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Howell Tong
The London School of Economics and Political Science

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ON THE USE OF THE DETERMINISTIC LYAPUNOV FUNCTION FOR THE ERGODICITY OF STOCHASTIC DIFFERENCE EQUATIONS

K. S. CHAN* AND H. TONG, * *The Chinese University of Hong Kong*

Abstract

We have shown that within the setting of a difference equation it is possible to link ergodicity with stability via the physical notion of energy in the form of a Lyapunov function.

STABILITY

1. Introduction

In this paper, we are exclusively interested in stochastic difference equations of the form

$$X_{n+1}=T(X_n)+e_{n+1}, \qquad n\geqq 0, \qquad T:\mathbb{R}^m\to\mathbb{R}^m. \tag{1.1}$$

X_n takes values in $\mathbb{R}^m$. Let $\mathscr{B}_m$ be the class of Borel sets of $\mathbb{R}^m$ and μ_m the Lebesgue measure. Then $(\mathbb{R}^m, \mathscr{B}_m, \mu_m)$ is the state space of (1.1). The random forcing terms, $\{e_{n+1}\}$, on the right-hand side of (1.1) are assumed to be of either one of the following forms:

(1.2a) i.i.d.; the marginal distribution is absolutely continuous and has a positive p.d.f. $f(\cdot)$ over $\mathbb{R}^m$;

(1.2b) $e_n=\begin{pmatrix} e'_n \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ with e'_n i.i.d., each having an absolutely continuous distribution and the p.d.f. $f(\cdot)$ is positive everywhere in $\mathbb{R}$.

Now, we assume (1.2a) holds. Let $A\in\mathscr{B}_m$ and $x\in\mathbb{R}_m$. Let $P(x, A)$ be the transition probability function. Then

$$P(x,A)=\int_{A-T(x)} f(t)\mu_m(dt). \tag{1.3a}$$

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* Postal address: Department of Statistics, University Science Centre, The Chinese University of Hong Kong, Shatin, N. T., Hong Kong.

Hence, $\{X_n\}$ with $P(x, A)$ defined in (1.3a) forms a Markov chain with the state space $(\mathbb{R}^m, \mathscr{B}_m, \mu_m)$.

Suppose (1.2b) holds. Let $A \in \mathscr{B}_m$ and $x \in \mathbb{R}^m$. Let

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}.$$

Define

$$A_x = \left\{ y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} \in \mathbb{R}^m : y_i = x_i,\ i \geqq 2,\ y \in A \right\}.$$

Let $\nu : \mathbb{R}^m \to \mathbb{R}$ be the projection map onto the first coordinate, i.e., $\nu(y) = y_1$. Then

$$P(x, A) = \int_{\nu(A_{Tx}) - (Tx)_1} f(t)\, dt, \tag{1.3b}$$

where $(Tx)_1$ is the first coordinate of Tx and we have written Tx for $T(x)$.

Again, $\{X_n\}$ with $P(x, A)$ defined in (1.3b) forms a Markov chain with state space $(\mathbb{R}^m, \mathscr{B}_m, \mu_m)$. (Note that in (1.3a) and (1.3b) we have abused the use of f, which represents two different entities.)

For the case where $T(X_n) = TX_n$, T being a companion matrix, it is well known that (1.1) defines an asymptotically *stationary* time series if the characteristic roots of T all lie inside the unit circle. It is equally well known that *exactly* the same condition ensures the *stability* of the solution of the deterministic equation associated with (1.1) in which e_{n+1} is replaced by the zero vector. *Is this situation merely a coincidence*?

At a deeper level, in the theory of a Markov chain over a general state space, it is known that *ergodicity* (in a suitable sense) of an irreducible chain may be established by identifying a 'centre' towards which there is a 'mean drift'—the so-called Foster condition. (See for example Tweedie (1975).) On the other hand, in the theory of a (non-linear) difference equation, *stability* (in a suitable sense) of the solution may be investigated by studying the behaviour of a generalized energy associated with the equation: roughly speaking, when the trajectory moves towards the asymptotic solution (cf. the centre), a dissipation of the generalised energy (cf. the mean drift) is essential for stability to be attained. (See, for example LaSalle (1976).) Again, the basic ideas seem rather strikingly similar. *Is this situation merely a coincidence again*?

In this paper, we aim to show that the Lyapunov function, which may be interpreted as a generalized energy, plays a significant role in studying not only the stability (a well-known fact) of a deterministic difference equation but also the ergodicity of a stochastic difference equation.

2. A simple criterion for non-null compact sets to be small

It is of interest to establish conditions under which (1.1) is geometric ergodic. We shall employ notations and terminology adopted in Tweedie (1983a). For the general theory, we refer to Tweedie (1975), (1976), (1983a), (1983b) and Nummelin and Tuominen (1982). Let $A, B, K \in \mathcal{B}_m$ with non-zero μ_m-measure. K always denotes a compact set. For non-null compact sets to be small, it is sufficient that T in (1.1) is continuous. However, we may relax this condition to some extent.

Consider (1.1) with e_n of the form (1.2a). Then $\{X_n\}$ is μ_m-irreducible and aperiodic. Suppose T is compact (i.e. T sends compact sets into relatively compact sets). Suppose $f(\cdot)$ is lower semi-continuous. Then, clearly,

$$\inf_{x \in K} P(x, A) > 0.$$

Hence, K is small.

We now consider the case where X_n satisfies (1.1) with e_n of the form (1.2b). Let $(\mathbb{R}^m, \mathcal{B}_m, \mu_m)$ be the state space. Then $P(x, A)$ is of the form (1.3b). We assume further that T is of a more restricted form, i.e.,

$$T(x) = \begin{pmatrix} h(x) \\ x_1 \\ \vdots \\ x_{m-1} \end{pmatrix}, \text{ where } \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \in \mathbb{R}^m$$

and h is a measurable function from $\mathbb{R}^m$ to $\mathbb{R}$. Suppose $m = 2$ and $A = (a_1, b_1) \times (a_2, b_2)$, an open interval in $\mathbb{R}^2$. Then

$$\begin{aligned} P^2(x, A) &= \int P(y, A) P(x, dy) \\ &= \int_{a_2}^{b_2} f(y_2 - h(x_1, x_2)) P(y, A) \, dy_2 \end{aligned}$$

where

$$y = \begin{pmatrix} y_2 \\ x_1 \end{pmatrix}.$$

However,

$$P(y, A) = \int_{a_1}^{b_1} f(y_1 - h(y_2, x_1)) \, dy_1.$$

Hence, by Fubini's theorem,

$$P^2(x, A) = \iint_A f(y_2 - h(x_1, x_2)) f(y_1 - h(y_2, x_1))\, dy_1\, dy_2. \tag{2.1}$$

Since for fixed $x \in \mathbb{R}^2$ both the right- and left-hand sides define a Borel measure on the Borel sets and they are equal over the rectangles, the equality holds for all $A \in \mathscr{B}_2$. Evidently, the same idea goes through in higher dimension and we have

$$p^m(x, A) = \int_A \prod_{j=1}^{m} f_j'(y) \mu_m(dy), \tag{2.2}$$

where $y = (y_1, y_2, \cdots, y_m)'$, $x = (x_1, x_2, \cdots, x_m)'$,

$$f_m'(y) = f(y_m - h(x_1, x_2, \cdots, x_m)),$$
$$f_i'(y) = f(y_i - h(y_{i+1}, \cdots, y_m, x_1, \cdots, x_i)), \qquad m > i \geqq 1.$$

Formula (2.2) is very useful. It is clear that $P^m(x, A) > 0$, $\forall x \in \mathbb{R}^m$. Therefore $\{X_n\}$ is μ_m-irreducible and aperiodic. Suppose h is compact and $f(\cdot)$ is lower semi-continuous. Then

$$\inf_{x \in K} P^m(x, A) > 0.$$

Hence, K is small.

3. From deterministic stability to ergodicity: a precursor

In this section, we apply the above framework to a particular case, the so-called SETAR model in non-linear time series analysis. (For a comprehensive introduction to this kind of model, see Tong (1983).)

$$h(x_1, x_2, \cdots, x_m) = c_i + \sum_{j=1}^{m} a_{ij} x_j \quad \text{if} \quad r_{i-1} \leqq x_d < r_i, \tag{3.1}$$

where $\{-\infty = r_0 < r_1 < \cdots < r_l = +\infty\}$ is an ordered partition of $\mathbb{R}$, $d \leqq m$, c_i and a_{ij} are constants. The function h of (3.1) is the autoregressive function for the full SETAR model. If we define $T: \mathbb{R}^m \to \mathbb{R}^m$ by

$$T(x) = \begin{pmatrix} h(x) \\ x_1 \\ \vdots \\ x_{m-1} \end{pmatrix} \quad \text{where} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \in \mathbb{R}^m,$$

then this T together with e_n of the form (1.2b) constitutes the Markovian state-space equation for the full SETAR model. Specifically, we have the following setup:

$$X_n \text{ takes values in } \mathbb{R}^m.$$

$$X_{n+1} = T(X_n) + e_n, \tag{3.2a}$$

$$\xi_n = (1, 0, \cdots, 0)X_n, \tag{3.2b}$$

where

$$T(x) = \begin{pmatrix} h(x) \\ x_1 \\ \vdots \\ x_{m-1} \end{pmatrix}$$

with $h(\cdot)$ as defined in (3.1) and

$$e_n = \begin{pmatrix} e'_n \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

e'_n's being i.i.d. zero-mean random variables and each having an absolutely continuous distribution, the density of which is lower semi-continuous and positive everywhere in $\mathbb{R}$.

It is then clear that $h(\cdot)$ is compact. From the results of Sections 1 and 2, we see that the chain $\{X_n\}$ is μ_m-irreducible and aperiodic, and non-null compact sets are small sets. Henceforth, let $\|.\|$ denote the Euclidean norm.

Lemma 3.1. If $\max_i \sum_j |a_{ij}| < 1$ and e'_n possesses first absolute moment, then (3.2a) is geometrically ergodic.

Proof. Let

$$z = \begin{pmatrix} z_1 \\ \vdots \\ z_m \end{pmatrix} \in \mathbb{R}^m.$$

As $\max_i \sum_j |a_{ij}| < 1$, $\exists p_1 > p_2 > \cdots > p_m > 0$ such that $\max_i \sum_j |a_{ij}| (p_1/p_j) < \theta < 1$ for some θ. Moreover, θ may be chosen such that $\theta > (p_{i+1}/p_i)$. Define $g : \mathbb{R}^m \to \mathbb{R}$ by $g(z) = 1 + \max_i |z_i| p_i$. Then,

$$\int g(z)P(x, dz) \leqq C + \theta g(x).$$

Since $m \|z\| \leqq g(z) \leqq M \|z\| + 1$, for some $0 < m < M$, therefore for $r \geqq$

$2C/m(1-\theta)$ and some $B>0$,

(i) $\int g(z)P(x, dz) < B < +\infty$, $\|x\| \leqq r$

(ii) $\int g(z)P(x, dz) < \frac{1}{2}(1+\theta)g(x)$, $\|x\| > r$.

By Theorem 4 of Tweedie (1983a), $\{X_n\}$ is geometrically ergodic.

4. Some general results

Suppressing the random forcing term e_{n+1} in (1.1), we obtain

$$X_{n+1} = T(X_n) \qquad n \geqq 0, \tag{4.1}$$

which will be called the associated deterministic difference equation or the deterministic part of the stochastic difference equation (1.1). The study of (4.1) may be viewed as a stepping-stone to, or even the 'bone' of, the study of (1.1). To start with, if the range of T is bounded, it is clear that (1.1) is ergodic. Boundedness is one form of stability of the dynamics of (4.1). Since the stability theory of (4.1) is well known, we omit all details but refer the readers to, for example, Kalman and Betram (1960), Halanay (1963) and LaSalle (1976).

Remark 4.1. It is known that the existence of a continuous Lyapunov function $V(x)$ near the origin implies the uniform asymptotic stability of the origin. For a precise statement, see for example Kalman and Bertram (1960). The converse is also true. Moreover, if T is Lipschitz-continuous near the origin, then the Lyapunov function constructed is also Lipschitz-continuous. However, it is easily seen from the proof given in Halanay (1963) that for the existence of a continuous Lyapunov function near the origin, T only needs to be continuous.

Theorem 4.2. Let $\{X_n\}$ satisfies (1.1). Let T be continuous and homogeneous (i.e. $T(cx) = cT(x)$, $\forall c > 0$, $x \in \mathbb{R}^m$). Let the origin, O, be a fixed point of T. In the case of e_n satisfying (1.2a), we assume that $\int \|t\| f(t)\mu_m(dt) < +\infty$. If e_n satisfies (1.2b), we assume that $\int |t| f(t)\, dt < +\infty$ and

$$T(x) = \begin{pmatrix} h(x) \\ x_1 \\ \vdots \\ x_{m-1} \end{pmatrix}.$$

Then the existence of a continuous Lyapunov function, V, in a neighbourhood of the origin implies the geometric ergodicity of (1.1).

Proof. Let $W \subseteq \mathbb{R}^m$. We denote the closure of W by $\bar{W}$ and its boundary by ∂W. Without loss of generality, let V be defined over the closure of the unit ball. Let $m_0 = \inf_{\|x\|=1} V(x)$. Let G be the maximal connected component of $\{x : V(x) < \frac{1}{2}m_0\}$ which contains the origin. Then $T(\bar{G}) \subset G$. Let

$$g(x) = \inf\{r \geqq 0, x \in rG\}, \qquad x \in \mathbb{R}^m,$$

where $rG = \{rx, x \in G\}$. Then $g(x)$ is well defined and it may be easily checked that g has the following properties:

(i) $g(cx) = cg(x), \forall c > 0$

(ii) $\exists 0 < m < M < +\infty$ such that $m\|x\| \leqq g(x) \leqq M\|x\|$

(iii) $(x/g(x)) \in \partial G$

(iv) $\exists \varepsilon > 0, 0 < \theta < 1$ such that $\forall x \in \partial G$, $y \in \mathbb{R}^m$

$$\|y - Tx\| < \varepsilon \Rightarrow y \in G \quad \text{and} \quad g(y) < \theta$$

$$(\text{i.e.} \quad \operatorname{dist}(y, T(\partial G)) < \varepsilon \Rightarrow g(y) < \theta).$$

Now, for model (1.1) with e_n satisfying (1.2a) and $\int \|t\| f(t)\mu_m(dt) < +\infty$, it holds that

$$\begin{aligned}\int g(y)P(x, dy) &= \int g(T(x) + t)f(t)\mu_m(dt) \\ &= \int_{\|t/g(x)\| < \varepsilon} g(x)g\left(T\left(\frac{x}{g(x)}\right) + \frac{t}{g(x)}\right)f(t)\mu_m(dt) \\ &\quad + \int_{\|t/g(x)\| \geqq \varepsilon} g(x)g\left(T\left(\frac{x}{g(x)}\right) + \frac{t}{g(x)}\right)f(t)\mu_m(dt).\end{aligned}$$

The first term is less than $\theta g(x)$. The second term is $C + g(x) \cdot \xi(x)$ for some $C > 0$.

Here, $|\xi(x)| < B < +\infty, \forall x$ and $\xi(x) \to 0$ as $\|x\| \to +\infty$. Let $h(x) = g(x) + 1$. Let r be such that

$$|\xi(x)| < \tfrac{1}{4}(1 - \theta) \quad \text{for} \quad \|x\| > r.$$

Then for $r_0 = \max(r, 4(C+1)/(1-\theta)m), \exists B'$, such that

(i) $\int h(y)P(x, dy) < B' < +\infty, \|x\| \leqq r_0$

(ii) $\int h(y)P(x, dy) < \frac{1}{2}(1+\theta)h(x)\ \|x\| > r_0$.

Hence, Theorem 4 in Tweedie (1983a) shows that $\{X_n\}$ is geometric ergodic. The case when e_n satisfies (1.2b) is similarly proved.

Corollary 4.3. Suppose $\{X_n\}$ satisfies (1.1) and the conditions in Theorem 4.2 hold. Define

$$\xi_n = (1, 0, \cdots, 0)'X_n.$$

Then ξ_n equipped with the marginal distribution of the first component of X_n is strictly stationary.

Corollary 4.4. Under the same conditions as in Corollary 4.3, if e_n possesses the kth absolute moment (i.e. $\int \|t\|^k f(t)\mu_m(dt) < +\infty$ if (1.2a) holds, or $\int |t|^k f(t)\, dt < +\infty$ if (1.2b) holds), then the stationary distribution of $\{\xi_n\}$ has finite kth absolute moment.

Proof. Use the test function $H(y) = (g(y))^k + 1$ and then it may be verified that the conditions in Theorem 3 of Tweedie (1983b) are satisfied.

On appealing to our earlier remark, we obtain the following result.

Theorem 4.5. Under the same conditions as in Theorem 4.2, the uniform asymptotic stability of the origin (and thus global uniform asymptotic stability) implies the existence of a continuous Lyapunov function near the origin and therefore the geometric ergodicity of $\{X_n\}$.

5. Some extensions and examples

Theorems 4.2 and 4.5 place some restrictions on T. The weakest assumption on T is that of homogeneity. We mention some easy extensions. Suppose T is merely compact but can be decomposed into two parts, namely,

$$T = T_h + T_d,$$

where T_h is homogeneous and continuous and T_d is of bounded range. We consider the 'component' of (1.1) given by

$$X_{n+1} = T_h(X_n) + e_n, \qquad n \geqq 0. \tag{5.1}$$

Then we can apply Theorems 4.2 and 4.5 to (5.1). It is clear that the conclusion then holds also for $\{X_n\}$ satisfying (1.1).

Since the stability of a deterministic dynamical system is a very well-researched area, the link which we have just established should prove useful in the study of ergodicity of the associated stochastic system. We now give some examples illustrating the use of Theorem 4.2.

We assume that $\{X_n\}$ satisfies (1.1) with e_n satisfying (1.2b). Moreover, let

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \in \mathbb{R}^m \quad \text{and} \quad T(x) = \begin{pmatrix} h(x_1, x_2, \cdots, x_m) \\ x_1 \\ \vdots \\ x_{m-1} \end{pmatrix}$$

where $h(\cdot)$ is from $\mathbb{R}^m$ to $\mathbb{R}$.

Example 1.

$$h(x) = \sum_{i=1}^{m} a_i x_i + l \tag{5.2a}$$

where the a_i's and l are constants.

Decompose $h(x)$ into the sum of $h_h(x)$ and $h_d(x)$, where

$$h_h(x)=\sum_i a_i x_i, \tag{5.2b}$$

and

$$h_d(x)=l. \tag{5.2c}$$

Clearly $h(x)$ is compact. Now 0 is uniformly asymptotically stable with respect to (5.2b) iff all the roots of the characteristic equation (i.e. $x^m - \sum a_i x^{m-i}=0$) have magnitudes less than 1. Thus, from Theorem 4.5, we see that $\{X_n\}$ satisfying (1.1) with the autoregressive function $h(x)$ defined by (5.2a) is geometrically ergodic if all the characteristic roots have magnitudes less than unity.

Example 2.

$$h(x)=\sum_{i=1}^{m}(a_i+b_i\exp(-\gamma x_i^2))x_i. \tag{5.3a}$$

Similarly, we have

$$h_h(x)=\sum_{i=1}^{m} a_i x_i \tag{5.3b}$$

$$h_d(x)=\sum_i b_i\exp(-\gamma x_i^2)x_i. \tag{5.3c}$$

It follows exactly as in Example 1 that $\{X_n\}$ satisfying (1.1) with the autoregressive function $h(x)$ defined in (5.3a) is geometrically ergodic if all the characteristic roots of (5.3b) have magnitude less than unity. This model is a variant of the exponential autoregressive model introduced by Ozaki (1980). His original model replaces each x_i in the exponent of Equation (5.3a) by x_1. It seems that the ergodicity of the latter model remains an open problem unless $m=1$.

Example 3. Consider the first-order SETAR model. Here

$$h(x)=\begin{cases} c_1+\phi_1 x & \text{if} \quad x>r_1,\\ c_2+\phi_2 x & \text{otherwise.}\end{cases} \tag{5.4a}$$

Then

$$h_h(x)=\begin{cases} \phi_1 x & \text{if} \quad x>0\\ \phi_2 x & \text{otherwise}\end{cases} \tag{5.4b}$$

$$h_d(x)=h(x)-h_h(x). \tag{5.4c}$$

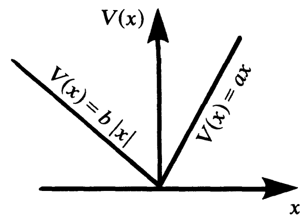

Figure 1. a and b are any real numbers such that $a>0$, $b>0$, $1>\phi_1>-a/b$, $1>\phi_2>-b/a$.

It is clear that a necessary and sufficient condition for the uniform asymptotic stability of the origin with respect to the deterministic difference equation corresponding to $h_h(x)$ of (5.4b) is

$$\phi_1<1, \qquad \phi_2<1, \qquad \phi_1\phi_2<1.$$

In fact, an appropriate Lyapunov function for (5.4b) is as shown in Figure 1.

Petruccelli and Woolford (1984) have proved that this condition is both necessary and sufficient for the ergodicity of the associated first-order threshold autoregressive model.

Example 4.

$$T(x)=\begin{cases} Ax & \text{if} \quad x\in\Omega, \\ Bx & \text{otherwise,} \end{cases} \tag{5.5}$$

where $\Omega\subseteq\mathbb{R}^m$, and is bounded, and $A=(a_{ij})$ and $B=(b_{ij})$ are matrices of constants. Some sufficient conditions for the uniform asymptotic stability of the origin with respect to (5.5) are given in LaSalle (1976). We give one easily checked condition. Let $|A|=(|a_{ij}|)$ and $|B|=(|b_{ij}|)$. If C is such that $c_{ij}\geqq \max(|a_{ij}|, |b_{ij}|)$ and the eigenvalues of C have magnitudes less than unity, then the origin is asymptotically stable. Thus, this condition is sufficient for the geometric ergodicity of (1.1) with T of the form (5.5) and e_n of the form (1.2a). For unbounded Ω, a Lyapunov function, V, may be constructed such that $V(x)=V(|x|)$ and $V(|x|)>V(|y|)$ if $|x|>|y|$, where $|x|=(|x_1|,\cdots,|x_m|)'$ (cf. LaSalle (1976), p. 17). From this V, geometric ergodicity follows on using standard arguments.

Another way to remove the restriction of the homogeneity of T in Theorems 4.2 and 4.5 is to set stronger conditions on the stability of the deterministic part. Suppose T is Lipschitz and the origin is exponential—asymptotically stable in the large. (For further details in this area, see for example, Yoshizawa (1966).)

Theorem 5.1. Suppose $\exists M>0$ such that $\forall x, y\in\mathbb{R}^m$

$$\|Tx-Ty\|\leqq M\,\|x-y\|.$$

Let $x(n; x_0)$ be the solution of (4.1) with x_0 as the starting point. Suppose $\exists c>0, K>0$ such that

$$\|x(n; x_0)\| \leqq K e^{-cn} \|x_0\|, \quad \forall n.$$

Moreover, e_n possesses appropriate moments, i.e. $\int \|t\| f(t)\mu_m(dt) < +\infty$ or $\int |t| f(t)\, dt < +\infty$.

Then $\{X_n\}$ satisfying (1.1) with e_n of the form (1.2a) or (1.2b) is geometrically ergodic.

Proof. As in Yoshizawa (1966), we define

$$g(x) = \left(\sup_{\tau \geqq 0} \|x(\tau; x)\| e^{qc\tau}\right) + \gamma$$

where $0<q<1$ and $\gamma>1$.

Then $g(x)$ satisfies

(i) $\|x\| + \gamma \leqq g(x) \leqq K\|x\| + \gamma$.

(ii) $|g(x) - g(y)| \leqq L\|x-y\|, \forall x, y \in \mathbb{R}^m$,

(iii) $g(Tx) - g(x) \leqq -\alpha g(x) + \gamma(1-(1/e^{qc}))$, for some L, α, positive constants.

It is easily seen that

$$\|x\| + \gamma \leqq g(x) \leqq \gamma + \sup_{\tau \geqq 0} K \,.\, \exp(-(1-q)c\tau)\|x\| \leqq \gamma + K\|x\|.$$

Now, for (ii) and the determination of L, let β be such that $K = \exp((1-q)c\beta)$. If $\tau \geqq \beta$, then $K \exp(-(1-q)c\tau)\|x\| \leqq \|x\|$ and hence $g(x) = \gamma + \sup_{0 \leqq \tau \leqq \beta} \|x(\tau; x)\| \exp(qc\tau)$. Therefore, if $x, y \in \mathbb{R}^m$

$$\begin{aligned} |g(x) - g(y)| &\leqq \sup_{0 \leqq \tau \leqq \beta} \|x(\tau; x) - x(\tau; y)\| e^{qc\tau} \\ &\leqq e^{qc\beta} M^{\beta} \|x-y\|, \end{aligned}$$

where M is the Lipschitz constant for T.

For (iii),

$$\begin{aligned} g(Tx) &= \gamma + \sup_{\tau \geqq 0} \|x(\tau; Tx)\| e^{qc\tau} \\ &= \gamma + \sup_{\tau \geqq 0} \|x(\tau+1; x)\| e^{qc(\tau+1)} \cdot \frac{1}{e^{qc}} \end{aligned}$$

$$\Rightarrow g(Tx) - g(x) \leqq -\alpha g(x) + \gamma\alpha, \quad \text{where} \quad \alpha = 1 - \frac{1}{e^{qc}}.$$

Then, as before, we conclude that $\{X_n\}$ is geometric ergodic.

We may note that e^{qc} is related to the (geometric) rate of convergence of $P^n(x, .)$ to the invariant measure.

6. Discussion

Results obtained so far encourage us to take the view that a systematic approach to prove (geometric) ergodicity of a stochastic difference equation via a Lyapunov function for its deterministic part is conceptually satisfying and practically useful. However, our results are quite modest. Deeper results should be possible. For example, we may consider replacing the random driving term e_{n+1} of (1.1) not merely by a zero vector but by a general deterministic 'control vector'. Even more generally, we may consider the wider class given by

$$X_{n+1} = T(X_n, e_{n+1}).$$

As far as stochastic differential equations are concerned, significant progress has already been made in this respect in the last two decades: see for example Arnold and Kliemann (1983). It is hoped that the significance of these results will be rendered more transparent to applied probabilists when they are 'translated' into the discrete-time case.

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