Chapter 1: Market Indexes, Financial Time Series and their Characteristics

- What is time series (TS) analysis?

  Observe the following two data sets:

Hang Seng Index	12877	12850	13023	$\cdots$
Date	30.8.04	31.8.04	01.9.04	$\cdots$

Student's Weights	130kg	200kg	45kg	$\cdots$
Students	A	B	C	$\cdots$

  What is the difference between these two data sets?

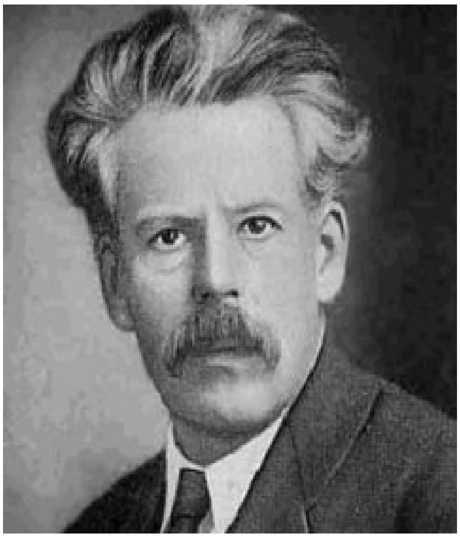
- **Definition**:

  A time series (TS) is a sequence of random variables labeled by time t.

  Time series data are observations of TS.

TSA History

- Linear TSA: The beginning/babyhood 1927

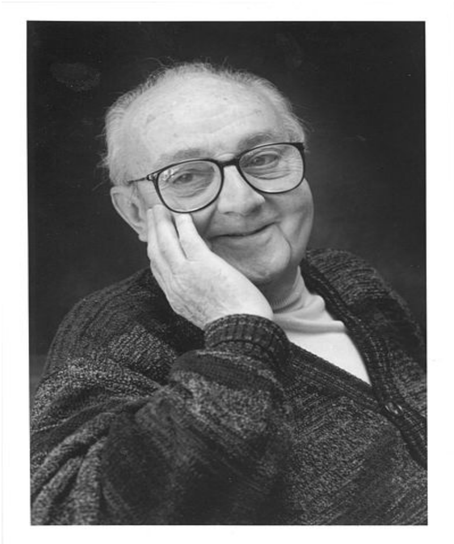



- George Udny Yule (1871-1951), a British statistician.

- Eugen Slutsky (1880-1948), a Russian/Soviet mathematical statistician, economist and political economist.

- Peter Whittle (1927-2021) (ARMA model)

- Linear TSA in 1970's



- George Edward Pelham Box (1919–2013), a British statistician (quality control, TSA, design of experiments, and Bayesian inference). He has been called "one of the great statistical minds of the 20th century".

- Box & Jenkins (1976) Time Series Analysis: Forecasting and Control

- Nonlinear TSA in 1950's

- Patrick Alfred Pierce Moran (1917–1988), an Australian statistician (probability theory, population and evolutionary genetics).

- Peter Whittle (1927-2021, New Zealand), stochastic nets, optimal control, time series analysis, stochastic optimisation and stochastic dynamics.

- Nonlinear TSA in 1980's

- Howell Tong (1944–, in Hong Kong) (TAR model).

- Robert Fry Engle III (1942–) is an American economist and the winner of the 2003 Nobel Memorial Prize.

- What is financial time series (FTS)?

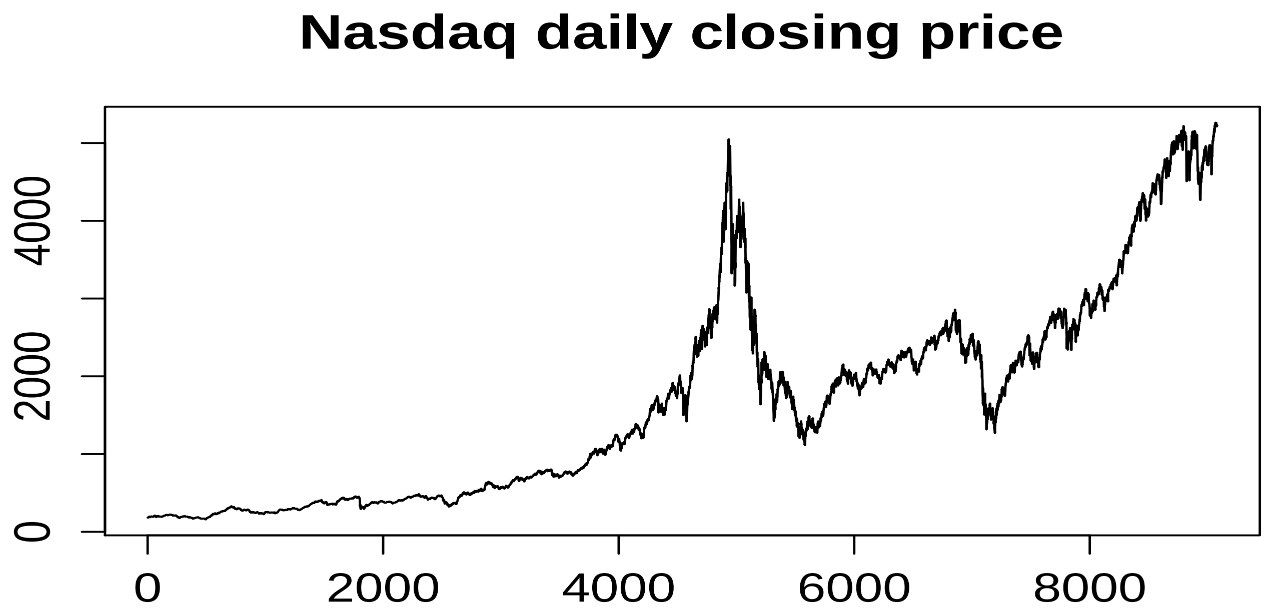
  Examples

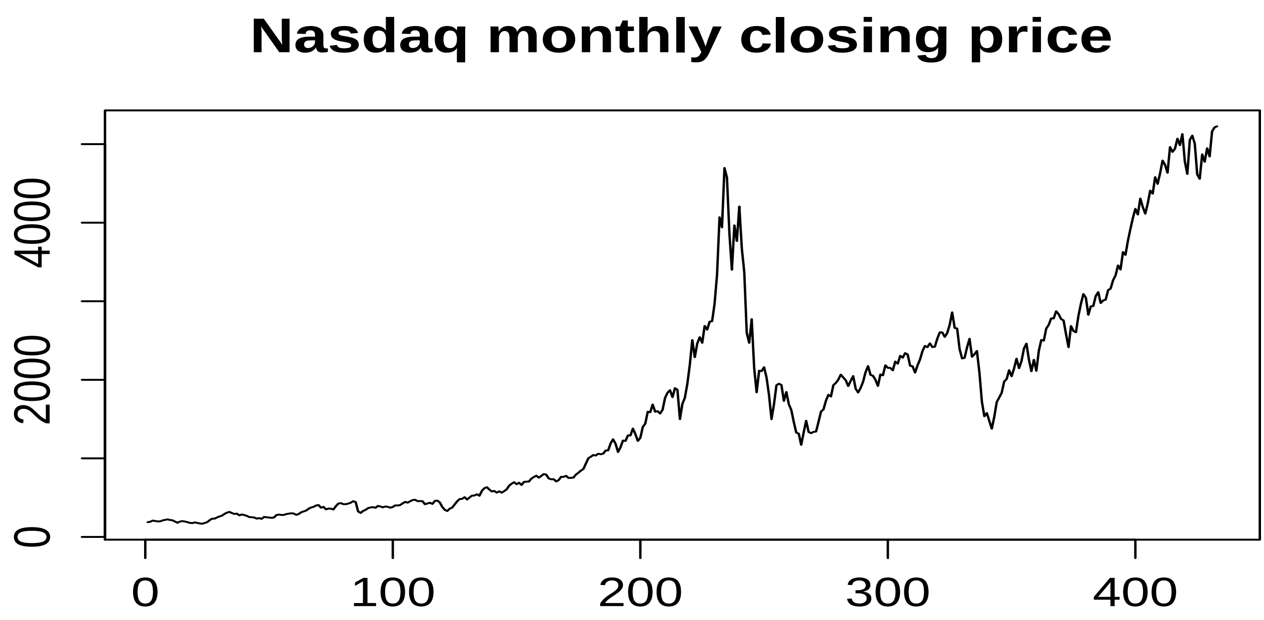
  1. Daily log returns of Hang Sang Index .

  2. Monthly log return of exchange rates of Japan-USA.

  3. China life daily stock data.

  4. HSBC daily stock data.





Nasdaq daily and monthly closing price from Sep 1, 1980 to Sep 1, 2016.

Special features of FTS

1. Theory and practice of asset valuation over time.

2. Added more uncertainty. For example, FTS must deal with the changing business and economic environment and the fact that volatility is not directly observed.

General objective of the course

to provide some basic knowledge of financial time series data

to introduce some statistical tools and econometric models useful for analyzing these series.

to gain empirical experience in analyzing FTS

to study methods for assessing market risk

to analyze high-dimensional asset returns.

Special objective of the course

Past data $\Longrightarrow$TS r.v. $Z_t \Longrightarrow$ future of TS.

(a) $\quad E\left(Z_{n+l}|Z_1,\cdots,Z_n\right),$

(b) $\quad P(a \leq Z_{n+l} \leq b|Z_1,\cdots,Z_n)$ for some $a < b$.

1.1 **Asset Returns**

Let P_t be the price of an asset at time t, and assume no dividend. One-period simple return or simple net return:

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}} = \frac{P_t}{P_{t-1}} - 1.$$

Gross return

$$1 + R_t = \frac{P_t}{P_{t-1}} \; or \;\; P_t = P_{t-1}(1 + R_t).$$

Multi-period simple return or the $k-$period simple net return:

$$R_t(k) = \frac{P_t - P_{t-k}}{P_{t-k}} = \frac{P_t}{P_{t-k}} - 1.$$

Gross return

$$\begin{aligned} 1 + R_t(k) &= \frac{P_t}{P_{t-k}} = \frac{P_t}{P_{t-1}} \times \frac{P_{t-1}}{P_{t-2}} \times \cdots \times \frac{P_{t-k+1}}{P_{t-k}} \\ &= (1 + R_t)(1 + R_{t-1}) \times \cdots \times (1 + R_{t-k+1}) \\ &= \prod_{j=0}^{k-1} (1 + R_{t-j}). \end{aligned}$$

Example: Suppose the daily closing prices of a stock are

Day	1	2	3	4	5
Price	37.84	38.49	37.12	37.60	36.30

1. What is the simple return from day 1 to day 2?

Ans: $R_2 = \frac{38.49-37.84}{37.84} = 0.017$.

2. What is the simple return from day 1 to day 5?

Ans: $R_5(4) = \frac{36.30-37.84}{37.84} = -0.041$.

3. Verify that

$1 + R_5(4) = (1 + R_2)(1 + R_3)(1 + R_4)(1 + R_5)$.

Time interval is important! Default is one year.

Annualized (average) return:

$$Annualized[R_t(k)] = \left[\prod_{j=0}^{k-1} (1 + R_{t-j})\right]^{1/k} - 1.$$

An approximation:

$$Annualized[R_t(k)] \approx \frac{1}{k} \sum_{j=0}^{k-1} R_{t-j}.$$

Continuous compounding

Assume that the interest rate of a bank deposit is 10% per annum and the initial deposit is \$1.00.

If the bank pays interest m times a year, then the interest rate for each payment is $10\%/m$, and the net value of the deposit become

$$\$1 \times \left(1 + \frac{0.1}{m}\right)^m.$$

Illustration of the power of compounding (int. rate 10% per annum):

Type	m(payment)	Int.	Net
Annual	1	0.1	\$1.10000
Semi-Annual	2	0.05	\$1.10250
Quarterly	4	0.025	\$1.10381
Monthly	12	0.0083	\$1.10471
Weekly	52	0.1/52	\$1.10506
Daily	365	0.1/365	\$1.10516
Continuously	∞		\$1.10517

In general, the net asset value A of the continuous compounding is

$$A = C\,\text{exp}(r \times n),$$

r is the interest rate per annum, C is the initial capital, n is the number of years, and and exp is the exponential function.

Present value:

$$C = A\,\text{exp}[-r \times n].$$

Continuously compounded (or log) return

$$r_t = \ln(1 + R_t) = \ln \frac{P_t}{P_{t-1}} = p_t - p_{t-1},$$

where $p_t = \ln(P_t)$

Multi-period log return:

$$\begin{aligned} r_t(k) &= \ln[1 + R_t(k)] \\ &= \ln[(1 + R_t)(1 + R_{t-1})(1 + R_{t-k+1})] \\ &= \ln(1 + R_t) + \ln(1 + R_{t-1}) \\ &\quad + \cdots + \ln(1 + R_{t-k+1}) \\ &= r_t + r_{t-1} + \cdots + r_{t-k+1}. \end{aligned}$$

Example (continued). Use the previous daily prices.

1. What is the log return from day 1 to day 2?

A: $r_2 = \ln(38.49) - \ln(37.84) = 0.017$.

2. What is the log return from day 1 to day 5?

A: $r_5(4) = \ln(36.3) - \ln(37.84) = -0.042$.

3. It is easy to verify $r_5(4) = r_2 + \cdots + r_5$.

Portfolio return:

Suppose that we have N assets with the $i-th$ asset price is P_{it} at time t.

Then price of Portfolio at $(t-1)$time is

$$P_{p,t-1} = \sum_{i=1}^{N} P_{i,t-1},$$

and the proportion of the $i-$asset in the whole Portfolio is

$$w_i = \frac{P_{i,t-1}}{P_{p,t-1}} \; and \; \sum_{i=1}^{N} w_i = 1.$$

At time t, the price of Portfolio is

$$P_{p,t} = \sum_{i=1}^{N} P_{i,t}.$$

Thus, the simple return of this Portfolio is

$$\begin{aligned} R_{pt} &= \frac{P_{p,t} - P_{p,t-1}}{P_{p,t-1}} = \sum_{i=1}^{N} \frac{P_{it} - P_{it-1}}{P_{p,t-1}} \\ &= \sum_{i=1}^{N} \frac{P_{i,t-1}}{P_{p,t-1}} \frac{P_{it} - P_{it-1}}{P_{it-1}} \\ &= \sum_{i=1}^{N} w_i R_{it}. \end{aligned}$$

Example: An investor holds stocks of IBM, Microsoft and Citi- Group. Assume that her capital allocation is 30%, 30% and 40%. The monthly simple returns of these three stocks are 1.42%, 3.37% and 2.20%, respectively. What is the mean simple return of her stock portfolio in percentage?

Answer:

$E(R_t) = 0.3 \times 1.42 + 0.3 \times 3.37 + 0.4 \times 2.20 = 2.32.$

The continuously compounded returns of a portfolio do not have the previous convenient property. When R_{it} is small in absolute value, we have

$$r_{p,t} \approx \sum_{i=1}^{N} w_i r_{it},$$

where r_{it} is the log-return of asset i.

Dividend payment: let D_t be the dividend payment of an asset between dates $t-1$ and t and P_t be the price of the asset at the end of period t.

$$R_t = \frac{P_t + D_t}{P_{t-1}} - 1, \;\; r_t = \ln(P_t + D_t) - \ln(P_{t-1}).$$

Excess return: (adjusting for risk)

$$Z_t = R_t - R_{0t}, z_t = r_t - r_{0t},$$

where r_{0t} denotes the log return of a reference asset (e.g. risk-free interest rate) such as short-term U.S. Treasury bill return, etc..

Relationship:

$$r_t = \ln(1 + R_t), R_t = e^{r_t} - 1.$$

If the returns are in percentage, then

$$r_t = 100 \times \ln(1 + \frac{R_t}{100}), R_t = [\exp(r_t/100) - 1] \times 100.$$

Temporal aggregation of the returns produces

$$1 + R_t(k) = (1 + R_t)(1 + R_{t-1}) \cdots (1 + R_{t-k+1}),$$

$$r_t(k) = r_t + r_{t-1} + \cdots + r_{t-k+1}.$$

These two relations are important in practice, e.g. obtain annual returns from monthly returns.

Example: If the monthly log returns of an asset are 4.46%, -7.34% and 10.77%, then what is the corresponding quarterly log return?

Answer: (4.46 - 7.34 + 10.77)% = 7.89%.

Example: If the monthly simple returns of an asset are 4.46%, -7.34% and 10.77%, then what is the corresponding quarterly simple return?

Answer: $R = (1+0.0446)(1-0.0734)(1+0.1077)-1 = 1.0721 - 1 = 0.0721 = 7.21\%$.

1.2 Distributional properties of returns

Is r_t a data or random variable?

What is the difference?

Key: What is the distribution of

$$(r_t : \; t = 1, \cdots, T)?$$

Review of theoretical statistics:

X is a random variable, $\{X \leq x\}$ is an event and

$$F_X(x) = P(\{X \leq x\}) = P(X \leq x),$$

is called its cumulative distribution function (CD-F).

The CDF is nondecreasing (i.e., $F_X(x_1) \leq F_X(x_2)$ if $x_1 \leq x_2$) and satisfies

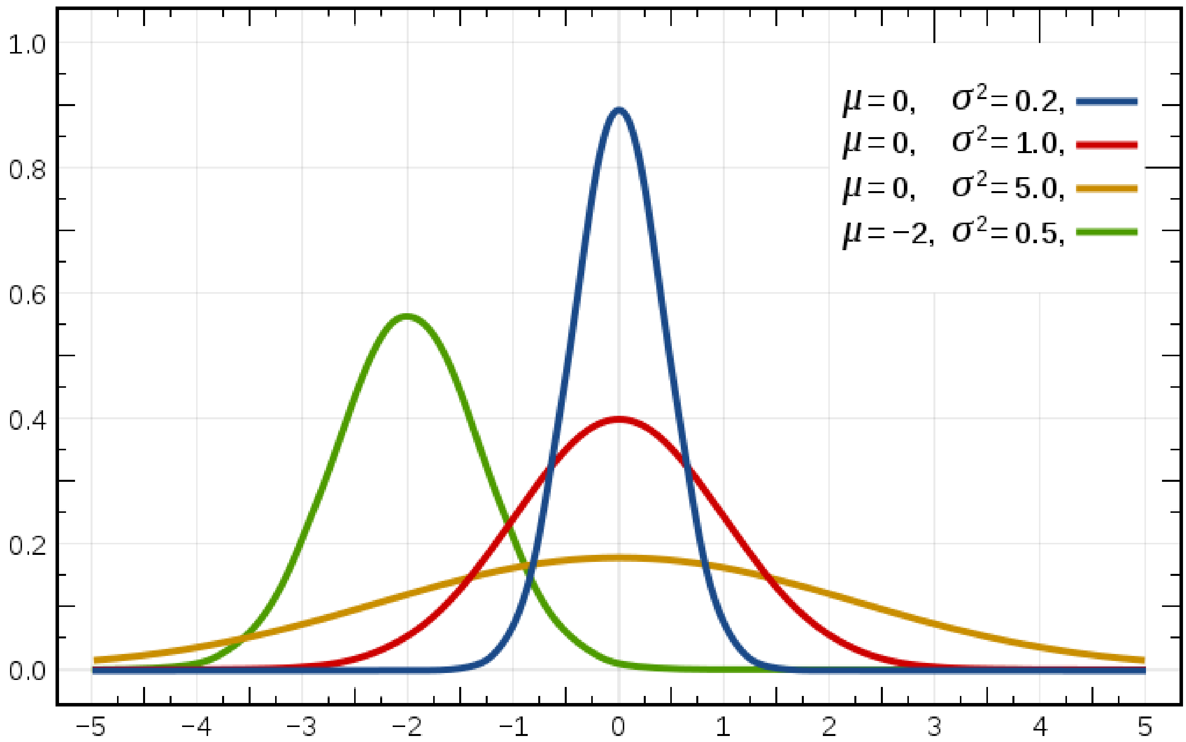
$$F_X(-\infty) = 0 \; and \; F_X(\infty) = 1.$$

$f(x) = F'(x)$ is called the density function of X.

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)dx.$$

Example:(Normal Distribution). Let $X \sim N(\mu, \sigma^2)$,

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \text{exp}\left(-\frac{(x-\mu)^2}{2\sigma^2} \right), \quad -\infty \leq x \leq \infty,$$



The density function of Normal distribution.

Quantile: For a given probability p,

$$x_p = \inf\{x | F_X(x) \geq p\}$$

is called the pth quantile of the random variable X. If $f(x)$ exists,

$$F_X(x_p) = \int_{-\infty}^{x_p} f(x)dx = p \text{ and } P(X \geq x_{1-p}) = p.$$

Moments of a random variable X:

Mean and variance:

$$\mu_x = E(X) = \int_{-\infty}^{\infty} x f(x) dx.$$

$$\sigma_x^2 = \text{Var}(X)$$

$$= E(X - \mu_x)^2 = \int_{-\infty}^{\infty} (x - \mu_x)^2 f(x) dx.$$

Why are mean and variance of returns important?

They are concerned with long-term return and risk, respectively.

Skewness (symmetry) and kurtosis (fat-tails)

$$S(x) = E\frac{(X - \mu_x)^3}{\sigma_x^3},$$

$$K(x) = E\frac{(X - \mu_x)^4}{\sigma_x^4}.$$

$K(x) - 3$: Excess kurtosis.

The $l-$th moment and $l-$th central moment:

$$m_l' = E(X^l) = \int_{-\infty}^{\infty} x^l f(x) dx.$$

$$m_l = E[(X - \mu_x)^l] = \int_{-\infty}^{\infty} (x - \mu_x)^l f(x) dx,$$

Why is symmetry of interest in financial study?

Symmetry has important implications in holding short or long financial positions and in risk management.

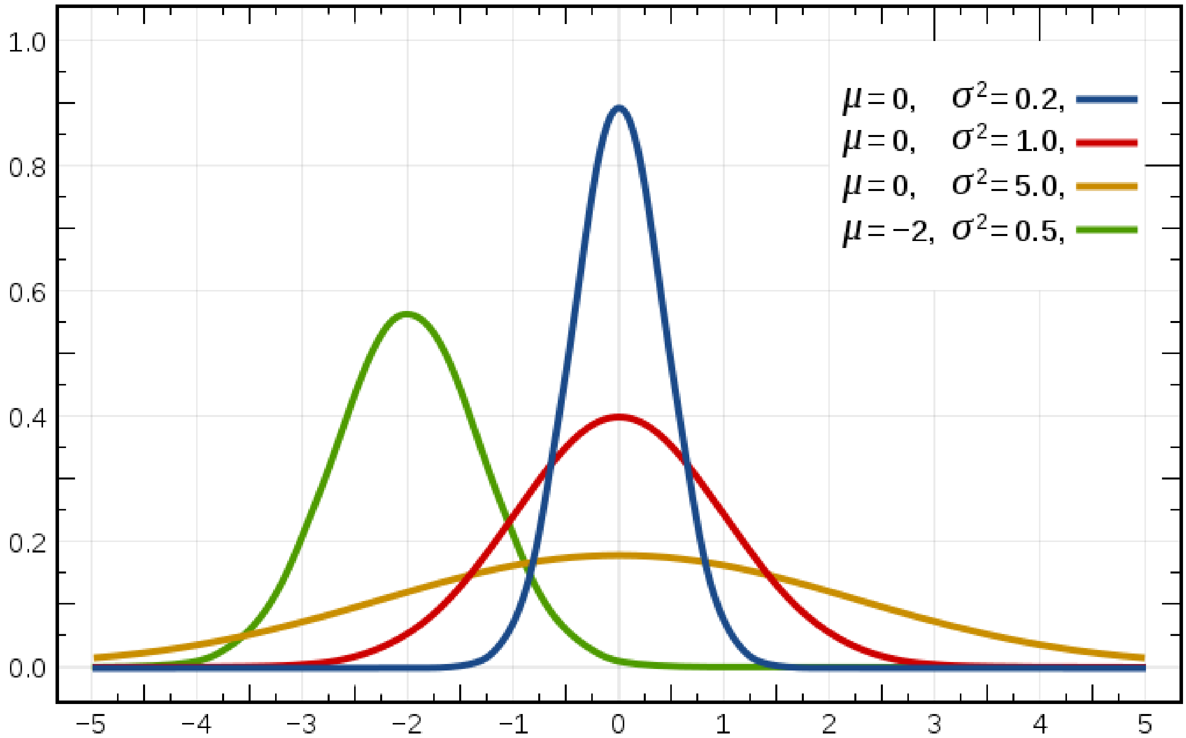
Why is kurtosis important?

Related to volatility forecasting, efficiency in estimation and tests, etc.

High kurtosis implies heavy (or long) tails in distribution.

Example:(Normal Distribution). Let $X \sim N(\mu, \sigma^2)$,

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \text{exp}\left(-\frac{(x-\mu)^2}{2\sigma^2} \right), \quad -\infty \leq x \leq \infty,$$



The density function of Normal distribution.

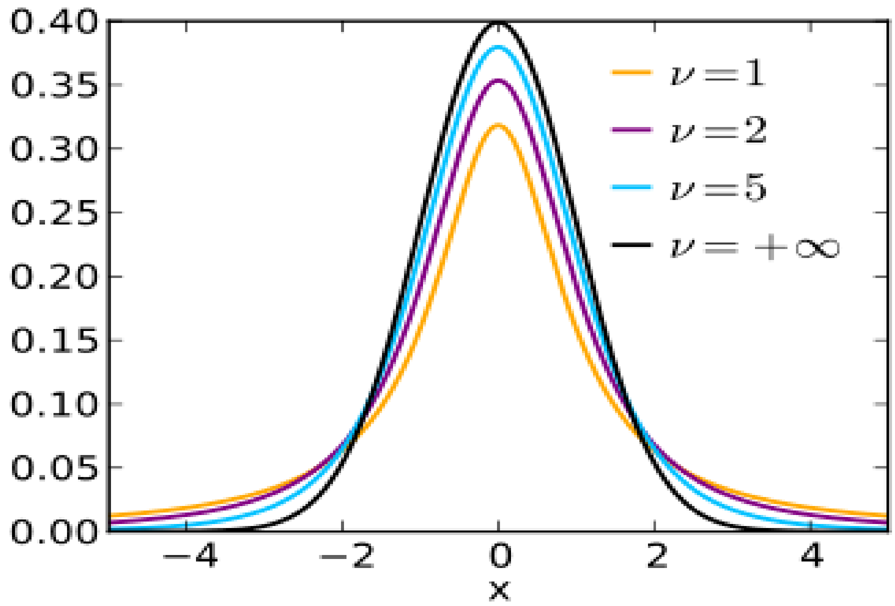
$$E(X) = \mu$$
$$\text{Var}(X) = \sigma^2$$
$$\text{S}(X) = 0$$
$$\text{K}(X) = 3$$
$$m_l = 0, \text{ for } l \text{ is odd.}$$

Example:(Student's-t distribution).

Let X follow Students t distribution with v degrees of freedom.

$$f(x) = \frac{\Gamma(\frac{v+1}{2})}{\sqrt{v\pi}\Gamma(\frac{v}{2})}(1 + \frac{x^2}{v})^{-\frac{v+1}{2}},$$

where $\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}ds$ is a gamma function.



The density function of Student's t-distribution.

Then

$$\begin{aligned} E(X) &= 0, \quad v > 1 \\ \text{Var}(X) &= \frac{v}{v-1}, \quad v > 2 \\ \text{S}(X) &= 0, \quad v > 3 \\ \text{K}(X) &= 3 + \frac{6}{v-4}, \quad v > 4. \end{aligned}$$

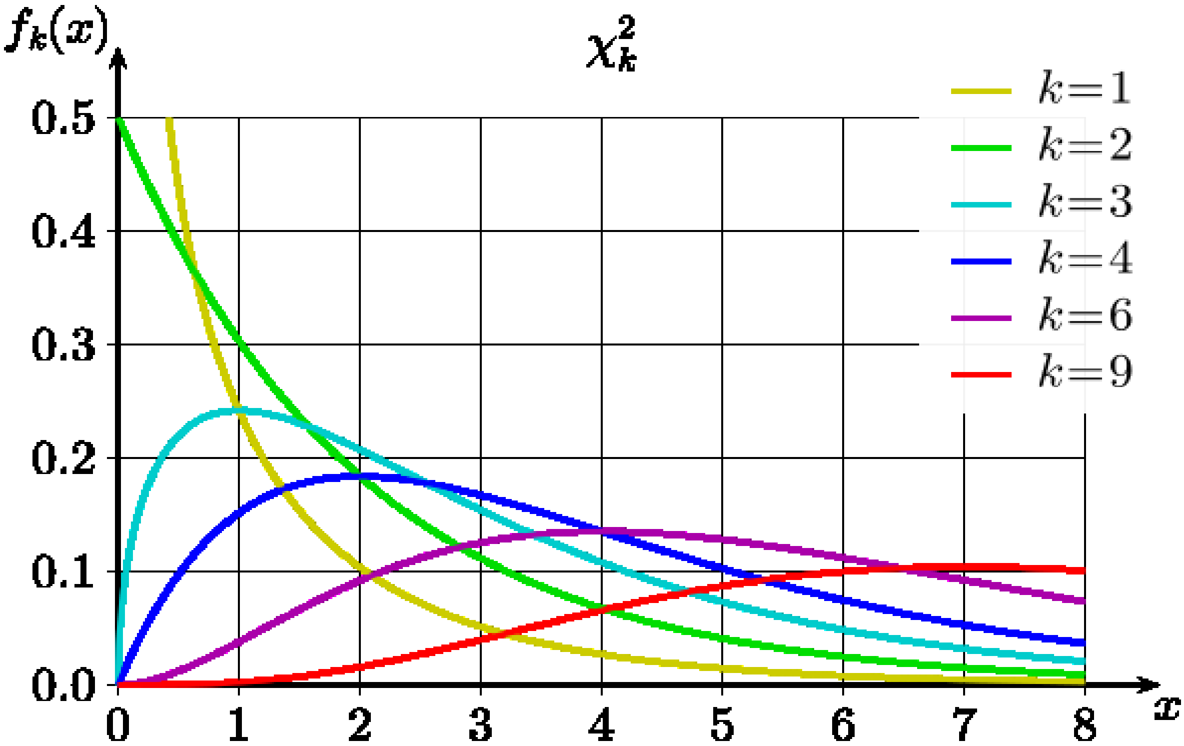
- Existence of moments depends on degrees of freedom (df) parameter v.

- Cauchy = Students-t with 1 df. Only density exists.

Example:(Chi-squared distribution).

Let X follow Chi-squared distribution with k degrees of freedom.

$$f(x) = \frac{x^{(k/2-1)} e^{-x/2}}{2^{k/2}\Gamma(\frac{k}{2})}, \quad x > 0.$$



The density function of Chi-squared distribution.

$$E(X) = k, \qquad \text{Var}(X) = 2k$$

Joint Distribution: The following is a joint distribution function of two variables: X and Y,

$$F_{X,Y}(x,y) = P(X \leq x, Y \leq y),$$

where $x \in R, y \in R$.

If the joint probability density function $f_{x,y}(x,y)$ of X and Y exists, then

$$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{x,y}(w,z) dz dw.$$

Marginal Distribution: The marginal distribution of X is given by

$$F_X(x) = F_{X,Y}(x, \infty).$$

Thus, the marginal distribution of X is obtained by integrating out Y . A similar definition applies to the marginal distribution of Y .

Conditional density function is

$$f_{x|y}(x) \equiv f_{x,y}(x,y)/f_y(y) \; or$$

$$f_{x,y}(x,y) = f_{x|y}(x) \times f_y(y)$$

X and Y are independent random vectors if and only if $f_{x|y}(x) = f_x(x)$. In this case,

$$f_{x,y}(x,y) = f_x(x) \times f_y(y).$$

Estimation:

Data: $\{x_1, \cdots, x_T\}$.

sample mean:

$$\widehat{\mu}_x = \frac{1}{T}\sum_{t=1}^{T} x_t,$$

sample variance:

$$\widehat{\sigma}_x^2 = \frac{1}{T-1}\sum_{t=1}^{T}(x_t - \widehat{\mu}_x)^2,$$

sample skewness:

$$\widehat{S}(x) = \frac{1}{(T-1)\widehat{\sigma}_x^3}\sum_{t=1}^{T}(x_t - \widehat{\mu}_x)^3,$$

sample kurtosis:

$$\widehat{K}(x) = \frac{1}{(T-1)\widehat{\sigma}_x^4}\sum_{t=1}^{T}(x_t - \widehat{\mu}_x)^4.$$

Random sample: $\{x_1, \cdots, x_T\}$.

$\widehat{\mu}_x$, $\widehat{\sigma}_x^2$, $\widehat{S}(x)$ and $\widehat{K}(x)$ are random.

If X_i's are iid $N(\mu, \sigma^2)$, we can show that

$$\widehat{S}(x) \sim N(0, \frac{6}{T}), \;\; \widehat{K}(x) - 3 \sim N(0, \frac{24}{T}).$$

Some simple tests for normality (for large T).

1. Test for H_0: symmetry v.s. H_1: asymmetry:

$$S^* = \frac{\widehat{S}(x)}{\sqrt{6/T}} \sim N(0, 1)$$

if normality holds.

Decision rule: Reject H_0 of a symmetric distribution if $|S^*| > Z_{\alpha/2}$ or p-value is less than α.

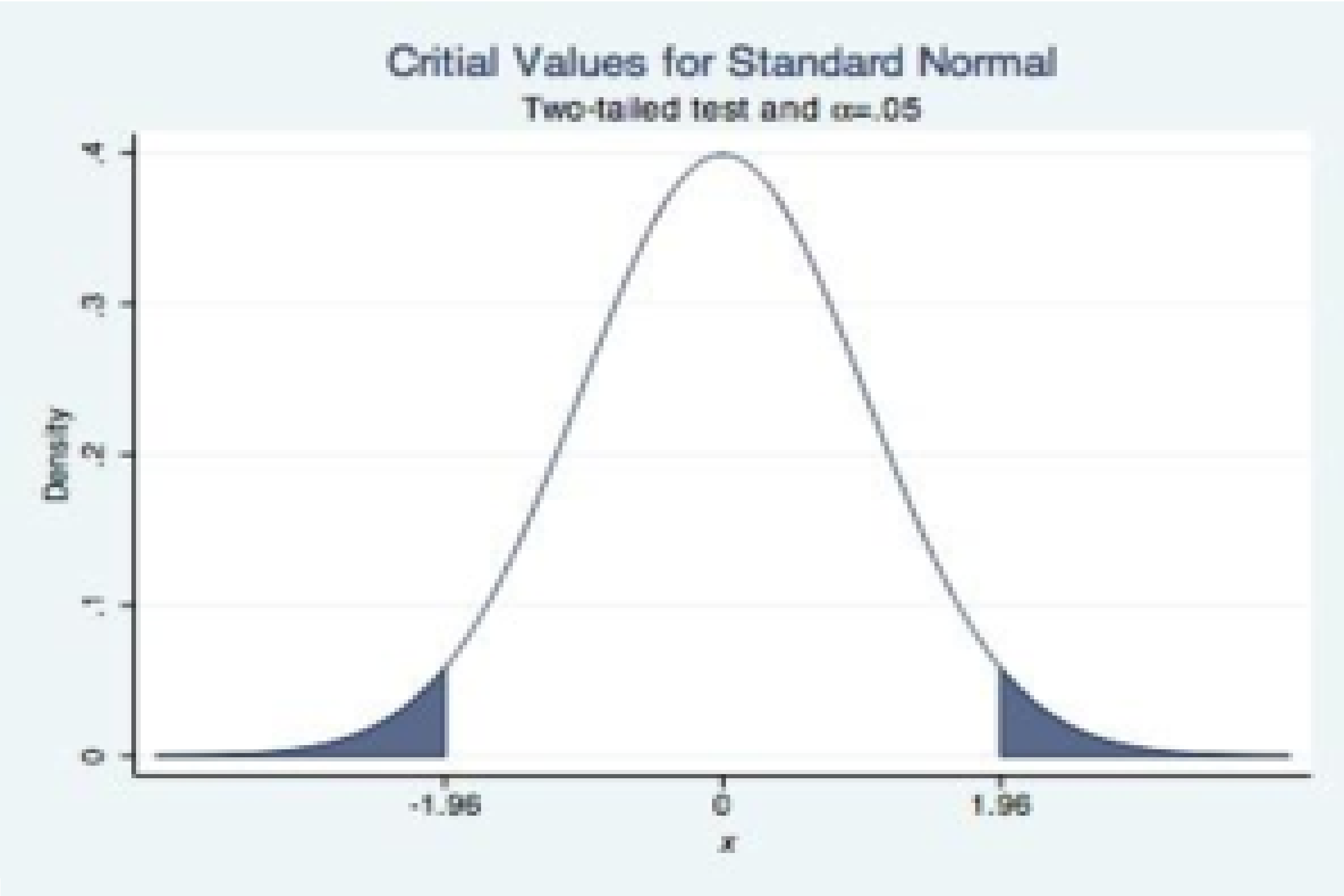
2. Test for

$$H_0 : K(x) = 3(\text{thick tail}) \; v.s. \; H_1 : K(x) \neq 3.$$

$$K^* = \frac{\widehat{K}(x) - 3}{\sqrt{24/T}} \sim N(0, 1)$$

if normality holds.

Decision rule: Reject H_0 of normal tails if $|K^*| > Z_{\alpha/2}$ or p-value is less than α



3. A joint test (Jarque-Bera test):

$$JB = (K^*)^2 + (S^*)^2 \sim \chi^2_2,$$

if normality holds, where χ^2_2 denotes a chi-squared distribution with 2 degrees of freedom.

$$H_0 : X \sim N(\mu, \sigma^2) \; v.s. H_1 : X \nsim N(\mu, \sigma^2).$$

Decision rule: Reject H_0 of normality if $JB > \chi^2_2(\alpha)$ or p-value is less than α.

4. Goodness-of-Fit Tests

Assume $r_1, \cdots, r_n$ are i.i.d with a common distribution function F(x). Order $\{r_1, \cdots, r_n\}$ from smallest to largest as $r_{(1)}, ..., r_{(n)}$. The empirical distribution function, $F_n(x)$, is defined as

$$F_n(x) = \frac{1}{n}\sum_{t=1}^{n} I\{r_t \le x\} = \frac{1}{n}\sum_{i=1}^{n} I\{r_{(i)} \le x\}$$
$$= \begin{cases} 0 & \text{if } x < r_{(1)} \\ \frac{i}{n} & \text{if } r_{(i)} \le x < r_{(i+1)},\ i = 1, \cdots, n-1 \\ 1 & \text{if } x \ge r_{(n)}. \end{cases}$$

$F_n(x)$ is an estimator of $F(x)$ and $F_n(x) \approx F(x)$.

PROC UNIVARIATE provides three EDF tests:

Kolmogorov-Smirnov (D)

Anderson-Darling ($A - sq$)

Cramr-von Mises ($W - sq$)

Kolmogorov D Statistic:

$$D_n = \sup_x |F_n(x) - F(x)|.$$

Empirical Examples of Returns

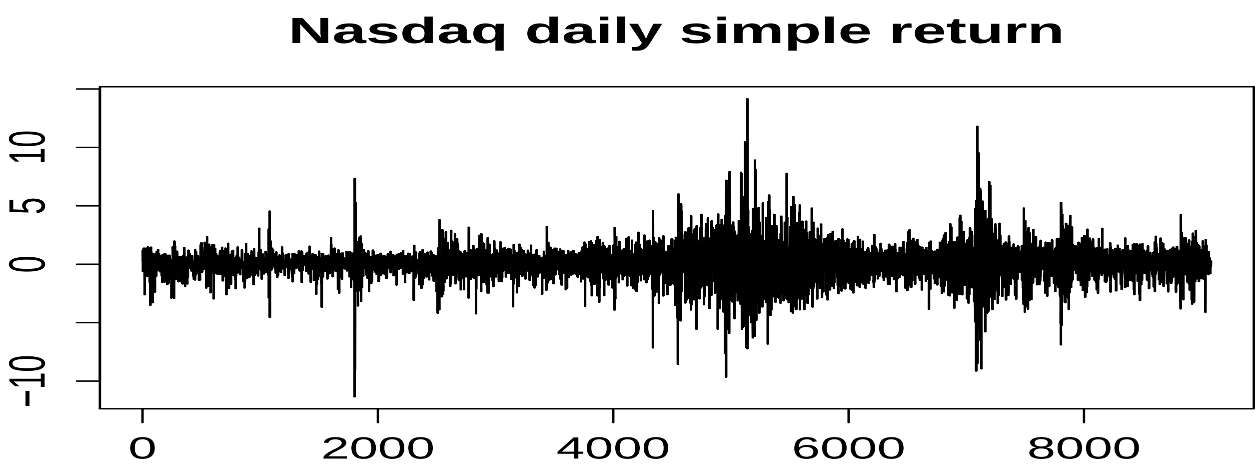
Data: Nasdaq daily and monthly closing price from Sep 1, 1980 to Sep 1, 2016.

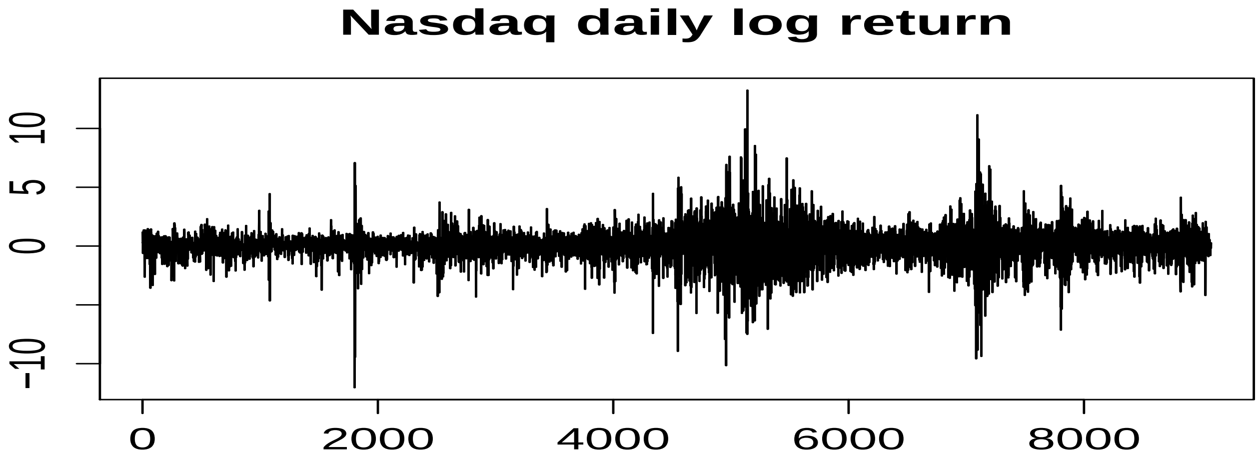
Descriptive Statistics of Nasdaq daily index with $n = 9081$

	Mean	SD	Skewness	Kurtosis
S. return	0.05	1.34	-0.05	8.77
Log return	0.04	1.34	-0.26	8.59

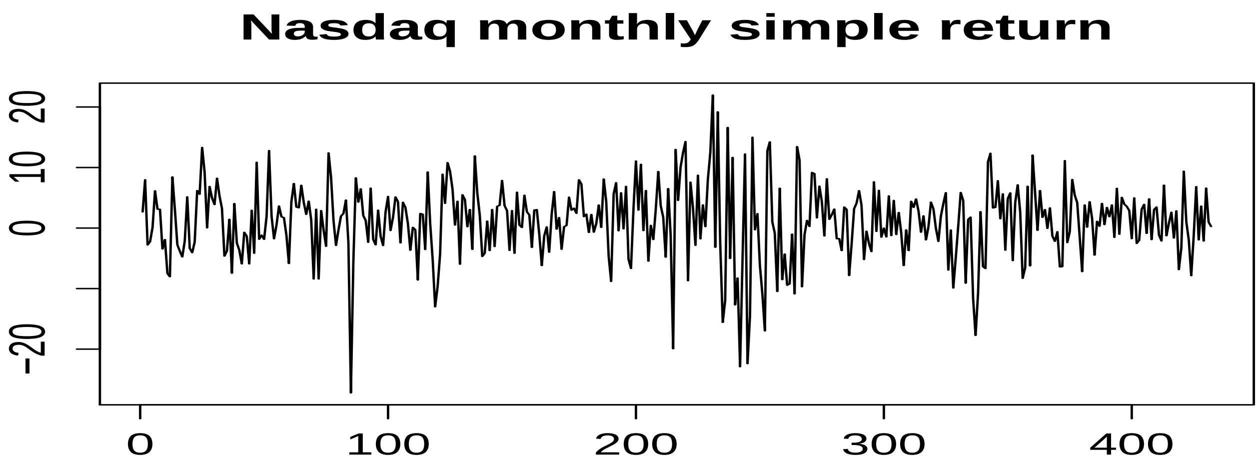
Descriptive Statistics of Nasdaq monthly index $n = 433$

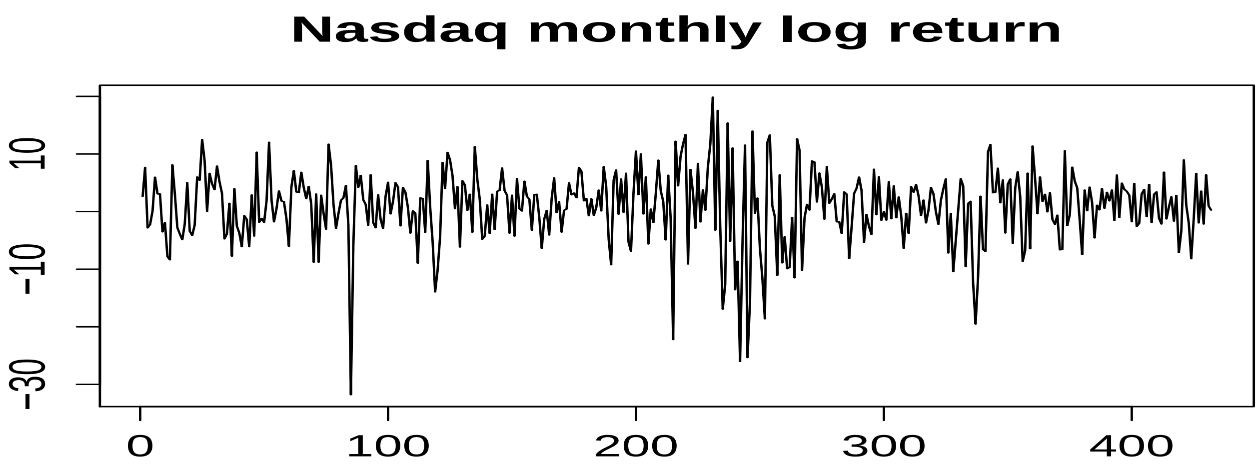
Monthly	Mean	SD	Skewness	Kurtosis
S. return	0.97	6.18	-0.52	2.00
Log return	0.77	6.27	-0.89	3.09



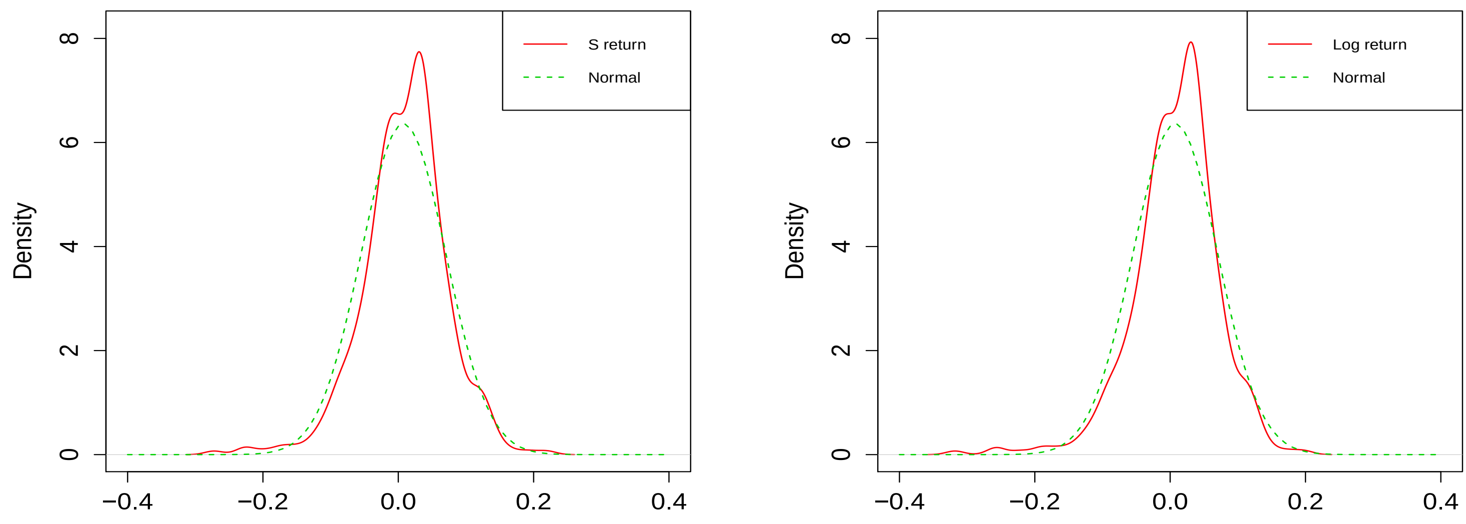


Nasdaq daily and monthly closing price from Sep 1, 1980 to Sep 1, 2016.





Nasdaq daily and monthly closing price from Sep 1, 1980 to Sep 1, 2016.



Comparison of empirical and normal densities for monthly simple and log returns of Nasdaq index.