Testing for Change-point ①

$H_0: \ \phi_{10} = \phi_{20}$

$H_a: \ \phi_{10} \neq \phi_{20}$ $\begin{cases} \phi_{10} = \phi, & t=1 \cdots k \\ \phi_{20} \neq \phi & t=k+1 \cdots n \end{cases}$ for some k.

$$L_n(\phi_1, \phi_2, k) = \sum_{t=1}^{k} (y_t - \phi_1 y_{t-1})^2 + \sum_{t=k+1}^{k} (y_t - \phi_2 y_{t-1})^2$$

$$\min_{\phi_1 \, \phi_2} L_n(\phi_1, \phi_2, k)$$

$$= L_n(\hat{\phi}_1(k), \hat{\phi}_2(k), k) \qquad \text{Under } H_a.$$

Under H_0,

$$\min L_n(\phi) = \sum_{t=1}^{n} (y_t - \phi y_{t-1})^2 .$$

$$\min_{\phi} L_n(\phi) = L_n(\hat{\phi}),$$

$$\hat{\phi} = \sum_{t=1}^{n} y_t y_{t-1} \Big/ \sum_{t=1}^{n} y_{t-1}^2 .$$

The Likelihood ratio test for a given k.

$$L_n(\hat{\phi}) - L_n(\hat{\phi}_1(k), \hat{\phi}_2(k), k)$$

$$= \sum_{t=1}^{n} (y_t - \hat{\phi} y_{t-1})^2 - \sum_{t=1}^{k} (y_t - \hat{\phi}_1(k) y_{t-1})^2 - \sum_{t=k+1}^{n} (y_t - \hat{\phi}_2(k) y_{t-1})^2$$

$$= \sum_{t=1}^{n} \left[(y_t - \hat{\phi} y_{t-1})^2 - \varepsilon_t^2 \right] - \sum_{t=1}^{k} \left[(y_t - \hat{\phi}_1(k) y_{t-1})^2 - \varepsilon_t^2 \right]$$

$$- \sum_{t=k+1}^{n} \left[(y_t - \hat{\phi}_2(k) y_{t-1})^2 - \varepsilon_t^2 \right]$$

$$= -\left(\sum_{t=1}^{n} y_{t-1}\varepsilon_t\right)^2 \Big/ \sum_{t=1}^{n} y_{t-1}^2$$

$$+ \left(\sum_{t=k+1}^{n} y_{t-1}\varepsilon_t\right)^2 \Big/ \sum_{t=k+1}^{n} y_{t-1}^2 + \left(\sum_{t=1}^{n} y_{t-1}\varepsilon_t\right)^2 \Big/ \sum_{t=1}^{n} y_{t-1}^2$$

$$\boxed{\frac{1}{k}\sum_{t=1}^{k} y_{t-1}^2 = E y_t^2 + o(1), \quad \frac{1}{n-k}\sum_{t=k+1}^{k} y_{t-1}^2 = E y_t^2 + o(1) \qquad \sigma_y^2 = E y_t^2.}$$

$$= +\frac{1}{\sigma_y^2}\left[\frac{1}{k}\left(\sum_{t=1}^{k} y_{t-1}\varepsilon_t\right)^2 + \frac{1}{n-k}\left(\sum_{t=k+1}^{n} y_{t-1}\varepsilon_t\right)^2 - \frac{1}{n}\left(\sum_{t=1}^{n} y_{t-1}\varepsilon_t\right)^2 \right] + o(1)$$

$$= \frac{1}{\sigma_y^2}\,\frac{\left[n\left(\sum_{t=1}^{k} y_{t-1}\varepsilon_t\right) - k\left(\sum_{t=1}^{n} y_{t-1}\varepsilon_t\right)\right]^2}{n k (n-k)}$$

$$= \frac{1}{\sigma_y^2}\,\frac{1}{\frac{k}{n}\left(1-\frac{k}{n}\right)} \cdot \left[\frac{1}{\sqrt{n}}\left(\sum_{t=1}^{k} y_{t-1}\varepsilon_t\right) - \frac{k}{n\sqrt{n}}\left(\sum_{t=1}^{n} y_{t-1}\varepsilon_t\right)\right]^2$$

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$$\frac{1}{\sqrt{n}} \sum_{t=1}^{[n\tau]} y_{t-1}\varepsilon_t \Longrightarrow B(\tau)\cdot\sigma_y \quad \text{in } D.$$

$$\max_{[n\tau_1] \le k \le n-[n\tau_1]} \left[L_n(\hat{\phi}) - L_n(\hat{\phi}_1(k), \hat{\phi}_2(k), k) \right]$$

$$\xrightarrow{d} \frac{\sigma^2}{\tau(\ } \max_{\tau_1 \le \tau \le 1-\tau_1} \frac{[B(\tau) - \tau B(1)]^2}{\tau(1-\tau)}$$

when $\tau_1 \in (0, \frac{1}{2})$.

Note that

$$\max_{0 \le k \le n} \left[L_n(\hat{\phi}) - L_n(\hat{\phi}_1(k), \hat{\phi}_2(k), k) \right] \xrightarrow{P} \infty$$

Lemma. Let $\{G_t : t = 1, 2, \ldots\}$ be i.i.d. with $E\,G_t = 0$ and $Var(G_t) = 1$. If $E|G_t|^{2+\varepsilon} < \infty$, for some $\varepsilon > 0$, then, for each $\mu \in (0,1)$, any x

$$P\left(\frac{1}{a_n} \left[\max_{\log n < k \le \mu n} \left| \frac{1}{\sqrt{k}} \sum_{t=1}^{k} G_t \right|^2 - b_n \right] \le x \right) \to e^{-e^{-\frac{x}{2}}},$$

where $a_n = \sqrt{b_n / (2 \log \log n)}$,

$$b_n = \frac{\{2 \log \log n + \log \log \log n / 2 - \log \Gamma(\tfrac{1}{2})\}^2}{2 \log \log n}.$$

$\Gamma(x)$ — gamma function.

Denote

~~$W_n = \max_{m < k < n-m}$~~

$$W_n(k) = L_n(\hat{\theta}) - L_n(\hat{\theta}_1(k), \hat{\theta}_2(k), k)$$

$$P\left(\max_{0 < k < n-1} \frac{W_n(k) - b_n}{a_n} \le x \right) \to e^{-2e^{-\frac{x}{2}}},$$

as $n \to \infty$.

Denote,

$$S_1(k)=\frac{1}{\sqrt{k}}\sum_{t=1}^{k} y_{t-1}\varepsilon_t, \qquad S_2(k)=\frac{1}{\sqrt{n-k}}\sum_{t=k+1}^{n} y_{t-1}\varepsilon_t.$$

To make it simple, assume that $\sigma_y^2=1$.

Let $\mu\in(0,0.5)$.

$$\left|\max_{\log n\le k\le \mu n} W_n(k)-\max_{\log n\le k\le \mu n} S_1^2(k)\right|$$

$$\le \max_{\log n\le k\le \mu n}\left|W_n(k)-S_1^2(k)\right|$$

$$=\max_{\log n\le k\le \mu n}\left|\frac{n}{n-k}\left(\frac{1}{\sqrt{n}}\sum_{t=k+1}^{n} y_{t-1}\varepsilon_t\right)^2 - \left(\frac{1}{\sqrt{n}}\sum_{t=1}^{n} y_{t-1}\varepsilon_t\right)^2\right|$$

$$\xrightarrow{\mathcal{L}} \max_{0\le\tau\le\mu}\left|\frac{[B(1)-B(\tau)]^2}{1-\tau}-B^2(1)\right|$$

$$\xrightarrow{P} 0 \quad \text{as} \quad \mu\to 0.$$

Thus, for $\forall\,\varepsilon>0$,

$$\lim_{\mu\to 0}\limsup_{n\to\infty} P\left(\left|\max_{\log n\le k\le \mu n} W_n(k)-\max_{\log n\le k\le \mu n} S_1^2(k)\right|>\varepsilon\right)=0.$$

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Similarly

$$\lim_{\mu \to 0} \limsup_{n \to \infty} P\left(\left| \max_{\log n \le n-k \le \mu n} W_n(k) - \max_{\log n < n-k \le n\mu} S_2^2(k) \right| > \varepsilon \right) = 0.$$

For $\forall \mu$,

$$\max_{\mu n \le k \le n - n\mu} W_n(k) \xrightarrow{d} \max_{\mu_0 \le \tau \le 1-\mu_0} \frac{[B(\tau) - \tau B(1)]^2}{\tau(1-\tau)}$$

$$P\left(\max_{\log n \le k \le n - \log n} \frac{W_n(k) - b_n}{b_n} \le x \right)$$

$$= P\left(\max\left\{ \max_{\log n \le k \le \cancel{\;\;} n\mu} \frac{W_n(k) - b_n}{a_n}, \max_{\log n \le n-k \le n\mu} \frac{W_n(k) - b_n}{a_n}, \max_{\mu n \le k \le n - n\mu} \frac{W_n(k) - b_n}{a_n} \right\} \le x \right)$$

$$= P\left(\max\left\{ \max_{l_{gn} \le k \le n\mu} \frac{W(k) - b_n}{a_n}, \max_{l_{gn} \le n-k \le n\mu} \frac{W(n) - b_n}{a_n} \right\} \le x \right)$$

$$+ o(\mu)$$

$$= P\left(\max\left\{ \max_{l_{gn} \le k \le n\mu} \frac{S_1^2(k) - b_n}{a_n}, \max_{l_{gn} \le n-k \le n\mu} \frac{S_2^2(k) - b_n}{a_n} \right\} \le x \right)$$

$$+ o_p(\mu)$$

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Strong Invariance Principle:

$$\frac{1}{\sqrt{k}} \sum_{t=1}^{k} \eta_{t-1} \varepsilon_t = \frac{1}{\sqrt{k}} \sum_{t=1}^{k} G_{1t} + o(k^{-\delta}) \text{ a.s.}$$

for some $\delta > 0$

$$\frac{1}{\sqrt{k}} \sum_{t=-k}^{-1} \eta_{t-1} \varepsilon_t = \frac{1}{\sqrt{k}} \sum_{t=1}^{k} G_{2t} + o(k^{-\delta}) \text{ a.s.},$$

where $\{G_{1t}\}$ and $\{G_{2t}\}$ are independent and

$G_{1t} \sim$ i.i.d $N(0, \mathrm{Var}(\eta_{t-1}\varepsilon_t))$,

$G_{2t} \sim$ i.i.d $N(0, \mathrm{Var}(\eta_{t-1}\varepsilon_t))$.

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$$\left| \max_{\log n \le k \le n\mu} S_1^2(k) - \max_{\log n \le k \le n\mu} \left[\frac{1}{\sqrt{k}} \sum_{t=1}^{k} G_{1,t} \right] \right|$$

$$\le \max_{\log n \le k \le n\mu} \left| \sqrt{\log\log k} \left[S_1(k) - \frac{1}{\sqrt{k}} \sum_{t=1}^{k} G_{1,t} \right] \times \frac{|S_1(k)| + \left|\frac{1}{\sqrt{k}} \sum_{t=1}^{k} G_{1,t}\right|}{\sqrt{\log\log k}} \right| = o_p(1)\,.$$

Similarly

$$\left| \max_{\log n \le n-k \le n\mu} S_2^2(k) - \max_{\log n \le n-k \le n\mu} \left[\frac{1}{\sqrt{n-k}} \sum_{t=1}^{n-k} G_{2,t+n} \right] \right| = o_p(1)\,.$$

Thus,

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$$= P\left(\max_{\log n \le k \le n\mu} \frac{\left|\frac{1}{\sqrt{k}} \sum_{t=1}^{k} G_{1,t}\right|^2 - b_n}{a_n} \le x, \right.$$

$$\left. \max_{\log n \le n-k \le n\mu} \frac{\left|\frac{1}{\sqrt{n-k}} \sum_{t=k+1}^{n} G_{2,t}\right|^2 - b_n}{a_n} \le x \right) + o(1)$$

$$= P\left(\max_{\log n \le k \le n\mu} \frac{\left| \frac{1}{\sqrt{k}} \sum_{t=1}^{k} G_{1t} \right|^2 - b_n}{a_n} \le x \right)$$

$$\times P\left(\max_{\log n \le n-k \le n\mu} \frac{\left| \frac{1}{\sqrt{n-k}} \sum_{t=1}^{n-k} G_{2,t+n} \right|^2 - b_n}{a_n} \le x \right)$$

$$\Rightarrow e^{-2e^{-\frac{x}{2}}} + o(\mu) \longrightarrow e^{-2e^{-\frac{\mu}{2}}}$$

Let $g_n = \log\log\log(\max\{e^e, n\})$

$$\max_{0 \le k \le \log n} [S_1(k)]^2 = \max_{0 \le k \le \log n}{}^{\log\log k} \left[\frac{\log}{\sqrt{k \log\log k}} \sum_{t=1}^{k} Y_{t} \varepsilon_t \right]^2$$

$$= O(\log\log\log n) = O(g_n).$$

$$a_n = 2\log\log n + \log\log\log n / 2 - \log \Gamma(\tfrac{1}{2})$$

$$\max_{0 \le k \le \log n} [S_1(k)]^2 / a_n = o(1).$$

Similarly

$$\max_{0 \le n-k \le \log n} [S_2(k)]^2 / a_n = o(1).$$