A DETERMINANT CONDITION FROM COMPLEX OSCILLATION THEORY

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Abstract. We give an explicit description of the solutions to the determinant condition on a third order complex linear differential equation studied by Chiang, Laine and Wang.

1. INTRODUCTION Content-Length: 12564

For complex differential equations, there is an interesting line of investigations on the distribution of the roots of the solutions. If a complex-valued function $f(z)$ has roots $z_1, z_2, \ldots$ (ordered by nondecreasing moduli), the *exponent of convergence* $\lambda(f)$ of f is defined as $\inf\{\lambda : \sum 1/|z_j|^\lambda < \infty\}$. The *oscillation theory* of complex differential equations (see [1] and [4]) investigates the exponents of convergence of the solutions, which yield informations on the distributions of the roots.

In [2], Bank and Laine studied the oscillation theory of certain second order differential equations with entire periodic coefficients and found explicit representations for the solutions with finite exponents of convergence. Recently, in [3], Chiang, Laine and Wang established the following result concerning the oscillation theory of a third order linear differential equation with entire periodic coefficients using Nevanlinna value distribution theory.

Theorem. *Let $K \in \mathbf{C}$ and suppose that*

$$f''' - Kf' + e^z f = 0 \tag{1.1}$$

admits a non-trivial solution f such that

$$\log^+ N(r, 1/f) = o(r) \qquad \text{as } r \to \infty. \tag{1.2}$$

Then there exist integers r and s such that $r + s \geq 0$ and

$$K = \frac{(r+s+1)^2}{9}. \tag{1.3}$$

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Moreover, if $n = r+s > 0$, then n satisfies the following tridiagonal $(n+1)\times(n+1)$-determinant condition:

$$\det \mathbf{A} = 0, \tag{1.4}$$

where the non-zero diagonals of $\mathbf{A}$ *are determined by*

$$\begin{cases} a_{j,j-1} := (j-1)j(j+1) - 2jn - jn^2, & j = 1,\dots,n, \\ a_{j,j} := -3j(j+1) + 2n + n^2, & j = 0,\dots,n, \\ a_{j,j+1} := 3(j+1), & j = 0,\dots,n-1. \end{cases} \tag{1.5}$$

Furthermore, f admits one of the following representations:

$$f_i(z) = e^{-(s+1)z/3}\psi(e^{z/3})\exp(c_i e^{z/3}), \tag{1.6}$$

where $c_i^3 + 27 = 0, i = 1,2,3$, and

$$\psi(\zeta) = \sum_{j=-r}^{s} d_j\zeta^j, \qquad d_{-r}d_s \neq 0. \tag{1.7}$$

Conversely, suppose K takes the form (1.3) and, if $n = r+s > 0$ and n satisfies (1.4) and (1.5), then there exist a rational function of the form (1.7) such that the three functions defined by (1.6) are linearly independent solutions of (1.1) each with exponent of convergence $\lambda(f_i) \leq 1$ for $i = 1,2,3$.

As observed in the concluding remarks of [3], the matrix $\mathbf{A}$ can be expressed as the product of three matrices $(\beta_{ij}), (\alpha_{ij})$ and (γ_{ij}), where (α_{ij}) is a diagonal matrix such that

$$\alpha_{0,0} = a_{0,0} = n(n+2) \tag{1.8}$$

and

$$\alpha_{j+1,j+1} = a_{j+1,j+1} - \frac{a_{j+1,j}a_{j,j+1}}{\alpha_{j,j}} = A_j + \frac{B_j}{\alpha_{j,j}}, \qquad j = 0,\dots,n-1, \tag{1.9}$$

with

$$A_j = a_{j+1,j+1} = n(n+2) - 3(j+1)(j+2) \tag{1.10}$$

and

$$B_j = -a_{j+1,j}a_{j,j+1} = 3(j+1)^2[n(n+2) - j(j+2)], \tag{1.11}$$

while (β_{ij}) is a lower triangular matrix such that $\beta_{j,j} = 1$ for all j, and (γ_{ij}) is an upper triangular matrix such that $\gamma_{j,j} = 1$ for all j. So the determinant condition (1.4) is equivalent to the product of $\alpha_{j,j}$, $j = 0,\dots,n$ being zero.

In [3], Chiang, Laine and Wang expressed that the determinant condition (1.4) in the theorem seemed to be equivalent to $n \not\equiv 2 \pmod 3$, which also meant K would be a ninth of a perfect square, but not an integer. Here, we will prove that this is indeed the case. In the next section, we will first give the proof of the easier half, namely the determinant condition implies $n \not\equiv 2 \pmod 3$. Then the remaining half will be established by analyzing the entries $\alpha_{j,j}$ using three recurrence relations.

2. PROOF OF EQUIVALENCE

First we present a proof of the easier half.

Proposition 2.1. *If* $\det \mathbf{A} = \prod_{j=0}^{n} \alpha_{j,j} = 0$, *then* $n \not\equiv 2 \pmod 3$.

Proof. Suppose $n \equiv 2 \pmod 3$, then $n(n+2) \equiv -1 \pmod 3$. We will show by induction that the entries $\alpha_{j,j}$ is either of the form $(3p_j+1)/(3q_j-1)$ or of the form $(3p_j-1)/(3q_j+1)$, where p_j and q_j are integers. Then we will get the contradiction

$$\det \mathbf{A} = \prod_{j=0}^{n} \alpha_{j,j} = \prod_{j=0}^{n} \frac{3p_j \pm 1}{3q_j \mp 1} \neq 0.$$

Now since $\alpha_{0,0} = n(n+2) \equiv -1 \pmod 3$, $\alpha_{0,0} = (3p_0 - 1)/1$. For the inductive step, suppose $\alpha_{j,j} = (3p_j+1)/(3q_j-1)$. Since $A_j \equiv -1 \pmod 3$ and $B_j \equiv 0 \pmod 3$, we have $A_j = 3r_j + 1$ and $B_j = 3s_j$ for some integers r_j, s_j and

$$\begin{aligned}\alpha_{j+1,j+1} = A_j + \frac{B_j}{\alpha_{j,j}} &= (3r_j - 1) + \frac{3s_j}{(3p_j+1)/(3q_j-1)} \\ &= \frac{3(r_jp_j - p_j + r_j - 3s_jq_j - s_j) - 1}{3p_j + 1}.\end{aligned}$$

Similarly, suppose $\alpha_{j,j} = (3p_j-1)/(3q_j+1)$, then

$$\alpha_{j+1,j+1} = A_j + \frac{B_j}{\alpha_{j,j}} = \frac{3(r_jp_j - p_j - r_j + 3s_jq_j + s_j) + 1}{3p_j - 1}.$$

This completes the induction. □

Next, we will examine the converse.

Proposition 2.2. *The sequences* $\{f_k\}, \{g_k\}, \{h_k\}$ *defined by*

$$f_0(x) = g_0(x) = h_0(x) = 1 \tag{2.1}$$

and for $k = 1, 2, 3, \ldots,$

$$f_k(x) = [x - 3(3k-1)3k]h_{k-1}(x) + 3(3k-1)^2 g_{k-1}(x), \tag{2.2}$$

$$\begin{aligned}[x - 3k(3k+2)]g_k(x) = {} & [x - 3(3k)(3k+1)]f_k(x) \\ & + 27k^2[x - (3k-1)(3k+1)]h_{k-1}(x),\end{aligned} \tag{2.3}$$

$$[x - (3k+1)(3k+3)]h_k(x) = [x - 3(3k+1)(3k+2)]g_k(x) + 3(3k+1)^2 f_k(x) \tag{2.4}$$

are all polynomials.

Assuming the truth of Proposition 2.2 for the moment, the converse of Proposition 2.1 follows easily.

Theorem 2.3. *Let $m = n(n+2)$, then*

$$\alpha_{j,j} = \begin{cases} [m - 3k(3k+2)]\dfrac{g_k(m)}{f_k(m)} & \text{if } j = 3k, \\ [m - (3k+1)(3k+3)]\dfrac{h_k(m)}{g_k(m)} & \text{if } j = 3k+1, \\ \dfrac{f_{k+1}(m)}{h_k(m)} & \text{if } j = 3k+2. \end{cases}$$

Consequently,

$$\det \mathbf{A} = \prod_{j=0}^{n} \alpha_{j,j} = \begin{cases} g_l(m) \times \displaystyle\prod_{\substack{j=0 \\ j \not\equiv 2 \pmod 3}}^{n} [m - j(j+2)] = 0 & \text{if } n = 3l, \\ h_l(m) \times \displaystyle\prod_{\substack{j=0 \\ j \not\equiv 2 \pmod 3}}^{n} [m - j(j+2)] = 0 & \text{if } n = 3l+1, \\ f_{l+1}(m) \times \displaystyle\prod_{\substack{j=0 \\ j \not\equiv 2 \pmod 3}}^{n} [m - j(j+2)] \neq 0 & \text{if } n = 3l+2. \end{cases}$$

Proof of Theorem 2.3. The three initial cases $\alpha_{0,0} = m, \alpha_{1,1} = m - 3, \alpha_{2,2} = m - 6$ follow easily from (2.1) and (2.2). Next, suppose the case $j = 3k - 1$ is true, then using (2.3), we have

$$\begin{aligned} \alpha_{3k,3k} &= A_{3k-1} + \frac{B_{3k-1}}{\alpha_{3k-1,3k-1}} \\ &= m - 3(3k)(3k+1) + 3(3k)^2[m - (3k-1)(3k+1)]\frac{h_{k-1}(m)}{f_k(m)} \\ &= [m - 3k(3k+2)]\frac{g_k(m)}{f_k(m)}. \end{aligned}$$

Suppose the case $j = 3k$ is true, then using (2.4), we have

$$\begin{aligned} \alpha_{3k+1,3k+1} &= A_{3k} + \frac{B_{3k}}{\alpha_{3k,3k}} \\ &= m - 3(3k+1)(3k+2) + \frac{3(3k+1)^2 f_k(m)}{g_k(m)} \\ &= [m - (3k+1)(3k+3)]\frac{h_k(m)}{g_k(m)}. \end{aligned}$$

Suppose the case $j = 3k+1$ is true, then using (2.2), we have

$$\begin{aligned} \alpha_{3k+2,3k+2} &= A_{3k+1} + \frac{B_{3k+1}}{\alpha_{3k+1,3k+1}} \\ &= m - 3(3k+2)(3k+3) + \frac{3(3k+2)^2 g_k(m)}{h_k(m)} \\ &= \frac{f_{k+1}(m)}{h_k(m)}. \end{aligned}$$

Therefore, the first statement follows by induction. As for the last statement, it follows from telescoping products, the factor $j = n$ and Proposition 2.1. $\square$

To complete the overall argument, we now give a proof of Proposition 2.2.

Proof of Proposition 2.2. Suppose f_j, g_j, h_j are polynomials for $j = 0, 1, \ldots, k-1$. Then (2.2) implies f_k is a polynomial. Now to show g_k is a polynomial, it is equivalent to showing the right side of (2.3) vanishes at $x = 3k(3k+2)$. Substituting $x_0 = 3k(3k+2)$ for x into the right side of (2.3) and simplifying, we see that we have to show

$$f_k(x_0) = 9kh_{k-1}(x_0). \tag{2.5}$$

Now by (2.2), we get

$$f_k(x_0) = -3k(6k-5)h_{k-1}(x_0) + 3(3k-1)^2 g_{k-1}(x_0). \tag{2.6}$$

By (2.4) with k replaced by $k-1$, we get (after cancelling a factor of 3)

$$4kh_{k-1}(x_0) = (3k-2)^2 f_{k-1}(x_0) - (6k^2 - 11k + 2)g_{k-1}(x_0). \tag{2.7}$$

Substituting (2.7) into (2.6), we get

$$4f_k(x_0) = 3(36k^3 - 60k^2 + 43k - 6)g_{k-1}(x_0) - 3(6k-5)(3k-2)^2 f_{k-1}(x_0). \tag{2.8}$$

Next if in (2.2) we replace k by $k+1$, multiply both sides by $x - (3k+1)(3k+2)$, and apply (2.4) to the right side, we will get (after grouping terms)

(2.9)
$$\begin{aligned}[x - (3k+1)(3k+3)]f_{k+1}(x) = {}& [x^2 - 3(3k+2)^2 x + 6(3k+1)(3k+2)^2(3k+3)] \\ & \times g_k(x) + 3(3k+1)^2[x - 3(3k+2)(3k+3)]f_k(x).\end{aligned}$$

If we replace k by $k-1$ in (2.4) and apply it to the right side of (2.3), we get

(2.10)
$$\begin{aligned}[x - 3k(3k+2)]g_k(x) = {}& [x - 9k(3k+1)]f_k(x) + \frac{27k^2[x - (3k-1)(3k+1)]}{x - (3k-2)(3k)} \\ & \times \left\{[x - 3(3k-2)(3k-1)]g_{k-1}(x) + 3(3k-2)^2 f_{k-1}(x)\right\}.\end{aligned}$$

Observe that (2.9) allows us to express g_k in terms of f_k and f_{k+1}. Replacing k by $k-1$, we can express g_{k-1} in terms of f_{k-1} and f_k. In (2.10), if we substitute g_k in terms of f_k and f_{k+1} and g_{k-1} in terms of f_{k-1} and f_k, then we will get a recurrence relation involving f_{k-1}, f_k and f_{k+1}. Setting $x = x_0 = 3k(3k+2)$ in this recurrence relation and simplifying, we will get

$$2k(6k-5)f_k(x_0) = 9(3k-1)(3k-2)^2 f_{k-1}(x_0). \tag{2.11}$$

If we solve for $f_{k-1}(x_0)$ in (2.11) and substitute that expression into the right side of (2.8), then after simplification, we will get

$$9(3k-1)g_{k-1}(x_0) = 2f_k(x_0). \tag{2.12}$$

If we solve for $g_{k-1}(x_0)$ in (2.12) and substitute that expression into the right side of (2.6), then after simplification, we will get (2.5). Therefore, g_k is a polynomial.

Now to show h_k is a polynomial, it is equivalent to showing the right side of (2.4) vanishes at $x = (3k+1)(3k+3)$. Substituting $x_1 = (3k+1)(3k+3)$ for x into the right side of (2.4), we see that we have to show

$$(2k+1)g_k(x_1) = (3k+1)f_k(x_1). \tag{2.13}$$

Now by (2.2), we get

$$f_k(x_1) = -3(6k^2 - 7k - 1)h_{k-1}(x_1) + 3(3k-1)^2 g_{k-1}(x_1). \tag{2.14}$$

By (2.3), we get (after cancelling a factor of 3)

$$(2k+1)g_k(x_1) = -(2k-1)(3k+1)f_k(x_1) + 36k^2(3k+1)h_{k-1}(x_1). \tag{2.15}$$

Next if we use (2.2) to substitute $f_k(x)$ in (2.4), we will get

(2.16)
$$\begin{aligned}[x-(3k+1)(3k+3)]h_k(x) = {} & [x-3(3k+1)(3k+2)]g_k(x) \\ & + [x-3(3k-1)(3k)]h_{k-1}(x) + 3(3k-1)^2 g_{k-1}(x).\end{aligned}$$

If we use (2.2) to substitute away $f_k(x)$ in (2.3), then after simplification, we will get

(2.17)
$$\begin{aligned}[x^2 - 27k^2x + 54k^2(9k^2-1)]h_{k-1}(x) = {} & [x-3k(3k+2)]g_k(x) \\ & - 3(3k-1)^2[x-9k(3k+1)]g_{k-1}(x).\end{aligned}$$

Replacing k by $k+1$ in (2.17), we can express h_k in terms of g_k and g_{k+1}. In (2.16), if we substitute h_{k-1} using (2.17) and h_k using the "k replaced by $k+1$" version of (2.17), we will get a recurrence relation involving g_{k-1}, g_k and g_{k+1}. Setting $x = x_1 = (3k+1)(3k+3)$ in this recurrence relation and simplifying, we will get

$$(4k^2-1)g_k(x_1) = 18k(3k-1)^2 g_{k-1}(x_1). \tag{2.18}$$

If we solve for $g_{k-1}(x_1)$ in (2.8) and substitute that expression into (2.14), we will get

$$f_k(x_1) = -3(6k^2-7k-1)h_{k-1}(x_1) + \frac{4k^2-1}{6k}g_k(x). \tag{2.19}$$

Multiplying both sides of (2.15) by $2k-1$ and making substitution to the right side of (2.19), we will get (after simplification)

$$h_{k-1}(x_1) = \frac{f_k(x_1)}{18k}. \tag{2.20}$$

Putting (2.20) into (2.14) and simplifying, we obtain (2.13). Therefore, h_k is a polynomial and the induction is complete. □

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References

[1] S. Bank and I. Laine, *On the Oscillation Theory of $f'' + Af = 0$ where A is Entire*, Trans. Amer. Math. Soc. **273** (1982), 351–363.

[2] S. Bank and I. Laine, *Representations of Solutions of Periodic Second Order Linear Differential Equations*, J. Reine Angew. Math. **344** (1983), 1-21.

[3] Y. M. Chiang, I. Laine and S. Wang, *An Oscillation Result of a Third Order Linear Differential Equation with Entire Periodic Coefficients*, Complex Variables (to appear).

[4] I. Laine, *Nevanlinna Theory and Complex Differential Equations*, Walter de Gruyter, Berlin, 1992.