Time-Series Econometric Theory

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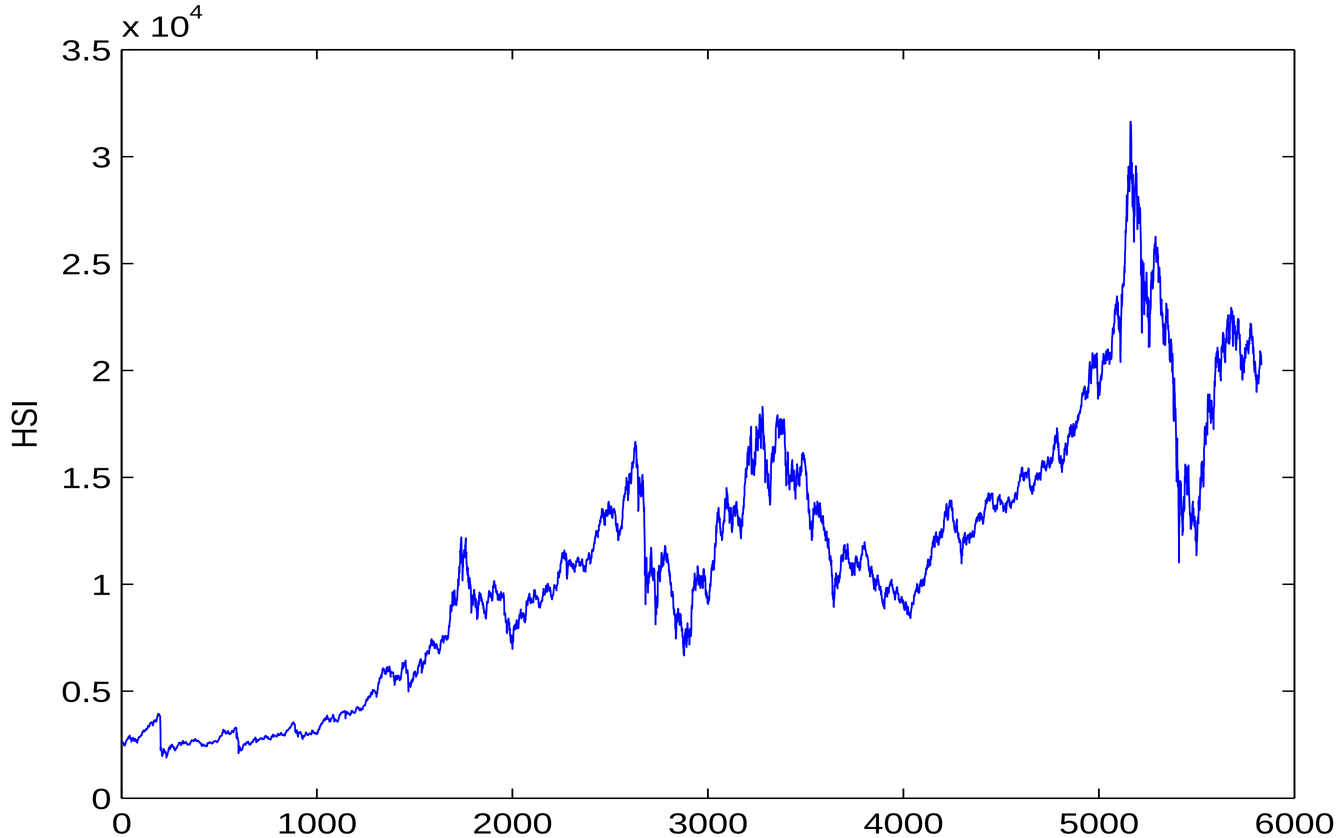
1 Introduction

- A time series is a sequence of observations ordered in time, which may be second, day, month or year etc.

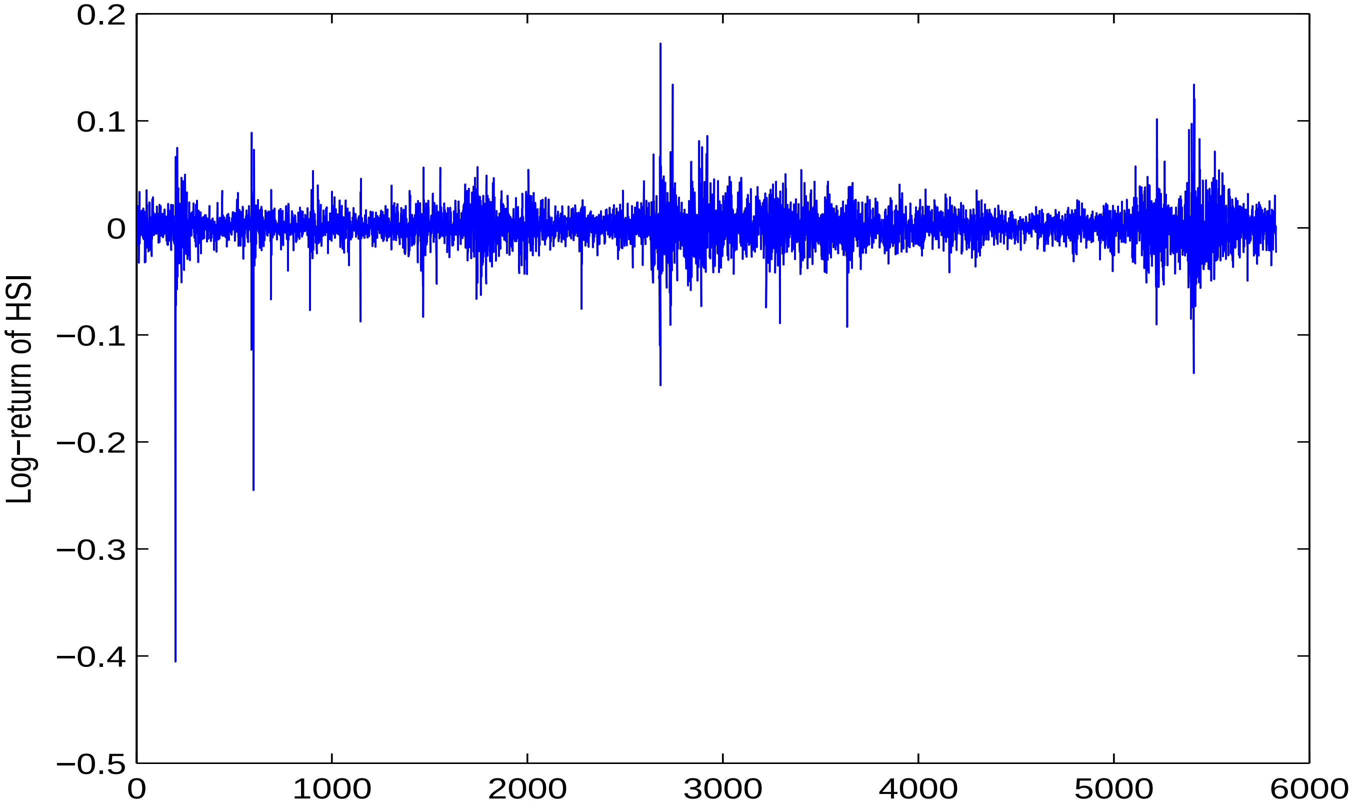
  For example, let y_t be the daily closing price of Hang Seng Index (HSI).

  Figure 1 is the plot of y_t from the year 1986-2010.

  Figure 2 is the plot of the corresponding log-return $x_t = \log y_t - \log y_{t-1}$.



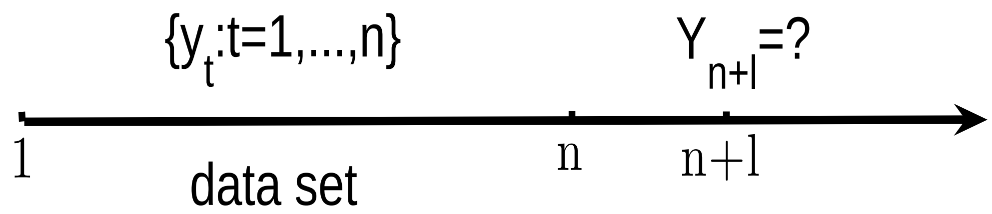
x 10^4
3.5
3
2.5
2
1.5
1
0.5
0
HSI
0
1000
2000
3000
4000
5000
6000


0.2
0.1
0
−0.1
−0.2
−0.3
−0.4
−0.5
Log−return of HSI
0
1000
2000
3000
4000
5000
6000

HSI in each day is random variable, denoted by Y_t, and the closing price is just an observation of Y_t.

We define a time series as a sequence of r.v.s ordered in time, i.e., $\{Y_t : t \in T\}$ where T is the range of time t.

In practice, we never can see $\{Y_t : t \in T\}$ rather than the data set $\{y_t : t \in T\}$ which is the collection of observations of $\{Y_t : t \in T\}$. Suppose we have a data set $\{y_t : t = 1, \cdots, n\}$, what is Y_{n+l} ?



Nobody can answer this question. However, we can try to look for

$$
\begin{array}{ll}
(a) & E\left(Y_{n+l} | y_1, \cdots, y_n\right), \\
(b) & P(a \leq Y_{n+l} \leq b) \text{ for some } a < b.
\end{array}
$$

This is the main target in the field of time series.

1.1 The general setup of time series

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space and $\{\mathcal{F}_t : t \in T\}$ be a sequence of $\sigma-$fields with $\mathcal{F}_t \subseteq \mathcal{F}_{t+1} \subseteq \cdots \subseteq \mathcal{F}$.

Let $\{Y_t : t \in T\}$ be a time series defined on $(\Omega, \mathcal{F}, \mathcal{P})$ and $Y_t \in \mathcal{F}_t$

Denote $\mathcal{F}_t^c = \mathcal{F}_t - \mathcal{F}_{t-1}$. Then, Y_t is a map from Ω to real line R, i.e.,

$$Y_t : \Omega \longrightarrow R.$$

$$\sigma(Y_t) = \mathcal{F}_{t-1} \cup \mathcal{F}_t^c$$

Given $\mathcal{F}_{t-1}$, the best predictor in mean square is

$$\mu_t = E\left(Y_t | \mathcal{F}_{t-1}\right).$$

Let $\varepsilon_t = Y_t - \mu_t$. Then, $E\left(\varepsilon_t | \mathcal{F}_{t-1}\right) = 0$, i.e., $\{\varepsilon_t\}$ is a sequence of martingale difference in terms of $\mathcal{F}_t$. Thus, we can decompose Y_t

as

$$Y_t = \mu_t + \varepsilon_t. \tag{1}$$

Furthermore, we denote the conditional variance of Y_t by h_t, i.e.,

$$h_t = E\left[(Y_t - \mu_t)^2 \, |\mathcal{F}_{t-1}\right] = E\left(\varepsilon_t^2|\mathcal{F}_{t-1}\right),$$

and $\eta_t = \varepsilon_t/\sqrt{h_t}$. Then, we can further decompose Y_t as

$$Y_t = \mu_t + \eta_t\sqrt{h_t}, \tag{2}$$

where $\{\eta_t\}$ is a sequence of uncorrelated r.v.s with

$$E\left(\eta_t|\mathcal{F}_{t-1}\right) = 0 \text{ and } E\left(\eta_t^2|\mathcal{F}_{t-1}\right) = 1.$$

To answer (1.1)-(1.2), we need to explore μ_t, h_t and η_t. We know that both μ_t and h_t are functions of $\mathcal{F}_{t-1}$. However, we never know what are the forms of these functions. In practice, we try to use some simple functions to approximate μ_t and h_t.

1.2. Some important models with $h_t \equiv \sigma^2$

- AR/MA/ARMA models.

  Assume that $\{Y_t\}$ satisfies the following equation:

$$Y_t = \phi_0 + \phi_1 Y_{t-1} + \cdots + \phi_p Y_{t-p} + \varepsilon_t, \tag{3}$$

where $\phi_i, i = 0, \cdots, p$, are coefficients and $\varepsilon_t = \eta_t \sigma$. We say that $\{Y_t\}$ follows an autoregressive (AR) model. If $\{Y_t\}$ satisfies the following equation:

$$Y_t = \theta_0 - \theta_1 \varepsilon_{t-1} - \cdots - \theta_q \varepsilon_{t-q} + \varepsilon_t, \tag{4}$$

where $\theta_i, i = 0, \cdots, q$, are coefficients. We say that $\{Y_t\}$ follows a moving average (MA) model. Let B be the back shift operator, i.e.,

$$BY_t = Y_{t-1}.$$

Then, model (3) and (4) can be written as

$$\Phi(B)Y_t = \phi_0 + \varepsilon_t \quad \text{and} \quad Y_t = \theta_0 + \Theta(B)\varepsilon_t,$$

respectively, where $\Phi(B) = 1 - \sum_{i=1}^{p} \phi_i B^i$ and
$\Theta(B) = 1 - \sum_{i=1}^{q} \theta_i B^i$.

Combining AR and MA model together, we have the so-called ARMA model:

$$\Phi(B)Y_t = \phi_0 + \Theta(B)\varepsilon_t. \tag{5}$$

- FARIMA model:

  Assume that $\{Y_t\}$ satisfies the following equation:

$$\Phi(B)(1-B)^d Y_t = \Theta(B)\varepsilon_t,$$

  where $(1-B)^d = \sum_{j=0}^{\infty} \psi_j B^j$ with $\psi_0 = 1$ and

$$\psi_j = \frac{d(d+1)\cdots(d+j-1)}{j!} = \frac{(d+j-1)!}{j!(d-1)!} \text{ for } j \geq 1.$$

  Here, $\psi_j \sim j^{d-1}/(d-1)!$ as $j \to \infty$.

- Threshold AR (TAR) model

  Assume that $\{Y_t\}$ satisfies the following equation:

$$Y_t = \begin{cases} \psi_0 + \sum_{i=1}^{p} \psi_i y_{t-i} + \varepsilon_t, & \text{if } Y_{t-d} \le r, \\ \\ \theta_0 + \sum_{i=1}^{p} \theta_i y_{t-i} + \varepsilon_t, & \text{if } Y_{t-d} > r, \end{cases}$$

  where $1 \le d \le p$ is the delay parameter and r is the threshold parameter. This model was proposed by Tong (1978).

- Bilinear (BL) model

  Assume that $\{Y_t\}$ satisfies the following equation:

$$Y_t = \phi_0 + \sum_{i=1}^{p} \phi_i Y_{t-i} + \sum_{j=1}^{r} \sum_{k=1}^{s} \psi_{jk} Y_{t-j} \varepsilon_{t-k} + \varepsilon_t.$$

- Factor AR model

  Assume that $\{Y_t\}$ satisfies the following equation:

$$Y_t = \phi_0 + \sum_{i=1}^{p} \phi_i Y_{t-i} + \sum_{j=1}^{q} \theta_j X_{t-j} + \varepsilon_t,$$

  where $\{X_t\}$ is another time series.

- Partially linear single-index model

  Assume that $\{Y_t\}$ satisfies the following equation:

$$Y_t = \sum_{i=1}^{p} \phi_i Y_{t-i} + g\left(\sum_{i=1}^{p} \theta_i Y_{t-i}\right) + \varepsilon_t,$$

  where g is a function on R.

1.3 Some important models with $\mu_t \equiv \mu$

- ARCH/GARCH model

  Assume that $\{Y_t\}$ satisfies the following equation:

$$\begin{cases} Y_t = \mu + \eta_t \sqrt{h_t}, \\ h_t = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \cdots + \alpha_r \varepsilon_{t-r}^2. \end{cases}$$

  This is the ARCH model proposed by Engle (1982). It was extended by Bollerslov to the GARCH model:

$$\begin{cases} Y_t = \mu + \eta_t \sqrt{h_t}, \\ h_t = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \cdots + \alpha_r \varepsilon_{t-r}^2 + \beta_1 h_{t-1} + \cdots + \beta_s h_{t-s}. \end{cases}$$

- Exponential GARCH model

$$\begin{cases} Y_t = \mu + \eta_t \sqrt{h_t}, \\ \ln h_t = \alpha_0 + \frac{1+\beta_1 B + \cdots + \beta_{s-1} B^{s-1}}{1-\alpha_1 B - \cdots - \alpha_r B^r} g(\eta_{t-1}), \end{cases}$$

where $g(\eta_t) = \theta \eta_t + \gamma \left[|\eta_t| - E|\eta_t| \right]$ with $E\left[g(\eta_t) \right] = 0$.

- Threshold GARCH (TGARCH) model

$$\begin{cases} Y_t = \mu + \eta_t \sqrt{h_t}, \\ h_t = \alpha_0 + \sum_{i=1}^{r} (\alpha_i + \nu_i N_{t-i}) \varepsilon_{t-i}^2 + \sum_{j=1}^{s} \beta_j h_{t-j}, \end{cases}$$

where N_{t-i} is an indicator variable, i.e.,

$$N_{t-i} = \begin{cases} 1, & \text{if } \varepsilon_{t-i} < 0, \\ 0, & \text{if } \varepsilon_{t-i} \geq 0. \end{cases}$$

1.4 No-constant μ_t and h_t models

- ARMA-GARCH model

  Assume that $\{Y_t\}$ satisfies the following equation:

$$\begin{cases} \Phi(B)Y_t = \phi_0 + \Theta(B)\varepsilon_t, \\ \varepsilon_t = \eta_t\sqrt{h_t}, \\ h_t = \alpha_0 + \sum_{i=1}^{r} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{s} \beta_j h_{t-j}. \end{cases}$$

- TAR-ARCH model

- Double AR model

  Assume that $\{Y_t\}$ satisfies the following equation:

$$Y_t = \sum_{i=1}^{p} \phi_i Y_{t-i} + \eta_t \sqrt{w + \sum_{i=1}^{p} \alpha_i Y_{t-i}^2}.$$

- Bilinear with GARCH model

There are still many other models in the literature.