ALGEBRAS ASSOCIATED WITH BLASCHKE PRODUCTS OF TYPE G

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Abstract. Let Ω (*resp.* Ω_{fi}) be the set of all interpolating Blaschke products of (*resp.* finite) type G. Let E (*resp.* E_{fi}) be the Douglas algebra generated by H^∞ and the complex conjugates of elements of Ω (*resp.* Ω_{fi}). Our main results are that the set of all invertible inner functions in E (*resp.* E_{fi}) is the set of all finite products of elements of Ω (*resp.* Ω_{fi}), which is also the closure of Ω (*resp.* Ω_{fi}) among the Blaschke products. Consequently, finite convex combinations of finite products of elements of Ω (*resp.* Ω_{fi}) are dense in the closed unit ball of the subalgebra of H^∞ generated by Ω (*resp.* Ω_{fi}).

1. Introduction

Let D be the open unit disk and T be the unit circle on the complex plane. Let H^∞ be the Banach algebra of bounded holomorphic functions on the open unit disk D. Via radial limits we can consider H^∞ as a closed subalgebra of L^∞, where L^∞ is the family of all essentially bounded measurable functions on T. Any function h in H^∞ with $|h| = 1$ *a.e.* on T is called an *inner* function. Let $\{z_n\}$ be a sequence in D with $\sum_n (1 - |z_n|) < \infty$, then the function

$$b(z) = \prod_n \frac{\bar{z}_n}{|z_n|} \frac{z_n - z}{1 - \bar{z}_n z} \quad \text{for } z \in D$$

is called a *Blaschke product* with roots $\{z_n\}$. Let

$$\delta(b) = \inf_k \prod_{n \neq k} \left| \frac{z_k - z_n}{1 - \bar{z}_n z_k} \right| .$$

If $\delta(b) > 0$, then b and $\{z_n\}$ are called *interpolating.* By [Ca], if $\delta(b) > 0$, then for every bounded sequence $\{a_n\}$, there exists f in H^∞ such that $f(z_n) = a_n$ for every n. If

$$\lim_{k \to \infty} \prod_{n \neq k} \left| \frac{z_k - z_n}{1 - \bar{z}_n z_k} \right| = 1,$$

then b and $\{z_n\}$ are called *thin* or *sparse.*

We denote by $M(H^\infty)$ the maximal ideal space of H^∞. A closed subalgebra B between H^∞ and L^∞ is called a *Douglas algebra.* We denote by $M(B)$ the maximal

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ideal space of the Douglas algebra B. For an interpolating Blaschke product b, we denote by $H^\infty[\bar{b}]$ the Douglas algebra generated by H^∞ and the complex conjugate of b. For a function f in H^∞, let

$$Z(f) = \{x \in M(H^\infty) : f(x) = 0\}$$

and for $0 < c \leq 1$,

$$\{|f| < c\} = \{x \in M(H^\infty) : |f(x)| < c\}.$$

For a point x in $M(H^\infty)$, there is a representing measure μ_x on $M(L^\infty)$, that is,

$$f(x) = \int_{M(L^\infty)} f \, d\mu_x$$

for every $f \in H^\infty$. We denote by $\operatorname{supp} \mu_x$ the support set for the representing measure μ_x.

By the Corona Theorem, D can be considered as a dense subset of $M(H^\infty)$. For points x, y in $M(H^\infty)$, let

$$\rho(x, y) = \sup\{|f(y)| : f \in H^\infty, \|f\|_\infty \leq 1, f(x) = 0\}$$

and put

$$P(x) = \{m \in M(H^\infty) : \rho(m, x) < 1\}.$$

The set $P(x)$ is called the *Gleason part containing* x. For $z, w \in D$, we have

$$\rho(z, w) = \left| \frac{z - w}{1 - \bar{w}z} \right|$$

and $P(0) = D$. We call $x \in M(H^\infty)$ a *trivial part* if $P(x) = \{x\}$. Let

$$G = \bigcup \{P(x) : x \in M(H^\infty), P(x) \neq \{x\}\},$$

then G is an open subset of $M(H^\infty)$.

By Hoffman's work [Ho], $Z(b) \subset G$ for every interpolating Blaschke product b and for each x in G, there exists an interpolating Blaschke product b such that $b(x) = 0$. Also by [Ho], for each $x \in G$, there exists a one-to-one and onto map $L_x : D \to P(x)$ such that $f \circ L_x \in H^\infty$ for every $f \in H^\infty$. The map L_x is given as follows. Let $\{z_\alpha\}$ be a net in D with $z_\alpha \to x$ and let $L_{z_\alpha}(z) = (z + z_\alpha)/(1 + \bar{z}_\alpha z)$, then

$$(f \circ L_x)(z) = \lim_\alpha (f \circ L_{z_\alpha})(z) \text{ for } f \in H^\infty \text{ and } z \in D.$$

A Blaschke product b is of *type* G if it is interpolating and $\{|b| < 1\} \subset G$. It is of *finite type* G if it is of type G and for every $x \in Z(b)$, the set $Z(b) \cap P(x)$ is finite.

A Blaschke product is *locally thin* if for each $x \in Z(b)$, there is an interpolating Blaschke product q such that

$$\lim_{\alpha}(1-|z_{n_\alpha}|^2)|q'(z_{n_\alpha})| = 1,$$

whenever $\{z_\alpha\}$ is a subnet of the root sequence $\{z_n\}$ of q that converges to x. Note q may be different from b. In fact, by [Go-Li-Mo], if $b = q$ for every $x \in z(b)$, then b is a thin Blaschke product. Blaschke products of type G, finite type G and locally thin Blaschke products are very important in the studies of Douglas algebras (see for example, [Go-Li-Mo], [Gu], [Gu-Iz-1] and [Gu-Iz-2]).

Let Ω be the family of all interpolating Blaschke products of finite type G and A be the closed subalgebra of H^∞ generated by Ω. Let B be the smallest (closed) C^*-subalgebra of L^∞ containing A. That is, B is generated by the ratio of interpolating Blaschke products of type G. Let E be the Douglas algebra generated by H^∞ and the complex conjugate of elements of Ω.

Our main results are that every inner function in E is a finite product of interpolating Blaschke products of type G, from which we are able to identify the closure of the interpolating Blaschke products of type G among the Blaschke products. As a consequence, we get $B = C_E$, where C_E denotes the C^*-subalgebra of L^∞ generated by the invertible inner functions in E and their complex conjugates. Another consequence is that the finite products of interpolating Blaschke products of type G are the only inner functions that are in $H^\infty \cap B$. Hence by Theorem 4.1 of [Ch-Ma], $A = H^\infty \cap B$ and finite convex combinations of finite products of interpolating Blaschke products of type G are dense in the closed unit ball of A. For Blaschke product of finite type G, we obtain similar results. In obtaining these results, we follow the approach in [He], but our proofs rely heavily on the results about type G and finite type G developed in [Gu-Iz-1] and [Gu-Iz-2].

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2. Results For Type G

We begin with a few useful propositions concerning basic properties of interpolating Blaschke product of type G.

Proposition 1. *If B is of type G and b is a subproduct of B, then b is of type G. If b_1, b_2 are of type G and b_1b_2 is an interpolating Blaschke product, then b_1b_2 is of type G. If b is of type G and $b_\lambda = (b-\lambda)/(1-\bar{\lambda}b)$ is an interpolating Blaschke product, then b_λ is of type G. The statements also hold if type G is replaced by finite type G.*

Proof. For type G, the first statement follows from $\{|b| < 1\} \subset \{|B| < 1\} \subset G$. The second statement follows from $\{|b_1b_2| < 1\} = \{|b_1| < 1\} \cup \{|b_2| < 1\} \subset G$. The third statement follows from $\{|b_\lambda| < 1\} = \{|b| < 1\} \subset G$.

For finite type G, the first statement follows from $Z(b) \cap P(m) \subset Z(B) \cap P(m)$. The second statement follows from $Z(b_1b_2) \cap P(m) = (Z(b_1) \cap P(m)) \cup (Z(b_2) \cap$

$P(m)$). The third statement follows from Theorem 3.2 (iii) of [Gu-Iz-2] and the fact $H^\infty[\bar{b}_\lambda] = H^\infty[\bar{b}]$ by considering their maximal ideal spaces. $\square$

We remark that not all of the statements in Proposition 1 are true for the family of thin Blaschke products.

Proposition 2. *Suppose b is an interpolating Blaschke product of type G with roots $\{z_n\}$ in D. Let q be an interpolating Blaschke product with roots $\{w_n\}$ in D such that $\rho(w_n, z_n) \le r$ for all n and for some $r < 1$, then q is of type G.*

Proof. Suppose $0 < \lambda < 1$ and $z \in D$ such that $|q(z)| < \lambda$. Then, by Lemma 1.4 and Corollary 1.3 on page 4 of [Gar],

$$|b(z)| = \prod_{n=1}^{\infty} \rho(z, z_n) \le \prod_{n=1}^{\infty} \Big(\frac{\rho(z, w_n) + r}{1 + r\rho(z, w_n)}\Big) \le \frac{|q(z)| + r}{1 + r|q(z)|} < \frac{\lambda + r}{1 + r\lambda} = \lambda' < 1.$$

Consequently,

$$\{|q| < 1\} = \bigcup_{0<\lambda<1} \{|q| < \lambda\} \subset \bigcup_{0<\lambda'<1} \{|b| < \lambda'\} = \{|b| < 1\} \subset G. \quad \square$$

Proposition 3. *Let $\mathcal{F} = \{x \in M(H^\infty) : x$ is in the closure of some interpolating sequence in D whose Blaschke product is of type $G\}$, then $\mathcal{F}$ is the union of a family of nontrvial Gleason parts. In fact, $\mathcal{F} = \bigcup\{P(m) : m \in Z(b)$ for some $b \in \Omega\}$.*

Proof. By the definition of $\mathcal{F}$, every point in $\mathcal{F}$ belongs to a nontrivial Gleason part. So let $m_0 \in \mathcal{F}$ and $m \in P(m_0)$. Then $m_0 \in \overline{\{z_n\}}$ for some interpolating sequence $\{z_n\}$ whose Blaschke product $b(z)$ is of type G. So there is a subnet $\{z_\alpha\}$ of $\{z_n\}$ converging to m_0. Since $m \in P(m_0)$, there is $\zeta \in D$ such that

$$\lim_\alpha L_{z_\alpha}(\zeta) = L_{m_0}(\zeta) = m.$$

Let

$$\zeta_n = L_{z_n}(\zeta) = \frac{\zeta + z_n}{1 + \bar{z}_n\zeta},$$

then $\rho(\zeta_n, z_n) = |\zeta| < 1$ for all n. By Corollary 1.6 on page 407 of [Gar], there is a factorization $b = b_1 b_2 \cdots b_k$ with

$$\delta(b_j) > \frac{2|\zeta|}{1 + |\zeta|^2} \text{ for } j = 1, 2, \dots, k.$$

By Lemma 5.3 on page 310 of [Gar], each $Z(b_j) \cap D = \{z_{j,n}\}$ is interpolating.

Since

$$\overline{\{z_n\}} = \bigcup_{j=1}^{k} \overline{Z(b_j)}$$

and the $Z(b_j)$'s have disjoint closures [Gar, p. 422], it follows that $m_0 \in \overline{Z(b_j)} = \overline{\{z_{j,n}\}}$ for some j and the net $\{z_\alpha\}$ is eventually in $Z(b_j)$ because

$$M(H^\infty) \setminus \bigcup_{i \neq j} \overline{Z(b_i)}$$

is an open neighborhood of $\overline{Z(b_j)}$ and $m_0 \in \overline{Z(b_j)}$.

By Proposition 1, each b_j is of type G. By Proposition 2, the Blaschke product with roots $\{\zeta_{j,n}\}$ is of type G. Finally,

$$\lim_\alpha \zeta_{j,\alpha} = \lim_\alpha L_{z_{j,\alpha}}(z) = L_{m_0}(\zeta) = m$$

and our assertion follows. □

Proposition 4. *An interpolating Blaschke product b of type G has modulus 1 on those Gleason parts of $M(H^\infty)$ that do not contain a point in $Z(b)$. Consequently, b has modulus 1 on $M(H^\infty) \setminus \mathcal{F}$.*

Proof. By Lemma 1.1 of [Gu-Iz-2] (or Theorem 1 of [Gu-Iz-1]) we have

$$\{|b| < 1\} = \bigcup_{m \in Z(b)} P(m).$$

Thus

$$|b| = 1 \quad \text{on} \quad M(H^\infty) \setminus \bigcup_{m \in Z(b)} P(m),$$

which contains $M(H^\infty) \setminus \mathcal{F}$ by Proposition 3. □

Corollary 5. *$M(E) = M(H^\infty) \setminus \mathcal{F}$.*

Proof. By Theorem 1.3 on page 375 of [Gar],

$$M(E) = \{m \in M(H^\infty) : |b(m)| = 1 \text{ for all } b \in \Omega\}.$$

Now the results follows immediately from Propositions 3 and 4. □

Corollary 6. *A is a proper subalgebra of H^∞.*

Proof. This follows because $M(H^\infty) \setminus (\mathcal{F} \cup M(L^\infty))$ is not empty. □

Theorem 7. *Every invertible inner function in E is a finite product of interpolating Blaschke products of type G.*

Proof. Let u be an arbitrary invertible inner function in E, then $\bar{u} = u^{-1}$ in $E \subset L^\infty$. So $H^\infty[\bar{u}] \subset E$. By Corollary 5 above and Theorem 1.3 on page 375 of [Gar]

$$M(H^\infty) \setminus \mathcal{F} = M(E) \subset M(H^\infty[\bar{u}]) = \{|u| = 1\}.$$

Hence $\{|u| < 1\} \subset \mathcal{F} \subset G$. This implies $Z(u)$ cannot contain any trivial part. By Corollary 24 of [McD-Su], $u = b_1 b_2 \cdots b_n$, where each b_j is an interpolating Blaschke product. Finally, for each j,

$$\{|b_j| < 1\} \subset \bigcup_{k=1}^n \{|b_k| < 1\} = \{|u| < 1\} \subset G. \quad \square$$

Corollary 8. *$B = C_E$, the C^*-subalgebra of L^∞ generated by the inner functions invertible in E and their complex conjugates.*

Corollary 9. *Let b be a finite product of interpolating Blaschke products of type G. If $f \in H^\infty$ is such that $\|f\|_\infty < 1$ and $\bar{f}b$ equals an H^∞ function g almost everywhere on T, then the function*

$$b_f(z) = \frac{b(z) - f(z)}{1 - g(z)} \quad \text{for } z \in D$$

is a finite product of interpolating Blaschke products of type G.

Proof. Just observe that b_f is an invertible inner function in E. □

In [Ch-Ma], Chang and Marshall showed that for an arbitrary Douglas algebra J, the closed unit ball of $H^\infty \cap C_J$ is the norm-closed convex hull of the Blaschke products in $H^\infty \cap C_J$, where C_J is the C^*-subalgebra of L^∞ generated by the invertible inner functions in J and their complex conjugates. They also showed that $J = H^\infty + C_J$ and that D is dense in the maximal ideal space of $H^\infty \cap C_J$. In our case $J = E$, $C_J = B$ and we have the following corollary. (Note that an inner function in $H^\infty \cap B$ is invertible in E.)

Corollary 10.

(1) *$A = H^\infty \cap B$, and finite convex combinations of finite products of interpolating Blaschke products of type G are dense in the closed unit ball of A.*
(2) *$E = H^\infty + B$.*
(3) *D is dense in the maximal ideal space $M(A)$ of A.*

3. Results For Finite Type G

Next we will turn to the main results for interpolating Blaschke product of finite type G analogous to those established in section 2. Let Ω_{fi} be the family of all interpolating Blaschke products of finite type G and let E_{fi} be the Douglas algebra generated by H^∞ and the complex conjugate of elements of Ω_{fi}. We will show that Theorem 7 holds if E is replaced by E_{fi}.

Theorem 11. *Every invertible inner function in E_{fi} is a finite product of interpolating Blaschke products of finite type G.*

Proof. We will first show that the analog of Proposition 2 is true for interpolating Blaschke products of finite type G. Let b be an interpolating Blaschke product of finite type G with zeros $\{z_n\}$ in D and let q be an interpolating Blaschke product with zeros $\{w_n\}$ in D such that $\rho(z_n, w_n) \le r$ for all n and some $r < 1$. The proof of Proposition 2 shows that $\{|q| < 1\} \subset \{|b| < 1\}$. Since b is of finite type G, by Theorem 2.1 of [Gu-Iz-2], there is a subproduct b_0 of b such that $\{|b_0| < 1\} = \{|q| < 1\}$. We will show that q is of finite type G.

For $x \in Z(q)$, $|b_0(x)| < 1$. By Lemma 1.1 of [Gu-Iz-2], there is an $x_0 \in Z(b_0)$ such that $x \in P(x_0)$. Suppose the set $Z(q) \cap P(x) = Z(q) \cap P(x_0)$ is infinite. Then, by Theorem 3.1(i) of [Gu-Iz-2], there exist y and y_0 in $Z(q)$ such that $\operatorname{supp} \mu_y \subsetneq \operatorname{supp} \mu_{y_0}$. Hence there are m and m_0 in $Z(b_0)$ such that $y \in P(m)$ and $y_0 \in P(m_0)$, but then $\operatorname{supp} \mu_m \subsetneq \operatorname{supp} \mu_{m_0}$ (because by page 143 of [Gam], $\operatorname{supp} \mu_y = \operatorname{supp} \mu_m$ and $\operatorname{supp} \mu_{y_0} = \operatorname{supp} \mu_{m_0}$). Since b_0 is of finite type G, this contradicts Theorem 3.2(ii) of [Gu-Iz-2]. Thus $Z(q) \cap P(x_0) = Z(q) \cap P(x)$ must be finite.

Next we remark that the analogs of Propositions 3 and 4 for finite type G also hold by the same reasoning because of the analog of Proposition 2 for finite type G.

Now let u be an invertible inner function in $E_{\text{fi}} \subset E$. By Theorem 7, $u = u_1 u_2 \cdots u_m$, where each u_i is of type G. Observe that if $\mathcal{F}_{\text{fi}}$ is the analog of $\mathcal{F}$ for finite type G, then

$$Z(u_i) \subset \{|u| < 1\} \subset \mathcal{F}_{\text{fi}} = \bigcup_{b \in \Omega_{\text{fi}}} \{|b| < \frac{1}{2}\}.$$

Let $\delta_i = \inf\{\rho(w,z) : w, z \in Z(u_i) \cap D, w \neq z\} > 0$. Since $Z(u_i)$ is compact,

$$Z(u_i) \subset \bigcup_{j=0}^{n_i} \{|b_j| < \frac{1}{2}\},$$

for some $b_1, b_2, \ldots, b_{n_i}$ of finite type G. Let

$$S_{ij} = Z(u_i) \cap \{|b_j| < \frac{1}{2}\} \cap D,$$

then

$$Z(u_i) \cap D = \bigcup_{j=1}^{n_i} S_{ij}.$$

By removing overlapping elements, we may assume the S_{ij}'s are disjoint.

Since b_j is of type G, by Lemma 2.1 of [Gu-Iz-2], there is $\delta < 1$ such that

$$S_{ij} \subset \{|b_j| < \frac{1}{2}\} \subset \bigcup_n \{z \in D : \rho(z, z_{j,n}) \leq \delta\},$$

where $\{z_{j,n}\}$ is the root sequence of b_j in D. For each disk $B(z_{j,n}, \delta) = \{z \in D : \rho(z, z_{j,n}) \leq \delta\}$, there are at most k_i elements of S_{ij} in $B(z_{j,n}, \delta)$, where k_i depends only on δ_i. So S_{ij} is the union of at most k_i sequences, each of which has at most one element in each $B(z_{j,n}, \delta)$. By the analog of Proposition 2 for finite type G, proved above, the Blaschke product with root sequence S_{ij} is a product of at most k_i interpolating Blaschke product of finite type G. So u_i is a finite product of at most $n_i k_i$ interpolating Blaschke product of finite type G. Therefore, u is a finite product of interpolating Blaschke product of finite type G. $\square$

In general, it is a difficult problem to determine the closure of an infinite set of interpolating Blaschke products among the family of all Blaschke products (see for example [Li]). However, for Blaschke products of type G and finite type G, their closures can be identified because of Proposition 1, Theorems 7 and 11.

Theorem 12. *Let $\mathcal{B}$ be the family of all Blaschke products with essential sup-norm. The closure of all interpolating Blaschke products of type G in $\mathcal{B}$ is the set of all finite products of interpolating Blaschke products of type G. Also, the closure of all interpolating Blaschke products of finite type G in $\mathcal{B}$ is the set of all finite products of interpolating Blaschke products of finite type G.*

Proof. Suppose B is in the closure of all interpolating Blaschke products of type G. Take B of type G such that $||B - b||_\infty = ||1 - B\bar{b}||_\infty < 1$. It follows that $B\bar{b}$ is invertible in E and so is $B = (B\bar{b})b$. By Theorem 7, B is a finite product of interpolating Blaschke products of type G.

For the converse, it suffices to show for B_1, B_2 of type G and $\varepsilon > 0$, there is B of type G such that $||B_1B_2 - B||_\infty < \varepsilon$. Let the root sequences of B_1 and B_2 be $\{z_n\}$ and $\{w_n\}$, respectively. Since each of these sequences are separated,

$$\delta_I = \inf\{\rho(z_m, z_n), \rho(w_m, w_n) : m \neq n\} > 0.$$

By Hoffman's lemma (see Lemma 1.4 on pages 404-5 of [Gar]), there are $\delta_0, \varepsilon_0 < \delta_I/3$ such that

$$V_n \subset \{z \in D : \rho(z, z_n) < \varepsilon_0\},$$

$$W_n \subset \{z \in D : \rho(z, w_n) < \varepsilon_0\}$$

and

$$\left|\left| z - \frac{z - (\delta_0/2)}{1 - (\delta_0/2)z} \right|\right|_\infty < \varepsilon,$$

where V_n and W_n are the components of $\{z \in D : |B_1(z)| < \delta_0\}$ and $\{z \in D : |B_2(z)| < \delta_0\}$ containing z_n and w_n, respectively. Since $3\varepsilon_0 < \delta_I$, we have

$$\rho(V_n, V_m) > \delta_I/3 \quad \text{and} \quad \rho(W_n, W_m) > \delta_I/3 \quad \text{for} \quad n \neq m.$$

Factor $B_2 = B_3B_4$ so that $w_n \in Z(B_3)$ if $\rho(w_n, Z(B_1)) \geq \delta_0/4$ and $w_n \in Z(B_4)$ if $\rho(w_n, Z(B_1)) < \delta_0/4$. Let

$$B_5 = \frac{B_4 - (\delta_0/2)}{1 - (\delta_0/2)B_4}$$

and $B = B_1B_3B_5$, then $||B_1B_2 - B||_\infty = ||B_4 - B_5||_\infty < \varepsilon$.

Next we will show B is an interpolating Blaschke product. Since $\rho(Z(B_1), Z(B_3)) \geq \delta_0/4$, B_1B_3 is an interpolating Blaschke product. By Lemma 1.4 on pages 404-5 of [Gar], B_5 is an interpolating Blaschke product.

To see $\rho(Z(B_1B_3), Z(B_5)) > 0$, let $w \in Z(B_5)$ and $z \in Z(B_1B_3)$. Then $w \in W_n$ for some n and $w_n \in Z(B_4)$. If $m \neq n$, then

$$\frac{\delta_0}{2} = \rho(B_5(w), B_5(w_n)) \leq \rho(w, w_n) < \varepsilon_0 < \rho(w, w_m).$$

Since $w_n \in Z(B_4)$, we have $\rho(w_n, z_k) < \delta_0/4$ for some $z_k \in Z(B_1)$.

In the case $z \in Z(B_1)$ and $\rho(z, Z(B_4)) < \varepsilon_0/4$, there is $w_m \in Z(B_4)$ such that $\rho(z, w_m) < \delta_0/4$ and so

$$\rho(w,z) \geq \rho(w, w_m) - \rho(w_m, z) \geq \frac{\delta_0}{2} - \frac{\delta_0}{4} = \frac{\delta_0}{4}.$$

In the case $z \in Z(B_1)$ and $\rho(z, Z(B_4)) \geq \delta_0/4$, we have $z \neq z_k$, hence $\rho(z, z_k) \geq \delta_I$. So

$$\rho(w,z) \geq \rho(z, z_k) - \rho(z_k, w_n) - \rho(w_n, w) \geq \delta_I - \frac{\delta_0}{4} - \varepsilon_0 \geq \frac{\delta_I}{2}.$$

In the case $z \in Z(B_3)$, we have

$$\rho(w,z) \geq \rho(z, w_n) - \rho(w_n, w) \geq \delta_I - \varepsilon_0 \geq \frac{2\delta_I}{3}.$$

So $\rho(Z(B_1B_3), Z(B_5)) > 0$. Therefore, $B = B_1B_3B_5$ is an interpolating Blaschke product.

Finally since B_1, B_2 are of type G, by Proposition 1, B is of type G. This completes the proof of the statement for the closure of type G. For the closure of finite type G, use Theorem 11 instead of Theorem 7 in the above proof. □

4. Questions

Let E_{loc} be the Douglas algebra generated by H^∞ and the complex conjugates of all locally thin Blaschke products.

(1) Does Theorem 7 hold if E is replaced with E_{loc}?
(2) Does $E = E_{\text{loc}}$ or does $E_{\text{loc}} = E_{\text{fi}}$ or neither?

If we let E^* be the Douglas algebra generated by H^∞ and the complex conjugates of all thin Blaschke products, the main result of Hedenmalm [He, Theorem 2.6] asserts that Theorem 7 holds for E^*. By Proposition 1.1 and Example 3.1 of [Gu-Iz-2], there exists a Blaschke product b of finite type G, which is not a finite product of thin Blaschke products. It follows that $E^* \subsetneq E_{\text{fi}}$ because otherwise $E_{\text{fi}} = E^*$ would contain $H^\infty[\bar{b}]$ forcing b to be a finite product of thin Blaschke products by [He, Theorem 2.6]. Also, by the proof of Theorem 2 of [Gu-Iz-1], there exists a Blaschke product of type G, but not of finite type G. So we have $E_{\text{fi}} \subsetneq E$. By Lemma 1 of [Gu], it can be shown that $E_{\text{fi}} \subseteq E_{\text{loc}}$, but it is not clear whether $E_{\text{loc}} \subseteq E$ or $E_{\text{loc}} \supseteq E$.

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