converging uniformly on each closed disk in Ω to the (holomorphic, by Weierstrass' theorem) function f. If f_n has no roots in Ω for all n, then either f has no roots in Ω or $f \equiv 0$. If f_n is one-to-one on Ω for all n, then either f is one-to-one on Ω or f is a constant function.

Proof. For the first assertion, suppose $f \not\equiv 0$ and f has a root at $w \in \Omega$. Then by the identity theorem, there is a circle $C = C(w,r)$ such that $f(z) \neq 0$ for $z \in C$ and $\overline{B(w,r)} \subset \Omega$. Let $\varepsilon = \min\limits_{z\in C}|f(z)|$, then $\varepsilon > 0$. Since f_n converges uniformly on $\overline{B(w,r)}$ to f, there is N such that $n \geq N \Rightarrow |f_n(z) - f(z)| < \varepsilon$ for all $z \in \overline{B(w,r)}$. Then for $z \in C$, $|f_n(z) - f(z)| < \varepsilon \leq |f(z)| \leq |f_n(z)| + |f(z)|$. By Rouché's theorem, it follows that f_n must have a root inside C, which is a contradiction.

For the second assertion, suppose $f(a) = f(b)$ for distinct $a, b \in \Omega$ and f is not the constant function $f(b)$. Then by the uniqueness theorem, there is a closed disk D centered at a in Ω such that $f(z) \neq f(b)$ for all $z \in D \setminus \{a\}$. In particular, $b \notin D$. Since f_n converges uniformly on D to f, it follows that $f_n(z) - f_n(b)$ converges uniformly on D to $f(z) - f(b)$. By the first assertion, since $f_n(z) - f_n(b)$ has no roots in D, $f(z) - f(b)$ cannot have any root in D, which is a contradiction to $f(a) = f(b)$. **QED.**

Exercises

1. Find all possible values of $I = \displaystyle\int_C \frac{dz}{1+z^2}$, where C is a curve with initial point 0 and final point 1 that does not meet the poles of $\dfrac{1}{1+z^2}$.

2. Is there a holomorphic function f on the closed unit disk which sends the unit circle with the counter-clockwise orientation into the unit circle with the clockwise orientation?

3. Determine the number of roots $17z^5 + z^2 + 1$ on the real and imasginary axis and in each quadrant.

4. Show that if α and $\beta \neq 0$ are real, the equation $z^{2n} + \alpha^2 z^{2n-1} + \beta^2 = 0$ has $n-1$ roots with positive real parts if n is odd, and n roots with positive real parts if n is even.

5. If $a > e$, show that the equation $e^z = az^n$ has n solutions inside the unit circle.

6. Suppose f is holomorphic and one-to-one on $\overline{B(0,r)} = \{z : |z-a| \leq r\}$. If f has a root in $B(a,r)$, show that the root is given by $\displaystyle\frac{1}{2\pi i}\int\limits_{|w-a|=r} \frac{wf'(w)}{f(w)}\,dw$.

7. Suppose f is entire and $f(z)$ is real if and only if z is real. Use the argument principle to show that f can have at most one root. [*Hint*: Let Γ be a large circle $|z| = R$, what is $n(f \circ \Gamma, 0)$?]

8. If f is holomorphic on and inside a simple closed curve Γ and f is one-to-one on Γ, prove that f is one-to-one inside Γ. [*Hint*: Is $f \circ \Gamma$ a simple closed curve? For $w \notin f \circ \Gamma$, let $g(z) = f(z) - w$, what is $n(g \circ \Gamma, 0)$?]

9. Let $P(z)$ be a polynomial of degree at least 2.
   (a) Show that $\displaystyle\int_{|z|=r} \frac{dz}{P(z)} = 0$ when all the roots of P are inside the circle $|z| = r$.
   (b) Suppose P has n distinct roots $z_1, \ldots, z_n$. Show that $\displaystyle\sum_{j=1}^{n} \frac{1}{P'(z_j)} = 0$.

10. Use Rouché's theorem to give another proof of the fundamental theorem of algebra.

11. Let $f(z) = z + \displaystyle\sum_{n=2}^{\infty} a_n z^n$. Suppose $\displaystyle\sum_{n=2}^{\infty} n|a_n| \leq 1$.
   (a) Prove that f is holomorphic on the open unit disk D, i.e. the power series converges for every $z \in D$.
   (b) Prove that f is one-to-one on D. [*Hint*: Use Rouché's theorem to show that $g(z) = f(z) - f(z_0)$ has exactly one solution in D for each fixed $z_0 \in D$.]

17. Applications of Residue Theory

If f has a pole of order k at z_0, then the Laurent expansion of $f(z)$ is $\frac{a_{-k}}{(z-z_0)^k}+\ldots+\frac{a_{-1}}{(z-z_0)}+a_0+a_1(z-z_0)+\ldots$. So the function $\varphi(z)=(z-z_0)^k f(z)=a_{-k}+\ldots+a_{-1}(z-z_0)^{k-1}+a_0(z-z_0)^k+\ldots$ is holomorphic at z_0. Then $\operatorname{Res}(f,z_0)=a_{-1}=\frac{\varphi^{(k-1)}(z_0)}{(k-1)!}=\lim_{z\to z_0}\frac{\varphi^{(k-1)}(z)}{(k-1)!}=\lim_{z\to z_0}\frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}[(z-z_0)^k f(z)]$. If f is holomorphic at z_0, $\operatorname{Res}(f,z_0)=0$ (because $a_n=0$ for $n<0$).

Theorem. If g,h are holomorphic near z_0, g has a root of order k at z_0 (i.e. $g(z_0)=\ldots=g^{(k-1)}(z_0)=0, g^{(k)}(z_0)\neq 0$) and h a root of order l at z_0 (i.e. $h(z_0)=\ldots=h^{(l-1)}(z_0)=0, h^{(l)}(z_0)\neq 0$), then $\frac{g(z)}{h(z)}$ has a $\left\{\begin{array}{c}\text{removable singularity}\\ \text{pole (of order } l-k)\end{array}\right\}$ at z_0 in case $\left\{\begin{array}{c}k\geq l\\ k<l\end{array}\right\}$.

Proof. Near z_0, $\frac{g(z)}{h(z)}=\frac{\sum_{n=k}^{\infty}\frac{g^{(n)}(z_0)}{n!}(z-z_0)^n}{\sum_{n=l}^{\infty}\frac{h^{(n)}(z_0)}{n!}(z-z_0)^n}=\frac{g^{(k)}(z_0)}{k!}\frac{l!}{h^{(l)}(z_0)}(z-z_0)^{k-l}+\ldots$. If $k-l\geq 0$, then z_0 is a removable singularity. If $k-l<0$, then it is a pole of order $l-k$. **QED.**

Special Cases: (1) If $f=\frac{g}{h}$, z_0 is a root of order k for g, a root of order $k+1$ for h, then z_0 is a simple pole (i.e. pole of order 1) and $\operatorname{Res}(f,z_0)=(k+1)\frac{g^{(k)}(z_0)}{h^{(k+1)}(z_0)}$.

(2) If $f=\frac{g}{h}$, $g(z_0)\neq 0$, $h(z_0)=h'(z_0)=0$, $h''(z_0)\neq 0$, then z_0 is a double pole (i.e. pole of order 2) and $\operatorname{Res}(f,z_0)=2\frac{g'(z_0)}{h''(z_0)}-\frac{2}{3}\frac{g(z_0)h'''(z_0)}{(h''(z_0))^2}$. If $f=\frac{g}{h}$, $g(z_0)=0$, $g'(z_0)\neq 0$, $h(z_0)=h'(z_0)=h''(z_0)=0$, $h'''(z_0)\neq 0$, then z_0 is a double pole and $\operatorname{Res}(f,z_0)=3\frac{g''(z_0)}{h'''(z_0)}-\frac{2}{3}\frac{g'(z_0)h^{(4)}(z_0)}{(h'''(z_0))^2}$.

Examples. (1) $\frac{z}{1+e^z}$ has a pole at πi. It is a simple pole. So,

$$\operatorname{Res}(\frac{z}{1+e^z},\pi i)=\lim_{z\to\pi i}(z-\pi i)\frac{z}{1+e^z}=\frac{\pi i}{e^{\pi i}}=-\pi i.$$

(2) $\frac{1}{z\sin z}$ has a pole at 0. It is a double pole. So,

$$\operatorname{Res}(\frac{1}{z\sin z},0)=\lim_{z\to 0}\frac{1}{1!}\left(z^2\frac{1}{z\sin z}\right)'=\lim_{z\to 0}\frac{\sin z-z\cos z}{\sin^2 z}=\lim_{z\to 0}\frac{z}{2\cos z}=0.$$

Alternatively, near 0, $\frac{1}{z\sin z}=\frac{1}{z(z-\frac{1}{6}z^3+\cdots)}=\frac{1}{z^2-\frac{1}{6}z^4+\cdots}=\frac{1}{z^2}+\frac{1}{6}+\cdots$. So, $\operatorname{Res}(\frac{1}{z\sin z},0)=0$.

(3) $\frac{\cot z}{z(z-1)}=\frac{\cos z}{z(z-1)\sin z}$ has a pole at 0. It is a double pole. Let $g(z)=\cos z$, $h(z)=z(z-1)\sin z$, then $g(0)=1$, $g'(0)=0$, $h''(0)=-2$, and $h'''(0)=6$. So, $\operatorname{Res}(\frac{\cot z}{z(z-1)},0)=2\frac{0}{-2}-\frac{2}{3}\frac{1\times 6}{(-2)^2}=-1$.

Alternatively, near 0,

$$\frac{\cos z}{z(z-1)\sin z} = \frac{1-\frac{z^2}{2}+\cdots}{(-z+z^2)\left(z-\frac{z^3}{6}+\cdots\right)} = \frac{1-\frac{z^2}{2}+\cdots}{-z^2+z^3+\cdots} = -\frac{1}{z^2}-\frac{1}{z}+\cdots.$$

So, $\mathrm{Res}(\dfrac{\cot z}{z(z-1)}, 0) = -1$.

(4) $z\cos\dfrac{1}{z+1}$ has an essential singularity at -1, since near -1,

$$z\cos\frac{1}{z+1} = [(z+1)-1]\left[1-\frac{1}{2(z+1)^2}+\cdots\right] = (z+1)-1-\frac{1}{2(z+1)}+\cdots.$$

So $\mathrm{Res}(z\cos\dfrac{1}{z+1}, -1) = -\dfrac{1}{2}$.

Recall the residue theorem implies that if f is holomorphic on and inside a *simple closed curve* C except at poles $a_1, \ldots, a_n$, then $\displaystyle\int_C f(z)\,dz = 2\pi i\sum_{j=1}^{n}\mathrm{Res}(f, a_j)$. (In view of the examples above, notice this formula said something amazing, namely the integral on a closed curve can be computed by doing some derivatives!) Now let us compute some nonelementary integrals.

Some Common Types of Integrals.

Type I. $\displaystyle\int_0^{2\pi} F(\sin k\theta,\ \cos m\theta)\,d\theta = \int_{|z|=1} F\left(\frac{z^k-z^{-k}}{2i}, \frac{z^m+z^{-m}}{2}\right)\frac{dz}{iz}$, where $z=e^{i\theta}, dz = ie^{i\theta}\,d\theta$.

Example. $\displaystyle\int_0^{2\pi}\frac{d\theta}{5+4\sin\theta} = \int_{|z|=1}\frac{1}{5+4\left(\frac{z-z^{-1}}{2i}\right)}\frac{dz}{iz} = \int_{|z|=1}\frac{dz}{(2z+i)(z+2i)} = 2\pi i\mathop{\mathrm{Res}}_{z=-\frac{i}{2}}\frac{1}{(2z+i)(z+2i)}$

$= \dfrac{2}{3}\pi$, where we observed that only $-\dfrac{i}{2}$ is inside $|z|=1$ and it is a simple pole.

Type II. $\displaystyle\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}\,dx$, where P, Q are polynomials and Q has no real roots.

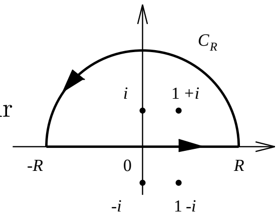
Remark. The improper integral $\displaystyle\int_{-\infty}^{\infty}$ is defined (in the Riemann sense) to be $\displaystyle\lim_{a\to-\infty}\int_a^0 + \lim_{b\to+\infty}\int_0^b$. If $\displaystyle\int_{-\infty}^{\infty}$ exists, it agrees with the limit $\displaystyle\lim_{R\to+\infty}\int_{-R}^{R}$ (which is called the *principal value* of $\displaystyle\int_{-\infty}^{\infty}$ and is denoted by P.V. $\displaystyle\int_{-\infty}^{\infty}$). If $\displaystyle\int_{-\infty}^{\infty}$ doesn't exist, sometimes P.V. $\displaystyle\int_{-\infty}^{\infty}$ may exist.

Example. $\displaystyle\int_{-\infty}^{\infty} x\,dx = \lim_{a\to-\infty}\int_a^0 x\,dx + \lim_{b\to+\infty}\int_0^b x\,dx$ doesn't exist, but P.V. $\displaystyle\int_{-\infty}^{\infty} x\,dx = 0$.

If $\deg Q(x) > \deg P(x)+1$, then $\displaystyle\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}\,dx$ exists (because near $\pm\infty$, it is like $\displaystyle\int_c^{\pm\infty}\frac{1}{x^n}\,dx$, $n>1$). In this case, we often compute its principal value instead. If $\deg Q(x) \le \deg P(x)+1$, then $\displaystyle\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}\,dx$ doesn't exist, but P.V. $\displaystyle\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}\,dx$ may exist.

Example. Find $\displaystyle\int_{-\infty}^{\infty} \frac{x^2-1}{(x^2+1)(x^2-2x+2)}\,dx \left(= \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx\right)$.

Factoring the denominator, we get $(x^2+1)(x^2-2x+2) = (x-i)(x+i)(x-1-i)(x-1+i)$.

Step 1. Consider the contour  where R is large.

By the residue theorem,

$$\int_{C_R+[-R,R]} f(z)\,dz = 2\pi i\left(\operatorname*{Res}_{z=i} f(z) + \operatorname*{Res}_{z=1+i} f(z)\right)$$
$$= 2\pi i\left(\lim_{z\to i}\frac{z^2-1}{(z+i)(z^2-2z+2)} + \lim_{z\to 1+i}\frac{z^2-1}{(z^2+1)(z-1-i)}\right) = \frac{\pi}{5}.$$

Step 2. Now $z\in C_R \Rightarrow |z| = R$, $\displaystyle\left|\int_{C_R} f(z)\,dz\right| \le ML = \frac{R^2+1}{(R^2-1)(R^2-2R-2)}\pi R \to 0$ as $R\to+\infty$.

Step 3. So, $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx = \lim_{R\to\infty}\left(\int_{C_R+[-R,R]} f(z)\,dz - \int_{C_R} f(z)\,dz\right) = \frac{\pi}{5}$.

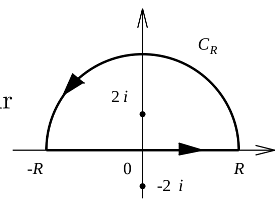
Type III. $\displaystyle\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}\begin{Bmatrix}\cos ax\\ \sin ax\end{Bmatrix}dx = \begin{Bmatrix}\mathrm{Re}\\ \mathrm{Im}\end{Bmatrix}\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}e^{iax}\,dx$, where P, Q are polynomials and Q has no real roots.

If $\deg Q(x) > \deg P(x)+1$, then $\displaystyle\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}\begin{Bmatrix}\cos ax\\ \sin ax\end{Bmatrix}dx$ exists (and equals its principal value). If $\deg Q(x) \le \deg P(x)+1$, then $\displaystyle\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}\begin{Bmatrix}\cos ax\\ \sin ax\end{Bmatrix}dx$ or its principal value may exist.

The following is a useful inequality when dealing with Type III integrals.

Jordan's Inequality. For $M>0$, $\displaystyle\int_0^{\pi} e^{-M\sin\theta}\,d\theta = 2\int_0^{\frac{\pi}{2}} e^{-M\sin\theta}\,d\theta \le 2\int_0^{\frac{\pi}{2}} e^{-M\frac{2\theta}{\pi}}\,d\theta < \frac{\pi}{M}$ (because on $[0,\frac{\pi}{2}]$, $\sin\theta \ge \frac{2\theta}{\pi}$ as can be seen from the graphs of $y=\sin\theta$ and $y=\frac{2\theta}{\pi}$).

Example. Find $\displaystyle\int_{-\infty}^{\infty}\frac{x\sin 9x}{x^2+4}\,dx \left(= \mathrm{Im}\int_{-\infty}^{\infty}\frac{ze^{i9z}}{z^2+4}\,dz\right)$. (Note $\dfrac{x\sin 9x}{x^2+4}$ is even. If the principal value of the integral exists, then $\displaystyle\int_{-\infty}^{0} = \frac{1}{2}\mathrm{P.V.}\int_{-\infty}^{\infty} = \int_0^{\infty}$ and the integral (in the Riemann sense) will exist.)

Step 1. Consider the contour  where R is large.

By the residue theorem, $\displaystyle\int_{C_R+[-R,R]}\frac{ze^{i9z}}{z^2+4}\,dz = 2\pi i\operatorname*{Res}_{z=2i}\frac{ze^{i9z}}{z^2+4} = 2\pi i\lim_{z\to 2i}\frac{ze^{i9z}}{z+2i} = e^{-18}\pi i$.

Step 2. As $R \to \infty$,

$$\left|\int_{C_R} \frac{ze^{i9z}}{z^2+4}\,dz\right| \le \int_{C_R} \left|\frac{z}{z^2+4}\right| \left|e^{i9z}\right| |dz| \le \frac{R}{R^2-4}\int_0^\pi e^{-9R\sin\theta} R\,d\theta \le \frac{R}{R^2-4} R\frac{\pi}{9R} \to 0,$$

where we have applied Jordan's inequality to get the last inequality.

Step 3. Therefore,

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{x\sin 9x}{x^2+4}\,dx = \text{Im}\left(\lim_{R\to\infty}\left(\int_{C_R+[-R,R]} \frac{ze^{i9z}}{z^2+4}\,dz - \int_{C_R} \frac{ze^{i9z}}{z^2+4}\,dz\right)\right) = e^{-18}\pi.$$

We extract from step 2 a useful fact.

Jordan's Lemma. If $\lim\limits_{R\to\infty} (\max\limits_{z\in C_R} |h(z)|) = 0$, then $\lim\limits_{R\to\infty} \displaystyle\int_{C_R} h(z)e^{iaz}\,dz = 0$ for $a > 0$.

Proof. As $R \to \infty$,

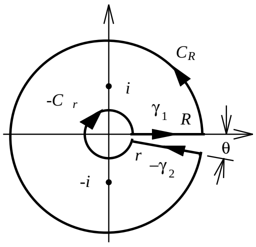
$$\left|\int_{C_R} h(z)e^{iaz}\,dz\right| \le \int_{C_R} |h(z)| \left|e^{iaz}\right| |dz| \le \max_{z\in C_R} |h(z)| \int_0^\pi e^{-aR\sin\theta} R\,d\theta \le \frac{\pi}{a} \max_{z\in C_R} |h(z)| \to 0.$$

QED.

Big *O*-Little *o* Notations. We say $f(z) = O(g(z))$ as $z \to a$ if there is M such that $\left|\dfrac{f(z)}{g(z)}\right| \le M$ as $z \to a$ (i.e. in a neighborhood of a). We say $f(z) = o(g(z))$ as $z \to a$ if $\dfrac{f(z)}{g(z)} \to 0$ as $z \to a$.

Type IV. $\displaystyle\int_0^\infty x^\alpha f(x)\,dx$ $(-1 < \alpha < 0)$ or $\displaystyle\int_0^\infty (\ln x)^k f(x)\,dx$ $(k = 1, 2, 3\ldots)$, where f is meromorphic on $\mathbb{C}$ and continuous at 0. (In the first integral, we require $f(z) = o(\dfrac{1}{|z|^{\alpha+1}})$ as $z \to \infty$ and in the second integral, we require $f(z) = O(\dfrac{1}{|z|^2})$ as $z \to \infty$. These conditions ensure the integrals will exist.)

Example. Find $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)} = \lim_{\substack{R\to\infty \\ r\to 0^+}} \int_r^R x^{-\frac{1}{2}} \frac{1}{x^2+1}\,dx.$

Step 1. Consider the contour  where r is small and R is large.

Here $\gamma_1(x) = x, \gamma_2(x) = xe^{i(2\pi-\theta)}, r \le x \le R$. Inside and on the contour, we have $0 \le \arg z < 2\pi$.

By the residue theorem, $\displaystyle\int_{C_R-\gamma_2-C_r+\gamma_1} z^{-\frac{1}{2}} \frac{1}{z^2+1}\,dz = 2\pi i\left(\operatorname*{Res}_{z=i} \frac{z^{-1/2}}{z^2+1} + \operatorname*{Res}_{z=-i} \frac{z^{-1/2}}{z^2+1}\right)$

$$= 2\pi i\left(\lim_{z\to i} \frac{z^{-1/2}}{z+i} + \lim_{z\to -i} \frac{z^{-1/2}}{z-i}\right) = \pi(i^{-1/2} - (-i)^{-1/2}) = \pi(e^{-i\pi/4} - e^{-3i\pi/4}) = \pi\sqrt{2}.$$

Step 2. We have $\displaystyle\int_{\gamma_1} \frac{z^{-1/2}}{z^2+1}\,dz = \int_r^R \frac{x^{-1/2}}{x^2+1}\,dx$, $\displaystyle\lim_{\theta\to 0^+}\int_{\gamma_2} \frac{z^{-1/2}}{z^2+1}\,dz = \int_r^R \frac{x^{-1/2}e^{-i\pi}}{x^2+1}\,dx = -\int_r^R \frac{x^{-1/2}}{x^2+1}\,dx$,
$\displaystyle\left|\int_{C_R} \frac{z^{-1/2}}{z^2+1}\,dz\right| \le ML = \frac{R^{-1/2}}{R^2-1}(2\pi-\theta)R \to 0$ as $R\to+\infty$ and
$\displaystyle\left|\int_{C_r} \frac{z^{-1/2}}{z^2+1}\,dz\right| \le ML = \frac{r^{-1/2}}{1-r^2}(2\pi-\theta)r \to 0$ as $r\to 0^+$.

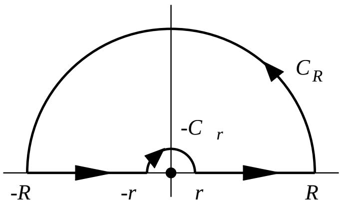
Step 3. We have $\displaystyle\pi\sqrt{2} = \lim_{r,\theta\to 0^+, R\to+\infty} \int_{C_R-\gamma_2-C_r+\gamma_1} z^{-1/2}\frac{1}{z^2+1}\,dz = 2\int_0^\infty \frac{x^{-1/2}}{x^2+1}\,dx.$

Therefore, $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)} = \frac{\pi\sqrt{2}}{2}.$

Miscellaneous Examples

Example. Find $\displaystyle\int_{-\infty}^\infty \frac{\sin x}{x}\,dx = \left(\mathrm{Im}\int_{-\infty}^\infty \frac{e^{ix}}{x}\,dx\right)$. (Note $\dfrac{\sin x}{x}$ is even.)

Step 1. Consider the contour



where r is small and R is large.

By the residue theorem, $\displaystyle\left(\int_{C_R} + \int_{-R}^{-r} - \int_{C_r} + \int_r^R\right)\frac{e^{iz}}{z}\,dz = 0$ (since there is no singularity inside the contour).

Step 2. Since $\displaystyle\lim_{R\to+\infty}\max_{z\in C_R}\left|\frac{1}{z}\right| = \lim_{R\to+\infty}\frac{1}{R} = 0$, Jordan's lemma implies $\displaystyle\lim_{R\to+\infty}\int_{C_R}\frac{e^{iz}}{z}\,dz = 0$. Now

$$\int_{C_r}\frac{e^{iz}}{z}\,dz = \int_{C_r}\frac{\left(1+iz+\dfrac{(iz)^2}{2}+\dots\right)}{z}\,dz = \int_{C_r}\left(\frac{1}{z}+g(z)\right)\,dz = \pi i + \int_{C_r} g(z)\,dz,$$

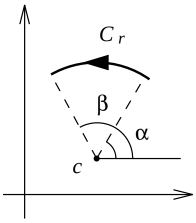
where $g(z)$ has a removable singularity (hence holomorphic) at 0, so it is bounded near 0 (say by M). Then, $\displaystyle\left|\int_{C_r} g(z)\,dz\right| \le ML = M\pi r \to 0$ as $r\to 0^+$.

Step 3. Therefore,

$$0 = \lim_{r\to 0^+, R\to+\infty}\left(\int_{C_R} + \int_{-R}^{-r} - \int_{C_r} + \int_r^R\right)\frac{e^{iz}}{z}\,dz = \int_{-\infty}^\infty \frac{e^{ix}}{x}\,dx - \pi i \Rightarrow \int_{-\infty}^\infty \frac{\sin x}{x}\,dx = \mathrm{Im}\int_{-\infty}^\infty \frac{e^{ix}}{x}\,dx = \pi.$$

We extract from step 2 the following useful rule for dealing with arcs around simple poles.

Rule. If $f(z)$ has an isolated simple pole at $z=c$ and C_r is the arc $z(\theta) = c + re^{i\theta}$, $\alpha\le\theta\le\beta$, then $\displaystyle\lim_{r\to 0^+}\int_{C_r} f(z)\,dz = i(\beta-\alpha)\operatorname*{Res}_{z=c} f(z)$.



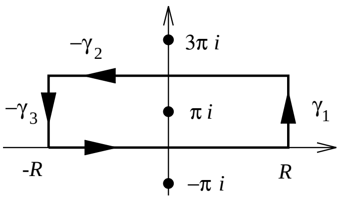
Proof. The Laurent expansion of f at c is $\dfrac{a_{-1}}{z-c}+\sum_{k=0}^{\infty}a_k(z-c)^k=\dfrac{a_{-1}}{z-c}+g(z)$ where $g(z)$ is *holomorphic* at c. So

$$\int_{C_r} f(z)\,dz=\int_{C_r}\left(\frac{a_{-1}}{z-c}+g(z)\right)dz=a_{-1}\int_\alpha^\beta \frac{1}{re^{i\theta}}ire^{i\theta}\,d\theta+\int_{C_r} g(z)\,dz$$
$$=i(\beta-\alpha)\operatorname*{Res}_{z=c} f(z)+\int_{C_r} g(z)\,dz.$$

Now g is bounded near c (say by M), then $\left|\int_{C_r} g(z)\,dz\right|\le ML=M(\beta-\alpha)r\to 0$ as $r\to 0^+$. **QED.**

Example. Find P.V.$\displaystyle\int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^x}\,dx$, where $0<a<1$.

<u>*Step 1*.</u> Consider the function $f(z)=\dfrac{e^{az}}{1+e^z}$, its poles are at $n\pi i$ (n odd integer).



Consider the contour on the left, where R is large.
$\gamma_1(t)=R+it, 0\le t\le 2\pi$,
$\gamma_2(t)=t+2\pi i, -R\le t\le R$,
$\gamma_3(t)=-R+it, 0\le t\le 2\pi$.

By the residue theorem, $\displaystyle\left(\int_{-R}^{R}+\int_{\gamma_1-\gamma_2-\gamma_3}\right)f(z)\,dz=2\pi i\operatorname*{Res}_{z=\pi i}f(z)=2\pi i\lim_{z\to\pi i}\frac{(z-\pi i)e^{az}}{1+e^z}=-2\pi i e^{\pi a i}$.

<u>*Step 2*.</u> We have $\displaystyle\left|\int_{\gamma_1}\frac{e^{az}}{1+e^z}\,dz\right|=\left|\int_0^{2\pi}\frac{e^{a(R+it)}i\,dt}{1+e^{R+it}}\right|\le ML=\frac{e^{aR}}{e^R-1}2\pi\to 0$ as $R\to+\infty$ (because $a<1$),

$\displaystyle\int_{\gamma_2}\frac{e^{az}}{1+e^z}\,dz=\int_{-R}^{R}\frac{e^{a(t+2\pi i)}}{1+e^{t+2\pi i}}\,dt=e^{2\pi ai}\int_{-R}^{R}\frac{e^{at}}{1+e^t}\,dt$ and

$\displaystyle\left|\int_{\gamma_3}\frac{e^{az}}{1+e^z}\,dz\right|=\left|\int_0^{2\pi}\frac{e^{a(-R+it)}i\,dt}{1+e^{-R+it}}\right|\le ML=\frac{e^{-aR}}{1-e^{-R}}2\pi\to 0$ as $R\to+\infty$ (because $0<a$).

<u>*Step 3*.</u> So, $\displaystyle -2\pi i e^{\pi ai}=\lim_{R\to+\infty}\left(\int_{-R}^{R}+\int_{\gamma_1-\gamma_2-\gamma_3}\right)f(z)\,dz=(1-e^{2\pi ai})\int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^x}\,dx$.

Therefore, $\displaystyle\int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^x}\,dx=\frac{-2\pi i e^{\pi ai}}{1-e^{2\pi ai}}=\pi\left(\frac{2i}{e^{\pi ai}-e^{-\pi ai}}\right)=\frac{\pi}{\sin\pi a}$.

The following example illustrates how the residue theorem can produce interesting series representations of some common functions.

Example. The poles of $\cot z=\dfrac{\cos z}{\sin z}$ are at $k\pi$, k any integer. For a fixed $z\ne k\pi$, let n be large so that z is inside the rectangle Γ_n with vertices $\pm(n+\dfrac{1}{2})\pi\pm i(n+\dfrac{1}{2})\pi$. Now for $w=(n+\frac{1}{2})\pi+iy$ on the right edge of Γ_n, $|\cot w|=\left|\dfrac{e^{iw}+e^{-iw}}{e^{iw}-e^{-iw}}\right|=\left|\dfrac{e^{-y}-e^{y}}{e^{-y}+e^{y}}\right|\le 1$ and for $w=x+i(n+\frac{1}{2})\pi$ on the top edge of Γ_n, $|\cot w|=\left|\dfrac{e^{2iw}+1}{e^{2iw}-1}\right|\le\dfrac{1+e^{-(2n+1)\pi}}{1-e^{-(2n+1)\pi}}\le\dfrac{1+e^{-\pi}}{1-e^{-\pi}}=C$. Since $|\cot w|=|\cot(-w)|$ and $1\le C$, so $|\cot w|\le C$ on Γ_n. By the M-L inequality, as $n\to\infty$,

$$\left|\int_{\Gamma_n}\frac{z\cot w\,dw}{w(w-z)}\right|\le\frac{|z|C}{(n+\frac{1}{2})\pi\left((n+\frac{1}{2})\pi-|z|\right)}(8n+4)\pi\to 0.$$

By the residue theorem,

$$0 = \lim_{n\to\infty} \frac{1}{2\pi i}\int_{\Gamma_n} \frac{z\cot w\,dw}{w(w-z)} = \lim_{n\to\infty}\left(\operatorname*{Res}_{w=z}\frac{z\cot w}{w(w-z)} + \sum_{k=-n}^{n}\operatorname*{Res}_{w=k\pi}\frac{z\cot w}{w(w-z)}\right)$$

$$= \cot z - \frac{1}{z} + \sum_{k\neq 0}\frac{z}{k\pi(k\pi - z)} = \cot z - \frac{1}{z} - \sum_{k=1}^{\infty}\frac{2z}{z^2-k^2\pi^2}.$$

Therefore, $\cot z = \dfrac{1}{z} + \displaystyle\sum_{k=1}^{\infty}\frac{2z}{z^2-k^2\pi^2} = \frac{1}{z} + \sum_{k=1}^{\infty}\left(\frac{1}{z-k\pi}+\frac{1}{z+k\pi}\right)$ for $z\neq k\pi$, k any integer.

Remarks. The last example can be modified to yield formulas for summing series. To be more precise, let f be meromorphic on $\mathbb{C}$ with finitely many poles z_p, $p = 1, 2, \ldots, k$ and $\lim\limits_{z\to\infty} zf(z) = 0$. Suppose C_n is the counterclockwise square with vertices $\pm(n+\frac{1}{2}) \pm i(n+\frac{1}{2})$, then by the residue theorem,

$$\int_{C_n} \pi f(z)\cot\pi z\,dz = 2\pi i\left[\sum_{\substack{j=-n\\ j\neq z_p}}^{n} f(j) + \sum_{p=1}^{k}\operatorname{Res}\left(\pi f(z)\cot\pi z, z_p\right)\right].$$

Since $\lim\limits_{z\to\infty} zf(z) = 0$ and $n+\frac{1}{2} \le |z| \le (n+\frac{1}{2})\sqrt{2}$ for all $z\in C_n$, we have $\lim\limits_{n\to\infty}(n+\frac{1}{2})\max\limits_{z\in C_n}|f(z)| = 0$. By the M-L inequality, as $n\to\infty$,

$$\left|\int_{C_n}\pi f(z)\cot\pi z\,dz\right| \le \pi\left(\max_{z\in C_n}|f(z)|\right)\frac{1+e^{-\pi}}{1-e^{-\pi}}(8n+4)\to 0.$$

It follows that

$$\sum_{\substack{j=-\infty\\ j\neq z_p}}^{\infty} f(j) = -\sum_{p=1}^{k}\operatorname{Res}\left(\pi f(z)\cot\pi z, z_p\right).$$

For instance, if we set $f(z) = \dfrac{1}{z^2}$, then

$$\sum_{j=1}^{\infty}\frac{1}{j^2} = \frac{1}{2}\sum_{\substack{j=-\infty\\ j\neq 0}}^{\infty}\frac{1}{j^2} = -\frac{1}{2}\operatorname{Res}\left(\frac{\pi\cot\pi z}{z^2}, 0\right) = \frac{\pi^2}{6}.$$

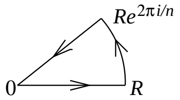
If we replace $\cot z$ by $\csc z$ in the above argument, then we get a similar formula

$$\sum_{\substack{j=-\infty\\ j\neq z_p}}^{\infty}(-1)^j f(j) = -\sum_{p=1}^{k}\operatorname{Res}\left(\pi f(z)\csc\pi z, z_p\right).$$

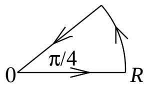
Exercises

1. Show that $\displaystyle\int_0^{\frac{\pi}{2}}\frac{d\theta}{1+\sin^2\theta} = \frac{\pi}{2\sqrt{2}}$ and $\displaystyle\frac{1}{2\pi}\int_0^{2\pi}(2\cos\theta)^{2n}\,d\theta = \frac{(2n)!}{n!n!}$ for every positive integer n.

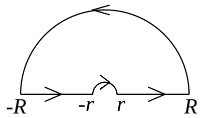
2. Find $\displaystyle\int_0^\infty \frac{dx}{1+x^n}$, where $n \geq 2$ is a positive integer. [*Hint*: This can be done following the example for Type II integrals. Alternatively, the contour below can be considered.]



3. Find $\displaystyle\int_{-\infty}^\infty \frac{\sin^2 x}{x^2}\,dx$, $\displaystyle\int_{-\infty}^\infty \frac{\sin^3 x}{x^3}\,dx$ and $\displaystyle\int_{-\infty}^\infty \frac{\sin^2 x}{1+x^2}\,dx$. [*Hint*: $4\sin^3 x = \mathrm{Im}(3e^{ix} - e^{3ix})$.]

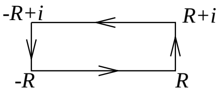
4. Find $\displaystyle\int_0^\infty \cos x^2\,dx$ and $\displaystyle\int_0^\infty \sin x^2\,dx$. [Hint: Use the contour 

.]

5. Find $\displaystyle\int_0^\infty \frac{\ln x}{x^2+1}\,dx$. [Hint: Use the contour 

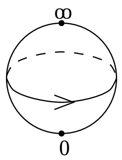
.]

6. Given that $\displaystyle\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$. Find $\displaystyle\int_0^\infty e^{-x^2}\cos 2x\,dx$.

[*Hint*: Use the contour 

and $f(z) = e^{-z^2}$.]

7. Suppose f is holomorphic on the annulus $\{z : r < |z| < \infty\}$. Define $\mathrm{Res}(f, \infty) = -\dfrac{1}{2\pi i}\displaystyle\int_{|z|=R>r} f(z)\,dz$.



(Here, the minus sign appears because the counterclockwise orientation around ∞ corresponds to the clockwise orientation around 0 as can be seen from stereographic projection.) Equivalently, if $f(z) = \displaystyle\sum_{k=-\infty}^{\infty} a_k z^k$ on $\{z : r < |z| < \infty\}$, then $\mathrm{Res}(f, \infty) = -a_{-1}$.

(a) If f is meromorphic on $\mathbb{C}$ with isolated poles at $a_1, \ldots, a_n$, show that $\displaystyle\sum_{j=1}^n \mathrm{Res}(f, a_j) + \mathrm{Res}(f, \infty) = 0$.

That is, the sum of all residues in $\mathbb{C} \cup \{\infty\}$ is 0.

(b) Show that $\mathrm{Res}(f(z), \infty) = \mathrm{Res}\left(-\dfrac{1}{z^2} f\left(\dfrac{1}{z}\right), 0\right)$.

(c) Find $\displaystyle\int_{|z|=1} \frac{dz}{\sin\left(\frac{1}{z}\right)}$.

8. Find $\displaystyle\sum_{j=1}^\infty \frac{1}{j^4}$, $\displaystyle\sum_{j=1}^\infty \frac{(-1)^{j+1}}{j^2}$ and $\displaystyle\sum_{j=1}^\infty \frac{1}{1+j^2}$.

18. Infinite Products

Suppose a polynomial $P(z)$ has roots at 0 (of order m), $z_1, z_2, \ldots, z_n$. Then

$$P(z) = Az^m \prod_{k=1}^{n} (z - z_k) = Bz^m \prod_{k=1}^{n} \left(1 - \frac{z}{z_k}\right),$$

where $B = A \times (-z_1) \times \cdots \times (-z_n)$ and can be determined from P, *e.g.* $B = \lim_{z \to 0} \dfrac{P(z)}{z^m}$.

Now $\sin z$ is entire and has simple roots at $k\pi$, k any integer. If we think of (the power series of) an entire function as an "infinite" polynomial, then we may conjecture that

$$\sin z = z \prod_{k=1}^{\infty} \left(1 - \frac{z}{k\pi}\right)\left(1 + \frac{z}{k\pi}\right) = z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2\pi^2}\right).$$

This formula is true for all complex numbers and was first discovered by Euler. Taking $z = \dfrac{\pi}{2}$ and transposing terms, we get

$$\frac{\pi}{2} = \prod_{k=1}^{\infty} \frac{4k^2}{4k^2 - 1} = \left(\frac{2 \times 2}{1 \times 3}\right)\left(\frac{4 \times 4}{3 \times 5}\right)\left(\frac{6 \times 6}{5 \times 7}\right)\left(\frac{8 \times 8}{7 \times 9}\right)\cdots,$$

which is known as *Wallis' Formula.* Also we have

$$z - \frac{z^3}{6} + \cdots = \sin z = z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2\pi^2}\right) = z\left[1 - \left(\sum_{k=1}^{\infty} \frac{1}{k^2\pi^2}\right) z^2 + \cdots\right].$$

By the uniqueness of the coefficients of z^3, we deduce that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

Now to prove Euler's formula for $\sin z$, we begin with the definition of infinite products.

Definitions. For a sequence $\{a_k\}$ of nonzero complex numbers, we define $\displaystyle\prod_{k=1}^{\infty} a_k = \lim_{n \to \infty} \prod_{k=1}^{n} a_k$. If the limit is a nonzero number, we say $\displaystyle\prod_{k=1}^{\infty} a_k$ *converges.* Otherwise, we say $\displaystyle\prod_{k=1}^{\infty} a_k$ *diverges.* For sequences with finitely many of the a_k's being 0 and $\displaystyle\prod_{k=1, a_k \neq 0}^{\infty} a_k$ converges, we also say $\displaystyle\prod_{k=1}^{\infty} a_k$ *converges to* 0.

Examples. (1) $\displaystyle\prod_{k=2}^{\infty} \left(1 - \frac{1}{k}\right) = \lim_{n \to \infty} \prod_{k=2}^{n} \frac{k-1}{k} = \lim_{n \to \infty} \frac{1}{n}$ diverges to 0.

(2) $\displaystyle\prod_{k=1}^{\infty} \left(1 + \frac{1}{k}\right) = \lim_{n \to \infty} \prod_{k=1}^{n} \frac{k+1}{k} = \lim_{n \to \infty} (n+1)$ diverges to ∞.

(3) $\displaystyle\prod_{k=2}^{\infty}\left(1-\frac{1}{k^2}\right)=\lim_{n\to\infty}\prod_{k=2}^{n}\frac{(k-1)(k+1)}{k^2}=\lim_{n\to\infty}\frac{n+1}{2n}=\frac{1}{2}$. (Note that $1-\dfrac{1}{k^2}=\left(1+\dfrac{1}{k}\right)\left(1-\dfrac{1}{k}\right)$. If we consider the factor $1-\dfrac{1}{k}$ as a "weight", then we see that an infinite product can be made to converge by putting proper weights on the terms.)

Term Test. If $\displaystyle\prod_{k=1}^{\infty} a_k$ converges, then

$$\lim_{k\to\infty} a_k=\lim_{k\to\infty}\left(\prod_{p=1,a_p\neq 0}^{k} a_p\Big/\prod_{p=1,a_p\neq 0}^{k-1} a_p\right)=\prod_{p=1,a_p\neq 0}^{\infty} a_p\Big/\prod_{p=1,a_p\neq 0}^{\infty} a_p=1.$$

Remark. For convergent products, the terms are close to 1. Since the logarithm of a finite product is the sum of the logarithms, we see easily that $\displaystyle\prod_{k=1}^{\infty} a_k$ converges if and only if $\displaystyle\sum_{k=1,a_k\neq 0}^{\infty}\operatorname{Log} a_k$ converges.

In the lemma below, it will be shown that for $|z|\leq\dfrac{1}{2}$, $|\operatorname{Log}(1-z)|\leq 2|z|$. So, by the absolute convergence test and the comparison test, $\displaystyle\sum_{k=1}^{\infty}|1-a_k|$ converges implies $\displaystyle\prod_{k=1}^{\infty} a_k$ converges.

Definition. An *elementary factor* is one of the following entire functions

$$E_0(z)=1-z\quad\text{and}\quad E_p(z)=(1-z)e^{z+\frac{z^2}{2}+\cdots+\frac{z^p}{p}}\quad\text{for }\ p=1,2,3,\ldots.$$

Lemma. If $|z|\leq\dfrac{1}{2}$, then $|\operatorname{Log}E_p(z)|\leq 2|z|^{p+1}$.

Proof. For $|z|\leq\dfrac{1}{2}$, let C be the line segment from 0 to z. Then

$$\operatorname{Log}(1-z)=-\int_C\frac{dw}{1-w}=-\int_C\sum_{k=0}^{\infty}w^k\,dw=-\sum_{k=0}^{\infty}\frac{z^{k+1}}{k+1}\quad\text{and}$$

$$|\operatorname{Log}E_p(z)|=\left|\sum_{k=p}^{\infty}\frac{z^{k+1}}{k+1}\right|\leq\sum_{k=p}^{\infty}\frac{|z|^{k+1}}{k+1}\leq\frac{|z|^{p+1}}{p+1}\sum_{k=p}^{\infty}\frac{1}{2^{k-p}}\leq 2|z|^{p+1}.$$

QED.

Theorem. Let $\{z_k\}$ be a sequence of nonzero complex numbers converging to ∞ and $\{m_k\}$ a sequence of nonnegative integers such that for any fixed $R>0$, $\displaystyle\sum_{k=1}^{\infty}\left(\frac{R}{|z_k|}\right)^{m_k}$ converges (*e.g.* $m_k=k$ because $\dfrac{R}{|z_k|}<\dfrac{1}{2}$ for k large). Then the product

$$\prod_{k=1}^{\infty}E_{m_k-1}\left(\frac{z}{z_k}\right)=\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)e^{\frac{z}{z_k}+\frac{z^2}{2z_k^2}+\cdots+\frac{z^{m_k-1}}{(m_k-1)z_k^{m_k-1}}}$$

converges to an entire function with roots z_k (repeated according to multiplicities.)

Proof. Fix $R > 0$. Let N be such that $k > N \Rightarrow 2R < |z_k|$. Then for $|z| \leq R$ and $k > N$, we have $\left|\frac{z}{z_k}\right| < \frac{1}{2}$. By the lemma,

$$\sum_{k=N+1}^{\infty} \left|\operatorname{Log} E_{m_k-1}\left(\frac{z}{z_k}\right)\right| \leq 2 \sum_{k=N+1}^{\infty} \left(\frac{R}{|z_k|}\right)^{m_k} < \infty.$$

By Weierstrass' M-test (and Weierstrass' theorem), $\displaystyle\sum_{k=N+1}^{\infty} \operatorname{Log} E_{m_k-1}\left(\frac{z}{z_k}\right)$ converges uniformly (to a holomorphic function) on $|z| \leq R$. Exponentiating this function and multiplying the first N terms, we see that $\displaystyle\prod_{k=1}^{\infty} E_{m_k-1}\left(\frac{z}{z_k}\right)$ is holomorphic on $|z| < R$. Since R is arbitrary, $\displaystyle\prod_{k=1}^{\infty} E_{m_k-1}\left(\frac{z}{z_k}\right)$ must be entire.

Clearly, the z_k's are roots. For $w \neq z_k$, $E_{m_k-1}\left(\frac{w}{z_k}\right) \neq 0$ and the convergence of $\displaystyle\prod_{k=1}^{\infty} E_{m_k-1}\left(\frac{w}{z_k}\right)$ implies $\displaystyle\lim_{K\to\infty} \prod_{k=K+1}^{\infty} E_{m_k-1}\left(\frac{w}{z_k}\right) = 1$. So for K large, $\displaystyle\prod_{k=K+1}^{\infty} E_{m_k-1}\left(\frac{w}{z_k}\right)$ is close to 1; in particular, it is nonzero. Then $\displaystyle\prod_{k=1}^{\infty} E_{m_k-1}\left(\frac{w}{z_k}\right) = \prod_{k=1}^{K} E_{m_k-1}\left(\frac{w}{z_k}\right) \prod_{k=K+1}^{\infty}\left(\frac{w}{z_k}\right) \neq 0$. Therefore, there cannot be any other roots. **QED.**

Weierstrass Factorization Theorem. Let f be an entire function with roots 0 (of order $m \geq 0$), $z_1, z_2, \ldots$ converging to ∞. Suppose there is a sequence $\{m_k\}$ of nonnegative integers as in the previous theorem, then there is an entire function g such that

$$f(z) \equiv e^{g(z)} z^m \prod_{k=1}^{\infty} \left(1 - \frac{z}{z_k}\right) e^{\frac{z}{z_k} + \cdots + \frac{z^{m_k-1}}{(m_k-1) z_k^{m_k-1}}}.$$

Proof. By the previous theorem, $h(z) = z^m \displaystyle\prod_{k=1}^{\infty} E_{m_k-1}\left(\frac{z}{z_k}\right)$ is entire and has the same roots (with the same multiplicities) as $f(z)$. Then $\frac{f}{h}$ is an entire function without any roots. By the logarithm theorem, $\frac{f}{h} = e^g$ for some entire function g. Therefore, $f = e^g h$. **QED.**

Finally, Euler's formula for $\sin z$ follows easily. Since $\displaystyle\sum_{k\neq 0} \left(\frac{R}{k\pi}\right)^2 < \infty$ for all $R > 0$, we may take $m_k = 2$ for all k. By the Weierstrass factorization theorem,

$$\sin z = e^{g(z)} z \prod_{k\neq 0} \left(1 - \frac{z}{k\pi}\right) e^{z/k\pi} = e^{g(z)} z \prod_{k=1}^{\infty} \left(1 - \frac{z}{k\pi}\right)\left(1 + \frac{z}{k\pi}\right).$$

Taking the logarithmic derivative, we get

$$\cot z = \frac{d}{dz}(\log \sin z) = g'(z) + \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2 - k^2\pi^2}.$$

From the last example of last section, we see that g must be constant. Now $e^g = \displaystyle\lim_{z\to 0} \frac{\sin z}{z} = 1$. Therefore,

$$\sin z = z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2\pi^2}\right).$$

Exercises

1. Prove that

   (a) $\displaystyle\prod_{n=1}^{\infty}\left(1+\frac{1}{n(n+2)}\right)=2;$ (b) $\displaystyle\prod_{n=2}^{\infty}\frac{n^3-1}{n^3+1}=\frac{2}{3}.$

2. Prove that $\displaystyle\prod_{n=1}^{\infty}\cos\frac{\alpha}{2^n}=\frac{\sin\alpha}{\alpha}$ and $\displaystyle\frac{2}{\sqrt{2}}\frac{2}{\sqrt{2+\sqrt{2}}}\frac{2}{\sqrt{2+\sqrt{2+\sqrt{2}}}}\cdots=\frac{\pi}{2}.$

3. Find $\displaystyle\prod_{n=2}^{\infty}\left(1+\frac{(-1)^{n-1}}{a_n}\right)$, where $\displaystyle a_n=\sum_{k=1}^{n-1}\frac{(-1)^{k-1}n!}{k!}.$

4. If $|z|<1$, prove that $\displaystyle\prod_{n=0}^{\infty}(1+z^{2^n})=\frac{1}{1-z}.$

5. Suppose $\displaystyle\sum_{n=1}^{\infty}|z_n|^2<\infty$. Is it necessary that $\displaystyle\prod_{n=1}^{\infty}\cos z_n$ must converge?

6. Show that $\displaystyle\sum_{k=1}^{\infty}(1-a_k)$ converges does not imply $\displaystyle\prod_{k=1}^{\infty}a_k$ converges by considering $\displaystyle a_k=1+\frac{(-1)^k}{\sqrt{k}}.$

Suggested Readings and References

There are a large number of textbooks on complex analysis. Some good texts for reference alongside this set of notes are:

[1] J. Bak and D. J. Newman, *Complex Analysis*, Springer-Verlag, 1982;
[2] R. P. Boas, *Invitation to Complex Analysis*, Random House, 1987;
[3] J. E. Marsden and M. J. Hoffman, *Basic Complex Analysis*, 2nd ed.,Freeman, 1987;
[4] Yu. V. Sidorov, M. V. Fedoryuk and M. I. Shabunin, *Lectures on the Theory of Functions of a Complex Variable*, Mir, 1985.

For students who like to have a simple glance at applications in science and engineering, we refer to

[5] S. Fisher, *Complex Variables*, 2nd ed., Brooks/Cole, 1990.

For *mature* students who like to read more about the subject, there are the classics:

[6] L. Ahlfors, *Complex Analysis*, 3rd ed., Mc Graw-Hill, 1979;
[7] S. Saks and A. Zygmund, *Analytic Functions*, 3rd ed., Elsevier, 1971;
[8] E. C. Titchmarsh, *The Theory of Functions*, 2nd ed., Oxford, 1939.

For students who like to do more problems, we recommend

[9] K. Knopp, *Problem Book in the Theory of Functions*, vol. I & II, Dover, 1948,1952;
[10] J. G. Krzyz, *Problems in Complex Variable Theory*, Elsevier, 1971;
[11] L. I. Volkovyskii, G. L. Lunts and I. G. Aramanovich, *A Collection of Problems on Complex Analysis*, Dover, 1991.

In passing, we will like to mention that there are many well-written historical accounts of the subject appeared in the periodical *The Mathematical Intelligencer*, published by Springer-Verlag. On the other hand, for those who prefer to read a book on the development of the subject with some history, the following is highly recommended:

[12] R. Remmert, *Theory of Complex Functions*, Springer-Verlag, 1991.

Among the references above, [1] and [4] are the best to read along with this set of notes. One of the many outstanding features of [4] is its large collection of examples, from simple to advanced. This reference also have excellent discussions on conformal mappings, multiple-valued analytic functions and asymptotic methods. On the other hand, [1] is a combination of beauty and elegance. There are interesting materials in [1] which do not appear anywhere else in the literature. Also, in [1], there are many ingenious ideas useful in different parts of mathematics and clever proofs of important theorems. For [2] and [3], the students will find interesting techniques, useful formulas and detailed discussions on specific materials. For example, the discussions on analytic continuation in [2] is very, very well presented, informative and worthy of any student's attention.

As for exercises, the problems in [9], [10] and [11] are very good for practice. Some are quite challenging and most will serve to strengthen the students' understanding.

For students who like to continue pursuing the subject, [8] should be a good place to start, then the second half of [6] and most of [7] would provide a solid foundation. Finally, those who plan to eventually specialize in analysis, we suggest the standard graduate texts:

[13] J. B. Conway, *Functions of One Complex Variable*, 2nd ed., Springer-Verlag, 1978;
[14] W. Rudin, *Real and Complex Analysis*, 3rd ed., Mc Graw-Hill, 1987.

Index