2.3. Autocorrelation Functions and White Noise

- Let $\{Y_t\}$ be a sequence of stationary time series

  If $EY_t^2 < \infty$, then

  $EZ_t = \mu$, a constant.

$$\gamma_k = \mathbf{cov}(Y_t, Y_{t+k}) = E[(Y_t - \mu)(Y_{t+k} - \mu)]$$

  only depends on k.

  γ_k is called **autocovariance** (ACV) of Y_t.

$$\rho_k \equiv \frac{\mathbf{cov}(Y_t, Y_{t+k})}{\sqrt{\mathbf{var}(Y_t)}\sqrt{\mathbf{var}(Y_{t+k})}} = \frac{\gamma_k}{\gamma_0}.$$

  only depends on k.

  ρ_k is called **autocorrelation function** (ACF) of Y_t.

Properties of γ_k and ρ_k:

(1). $\gamma_0 = \sigma^2, \quad \rho_0 = 1.$

(2). $\gamma_k = \gamma_{-k}, \quad \rho_k = \rho_{-k}.$

(3). $\gamma_k \leq \gamma_0, \quad \rho_k \leq \rho_0.$

Important point:

The smaller ρ_k, the less dependency between Y_t and Y_{t+k}.

Intuitively, as $k \to \infty, \rho_k \to 0$, generally.

In general, $\rho_k \neq 0$, this is an important feature of time series.

For a time series $\{Y_t\}$ which may not be strict stationary, if its mean, variance and ρ_k do not depend on time t, it is called weak stationary.

Partial Autocorrelation function (PACF).

Let Y_t be a stationary time series. The conditional correlation

$$
\begin{aligned}
& Corr(Y_t, Y_{t+k} | Y_{t+1}, \cdots, Y_{t+k-1}) \\
& \quad = \frac{Cov[(Y_t - \hat{Y}_t)(Y_{t+k} - \hat{Y}_{t+k})]}{\sqrt{Var(Y_t - \hat{Y}_t) Var(Y_{t+k} - \hat{Y}_{t+k})}}
\end{aligned}
$$

is called the PACF of Y_t and Y_{t+k}, denoted by ϕ_{kk}, where $\hat{Y}_t = E(Y_t | Y_{t+1}, \cdots, Y_{t+k-1})$.

Formula: $\phi_{11} = \rho_1$,

$$\phi_{kk} = \frac{\begin{vmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{k-2} & \rho_1 \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{k-3} & \rho_2 \\ & & \cdots & & & \\ & & \cdots & & & \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & \cdots & \rho_1 & \rho_k \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{k-2} & \rho_{k-1} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{k-3} & \rho_{k-2} \\ & & \cdots & & & \\ & & \cdots & & & \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & \cdots & \rho_1 & 1 \end{vmatrix}}$$

- White noise process

  A process $\{a_t\}$ is called a white noise process if

$$
\begin{aligned}
&Ea_t = 0, \\
&\mathbf{var}(a_t) = \sigma_a^2, \\
&\gamma_k = \mathbf{cov}(a_t, a_{t+k}) = 0, \quad \text{if } k \neq 0.
\end{aligned}
$$

  Properties of the white noises:

  if a_t is a white noise, then

$$
(1).(\text{ACF}) \quad \rho_k = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases},
$$

$$
(2).(\text{PACF}) \quad \phi_{kk} = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}.
$$

- Estimation of ACV and ACF

  Given $Y_1, Y_2, \cdots, Y_n$, how to estimate μ, σ^2, γ_k and ρ_k?

  Sample mean:

$$\overline{Y} = \frac{1}{n} \sum_{t=1}^{n} Y_t$$

  is called the sample mean of Y_t. $\overline{Y}$ is the estimator of the mean μ.

  By the ergodic theorem,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} Y_t = \mu \,,$$

  almost surely, if $E|Y_t| < \infty$.

Sample ACV:

$$\widehat{\gamma}_k = \frac{1}{n}\sum_{t=1}^{n-k}(Y_t - \overline{Y})(Y_{t+k} - \overline{Y}), \quad \text{or}$$

$$\widehat{\widehat{\gamma}}_k = \frac{1}{n-k}\sum_{t=1}^{n-k}(Y_t - \overline{Y})(Y_{t+k} - \overline{Y})$$

are called the sample ACV of Y_t.

$\widehat{\gamma}_k$ and $\widehat{\widehat{\gamma}}_k$ are the estimators of γ_k.

By the ergodic theorem,

$$\lim_{n\to\infty} \widehat{\gamma}_k = \gamma_k, \quad \lim_{n\to\infty} \widehat{\widehat{\gamma}}_k = \gamma_k$$

if $EY_t^2 < \infty$.

In particular,

$$\widehat{\sigma}_n^2 = \frac{1}{n}\sum_{t=1}^{n}(Y_t - \overline{Y})^2 = \frac{1}{n}\sum_{t=1}^{n} Y_t^2 - \overline{Y}^2$$

is called the sample variance of Y_t.

$\widehat{\sigma}_n^2$ is an estimator of σ^2.

The estimator is consistent, i.e.

$$\lim_{n\to\infty} \widehat{\sigma}_n^2 = \sigma^2,$$

if $EY_t^2 < \infty$.

- Sample ACF

$$\widehat{\rho}_k = \frac{\widehat{\gamma}_k}{\widehat{\gamma}_0} \quad \text{or} \quad \widehat{\rho}_k = \frac{\widehat{\widehat{\gamma}_k}}{\widehat{\widehat{\gamma}}_0}$$

is called the sample ACF of Y_t.

$\widehat{\rho}_k$ is the estimator of ρ_k.

$\widehat{\rho}_k$ is the consistent estimator of ρ_k, i.e.

$$\lim_{n\to\infty} \widehat{\rho}_k = \rho_k,$$

if $EY_t^2 < \infty$.

How to check whether Y_t is a white noise or not?

$$S_{\widehat{\rho}_k} \equiv \sqrt{\frac{1}{n}(1 + 2\widehat{\rho}_1^2 + \cdots + 2\widehat{\rho}_m^2)}\ ,$$

where m is a fixed integer.

If Y_t is a white noise, $S_{\widehat{\rho}_k} \approx \sqrt{\frac{1}{n}}$.

Ljung and Box (1978) test:

$$\mathcal{Q}_2(M) = n(n+2)\sum_{k=1}^{M} \frac{\hat{\rho}_k^2}{n-k} \sim \chi^2(M).$$

- Linear Process

  Moving average representation of Y_t:

$$Y_t = \mu + a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \cdots$$

where a_t is a white noise, $\sum_{j=0}^{\infty} \psi_j^2 < \infty$. (called Wold's representation or linear process)

Some properties:

$$EY_t = \mu\,,$$

$$\mathbf{Var}(Y_t) = \sigma_a^2 \sum_{j=0}^{\infty} \psi_j^2\,,$$

$$E(a_t Y_{t-j}) = \begin{cases} \sigma_a^2 & \text{for} \quad j = 0 \\ 0 & \text{for} \quad j > 0 \end{cases}\,,$$

$$\gamma_k = \sigma_a^2 \sum_{i=0}^{\infty} \psi_i \psi_{i+k} \, ,$$

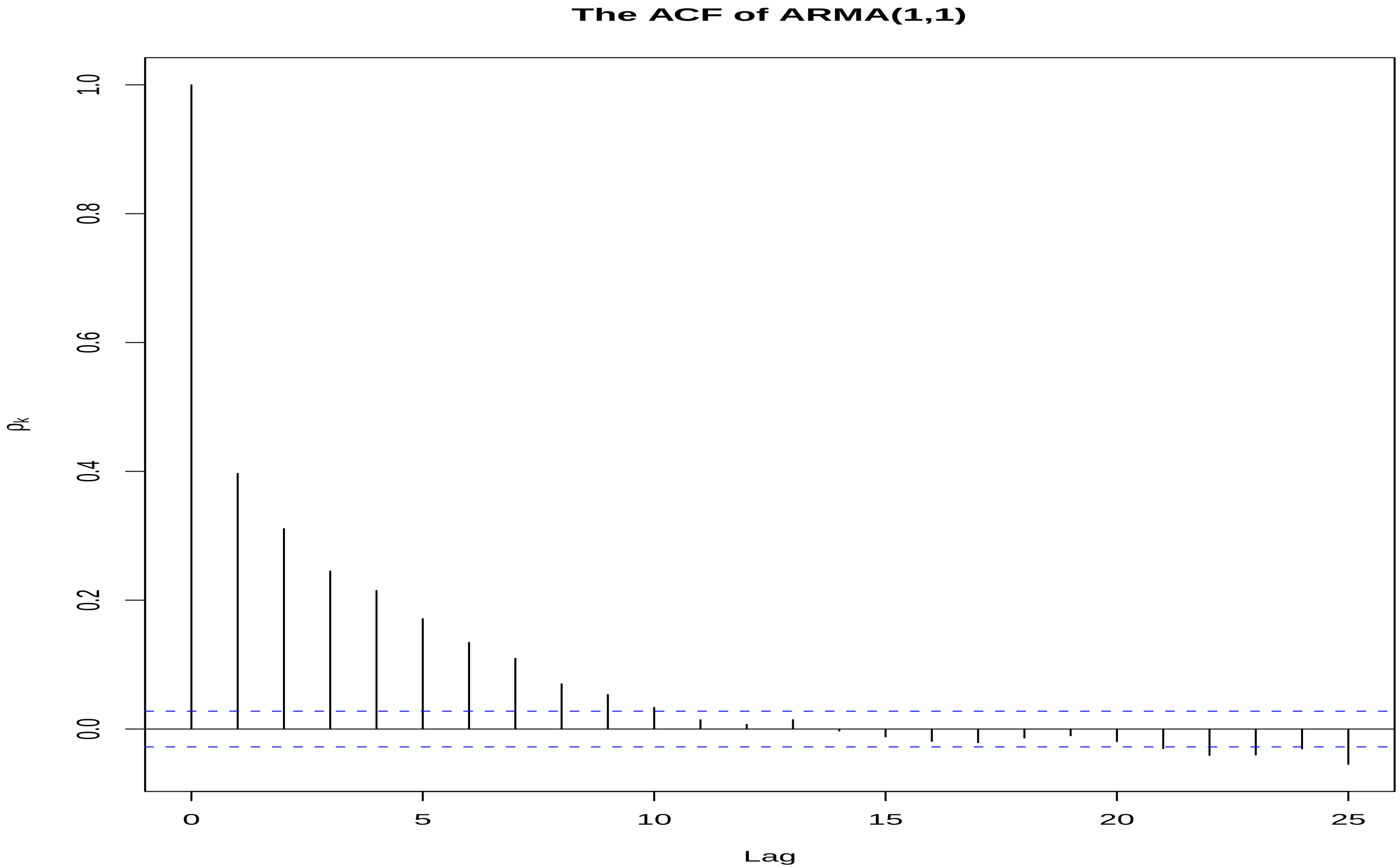
$$\rho_k = \frac{\sum_{i=0}^{\infty} \psi_i \psi_{i+k}}{\sum_{j=0}^{\infty} \psi_j^2} .$$

Autoregressive representations of Y_t:

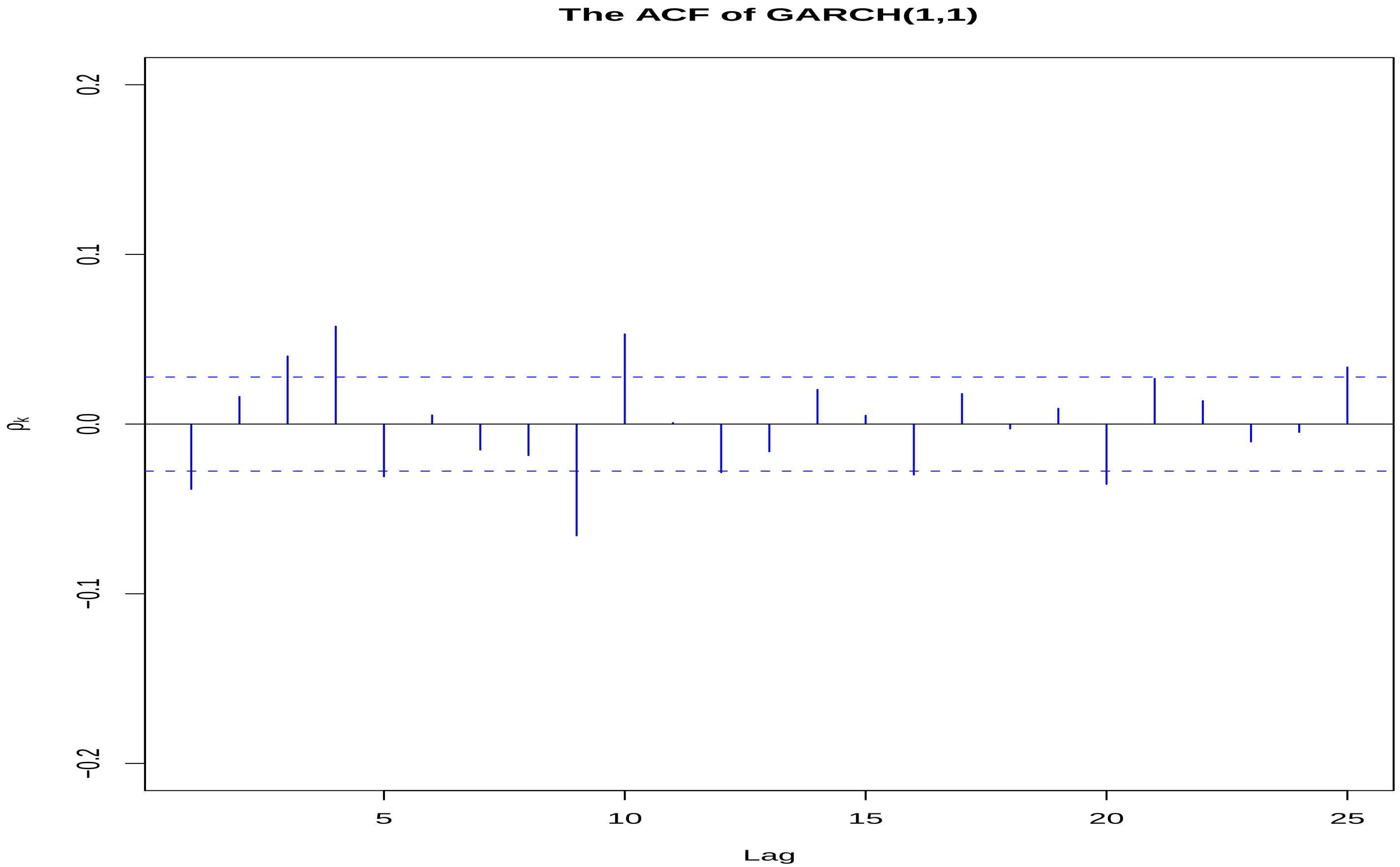
$$Y_t = \mu + \pi_1 Y_{t-1} + \pi_2 Y_{t-2} + \cdots + a_t$$

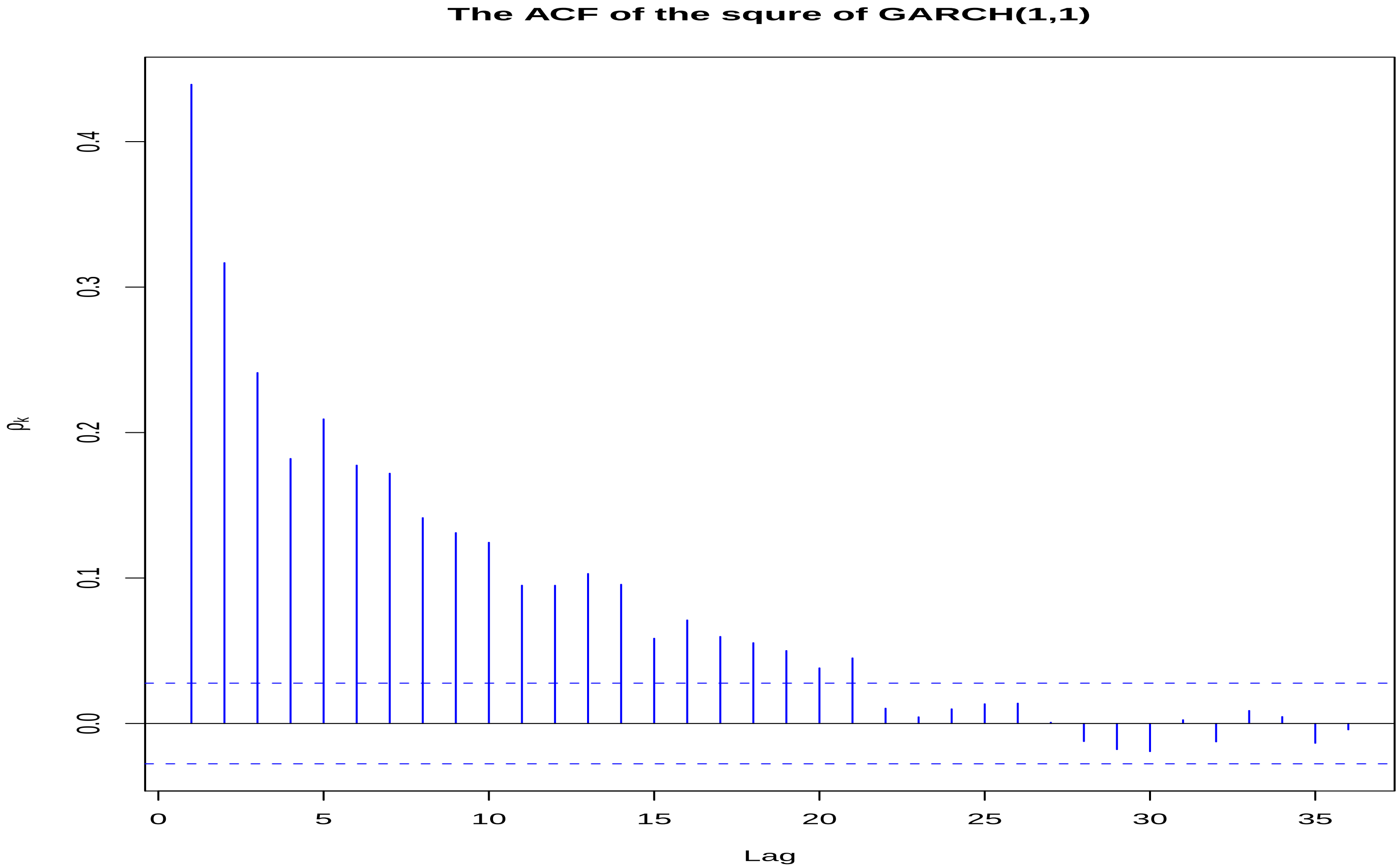
where $1 + \sum_{j=0}^{\infty} |\pi_j| < \infty$.

- ACF of ARMA(1,1) model:



- ACF of GARCH(1,1) model:





The ACF of the squre of GARCH(1,1)
0.0
0.1
0.2
0.3
0.4
ρ_k
0
5
10
15
20
25
30
35
Lag

- ACF of DAR(1) model: