Ⓑ

$$y_t = \begin{cases} \phi_{10} y_{t-1} + \varepsilon_t & \text{if } t = 1, \dots, k_0 \\ \phi_{20} y_{t-1} + \varepsilon_t & \text{if } t = k_0+1, \dots, n. \end{cases}$$

$$L_n(\phi_1, \phi_2, k) = \sum_{t=1}^{k} (y_t - \phi_1 y_{t-1})^2 + \sum_{t=k+1}^{n} (y_t - \phi_2 y_{t-1})^2$$

$k = 1, 2, \dots, n-1.$

$$\hat{\phi}_1^{(k)} = \sum_{t=1}^{k} y_t y_{t-1} \Big/ \sum_{t=1}^{k} y_{t-1}^2$$

$$\hat{\phi}_2^{(k)} = \sum_{t=k+1}^{n} y_t y_{t-1} \Big/ \sum_{t=k+1}^{n} y_{t-1}^2$$

$$\hat{k}_n = \arg\min_{k \in \{1, \dots, n-1\}} L_n(\hat{\phi}_1(k), \hat{\phi}_2(k), k)$$

$$\hat{\phi}_1 = \hat{\phi}_1(\hat{k}_n), \quad \hat{\phi}_2 = \hat{\phi}_2(\hat{k}_n).$$

$$L_n(\hat{\phi}_1, \hat{\phi}_2, \hat{k}_n) - L_n(\phi_{10}, \phi_{20}, k_0)$$

$$= \min_{k} \min_{\phi_1, \phi_2} \left[L_n(\phi_1, \phi_2, k) - L_n(\phi_{10}, \phi_{20}, k_0) \right]$$

$$\leq 0.$$

We will prove that

$$\hat{k}_n - k_0 = O_p(1)$$

Assume $\hat{k}_n > k_0$ and consider the case with $k > k_0$

(2)

$$L_n(\phi_1, \phi_2, k) - L_n(\phi_{10}, \phi_{20}, k_0)$$

$$= \sum_{t=1}^{k_0} \left[(y_t - \phi_1 y_{t-1})^2 - \varepsilon_t^2 \right]$$

$$+ \sum_{t=k+1}^{n} \left[(y_t - \phi_2 y_{t-1})^2 - \varepsilon_t^2 \right]$$

$$+ \sum_{t=k_0+1}^{k} \left[(y_t - \phi_1 y_{t-1})^2 - \varepsilon_t^2 \right]$$

$$\equiv S_1(\phi_1, k_0) + S_2(\phi_2, k) + D(\phi, k).$$

$$\min_{\phi_1} S_1(\phi_1, k_0) = -\left(\sum_{t=1}^{k_0} y_{t-1} \varepsilon_t \right)^2 \Big/ \sum_{t=1}^{k_0} y_{t-1}^2$$

$$= -\left(\frac{1}{\sqrt{n}} \sum_{t=1}^{k_0} y_{t-1} \varepsilon_t \right)^2 \Big/ \left[\frac{1}{n} \sum_{t=1}^{k_0} y_{t-1}^2 \right] = O_p(1).$$

$$\equiv A_n.$$

$$\min_{\phi_2} S_2(\phi_2, k) = -\left(\sum_{t=k+1}^{n} y_{t-1} \varepsilon_t \right)^2 \Big/ \left(\sum_{t=k+1}^{n} y_{t-1}^2 \right) \equiv \cancel{B_n(k)}$$

$$B_n \equiv \min_{k \in \{k_0+1, \cdots, n-1\}} \underbrace{\min_{\phi_2} S_2(\phi_2, k)}_{B_n(k)} = O_p(1) \quad (?)$$

$$P(\hat{k}_n - k_0 > M)$$

$$\cancel{= P(\hat{k}_n - k_0 > M,\ L_n(\hat{\phi}_1, \hat{\phi}_2, \hat{k}_n) - L_n(\phi_{10}, \phi_{20}, k_0) \le 0)}$$

$k = \lceil n\tau \rceil$, $\quad k_0 = \lceil n\tau_0 \rceil$, $\quad \hat{k}_n = \lceil n\hat{\tau}_n \rceil$.

Lemma. For $\forall$ given $\tilde{\tau}, \hat{\tau}_1 \in (0,1)$, $\tau_0 \in (\tilde{\tau}, \tilde{\tau}_1)$, it follows that

$$\lim_{n\to\infty} P(\hat{\tau}_n \notin [\tilde{\tau}, \hat{\tau}_1]) = 0.$$

$\exists M$, ~~for $\forall \delta > 0$~~ and $\forall \delta > 0$

$$P(\hat{k}_n - k_0 > M, |\hat{\phi}_1 - \phi_{10}| \geq \delta, \hat{\tau}_n \in [\tilde{\tau}, \tilde{\tau}_1]) < \varepsilon \quad ①$$

as $n \to \infty$. $\exists M$ and $\delta > 0$.

$$P(\hat{k}_n - k_0 > M, |\hat{\phi}_1 - \phi_{10}| < \delta, \hat{\tau}_n \in [\hat{\tau}, \tilde{\tau}_1]) < \varepsilon \quad ②$$

By Lemma, ① + ②, $\exists M$ s.t.

$$P(\hat{k}_n - k_0 > M) < \varepsilon \quad \text{as } n \to \infty.$$

That is,

$$\hat{k}_n - k_0 = O_p(1)$$

$\square$

To 2

$$P\left(\hat{k}_n - k_0 > M,\ |\hat{\phi}_1 - \phi_{10}| < \delta,\ \hat{\tau}_n \in [\hat{\tau}, \hat{\tau}_1]\right)$$

$$= P\left(\begin{array}{l} \hat{k}_n - k_0 > M,\ |\hat{\phi}_1 - \phi_{10}| < \delta,\ \hat{\tau}_n \in [\hat{\tau}, \hat{\tau}_1] \\ \min_{k} \min_{\phi_1, \phi_2} \left[L_n(\phi_1, \phi_2, k) - L_n(\phi_{10}, \phi_{20}, k_0) \right] \le 0 \end{array}\right)$$

$$\le P\left(\begin{array}{l} \min_{\substack{[n\hat{\tau}] \le k \le [n\hat{\tau}_1] \\ k - k_0 > M}} \ \min_{|\phi_1 - \phi_{10}| < \delta} \ \min_{\phi_2} \left[L_n(\phi_1, \phi_2, k) \right. \\ \qquad \left. - L_n(\phi_{10}, \phi_{20}, k) \right] \le 0 \end{array}\right)$$

$$\le P\left(\min_{k - k_0 > M} \ \min_{|\phi_1 - \phi_{10}| < \delta} \left[\frac{M}{k - k_0} D(\phi_1, k) \right] + A_n + B_n \le 0 \right)$$

$$\le P\left(\min_{k - k_0 > M} \ \min_{|\phi_1 - \phi_{10}| < \delta} \left[\frac{1}{k - k_0} D(\phi_1, k) \right] \le \varepsilon \right) + \varepsilon$$

as M is large.

①

Let $X_t(\phi_1) = (y_t - \phi_1 y_{t-1})^2 - \varepsilon_t^2$. Then

$$E X_t(\phi) = (\phi_1 - \phi_{20})^2 E y_t^2 \geq \left(|\phi_1 - \phi_{10}| - |\phi_{10} - \phi_{20}|\right)^2 E y_t^2$$

$$\geq \frac{1}{4} |\phi_{10} - \phi_{20}|^2 E y_t^2 \quad \text{when } \delta = \frac{|\phi_{10} - \phi_{20}|}{2}$$

$$\equiv C. \quad \text{and } |\phi_1 - \phi_{10}| < \delta.$$

$$\min_{|\phi_1 - \phi_{10}| < \delta} \left[\frac{D(\phi_1, k)}{k - k_0} \right]$$

$$= \frac{1}{k - k_0} \min_{|\phi_1 - \phi_{10}| < \delta} \left[\sum_{t=k_0+1}^{k} [X_t(\phi_1) - E X_t(\phi_1)] + (k - k_0) E X_t(\phi_1) \right]$$

$$\geq C - \max_{\phi_1} \left| \sum_{t=k_0+1}^{k} [X_t(\phi_1) - E X_t(\phi_1)] \right| / (k - k_0)$$

Let $k - k_0 = \tilde{k}$. Then, for $\forall \varepsilon > 0$

$$P\left(\max_{k - k_0 > M} \max_{\phi_1} \left[\frac{1}{k - k_0} \left| \sum_{t=k_0+1}^{k} [X_t(\phi_1) - E X_t(\phi_1)] \right| \right] > \varepsilon \right)$$

$$= P\left(\max_{\tilde{k} > M} \left[\frac{1}{\tilde{k}} \max_{\phi_1} \left| \sum_{t=1}^{\tilde{k}} X_t(\phi_1) - E X_t(\phi_1) \right| \right] > \varepsilon \right)$$

③

For each ϕ, we have

$$\frac{1}{\tilde{k}} \left| \sum_{t=1}^{\tilde{k}} [X_t(\phi_1) - E X_t(\phi)] \right| \to 0 \quad a.s.$$

Since $\phi_1 \in [-1, 1]$, by point-wise argument,

We can show that

$$\frac{1}{\tilde{k}} \max_{\phi_1} \left| \sum_{t=1}^{\tilde{k}} [X_t(\phi) - E X_t(\phi_1)] \right| \to 0 \quad a.s.$$

Thus,

$$③ < \varepsilon \quad a.s \quad M \to \infty.$$

$\Longrightarrow$ ② holds. $\square$

To ①,

$$P\left(\hat{k}_n - k_0 > M,\ |\hat{\phi}_1 - \phi_{10}| \geq \delta,\ \hat{\tau}_n \in [\tilde{\tau}, \tilde{\tau}_1]\right)$$

$$= P\left(\begin{array}{l} \hat{k}_n - k_0 > M,\ |\hat{\phi}_1 - \phi_{10}| \geq \delta,\ \hat{\tau}_n \in [\tilde{\tau}, \tilde{\tau}_1] \\ \min_{k, \phi_1, \phi_2} [L_n(\phi_1, \phi_2, k) - L_n(\phi_{10}, \phi_{20}, k_0)] \leq 0 \end{array}\right)$$

$$\leq P\left(\min_{\substack{[n\tilde{\tau}] < k \leq [n\tilde{\tau}_1] \\ k - k_0 > M}} \min_{|\phi_1 - \phi_{10}| > \delta} \min_{\phi_2} [L_n(\phi_1, \phi_2, k) - L_n(\phi_{10}, \phi_{20}, k_0)] \leq 0\right)$$

$$\le P\left(\min_{|\phi_1-\phi_{10}|>\delta} \sum_{t=1}^{k_0} X_t(\phi_1) + \min_{|\phi_1-\phi_{10}|>\delta} \min_{k-k_0>M} \sum_{t=k_0+1}^{k} X_t(\phi_1) + \min_{\substack{k-k_0>M \\ [n\tau_1] \le k \le [n\bar{\tau}_1]}} \min_{\phi_2} \sum_{t=k+1}^{r} X_t(\phi_2) \le 0 \right)$$ (0)

$$\frac{1}{n} \min_{|\phi_1-\phi_{10}|>\delta} \sum_{t=1}^{k_0} X_t(\phi_1) = \frac{1}{n} \min_{|\phi_1-\phi_{10}|>\delta} \left[\sum_{t=1}^{k_0} [X_t(\phi_1) - E X_t(\phi_1)] + k_0 E X_t(\phi_1) \right]$$

$$\ge \frac{k_0}{n} \min_{|\phi_1-\phi_0|} E X_t(\phi_1) - \frac{1}{n} \max_{\Theta} \left| \sum_{t=1}^{k_0} [X_t(\phi_1) - E X_t(\phi_1)] \right|$$

$$\ge \tau_0 c + o_p(1)$$ (4)

$$\frac{1}{n} \min_{|\phi_1-\phi_{10}|>\delta} \min_{k-k_0>M} \sum_{t=k_0+1}^{k} X_t(\phi_1)$$

$$= \frac{1}{n} \min_{|\phi_1-\phi_{10}|>\delta} \min_{k-k_0>M} \left[\sum_{t=k_0+1}^{k} [X_t(\phi_1) - E X_t(\phi_1)] + (k-k_0) E X_t(\phi_1) \right]$$

$$\ge \frac{M}{n} \min_{\Theta_1} E X_t(\phi_1) - \frac{1}{n} \max_{k-k_0>M} \max_{\Theta_1} \left| \sum_{t=k_0+1}^{k} [X_t(\phi_1) - E X_t(\phi_1)] \right|$$

$$= o_p(1).$$

(I)

(8)

$$\frac{1}{n} \min_{\substack{k-k_0 \geq M \\ \lceil n\hat{\tau}\rceil \leq k \leq \lceil n\hat{\tau}_1\rceil}} \min_{\phi_2} \left[\sum_{t=k+1}^{n} X_t(\phi_2) \right]$$

$$= \frac{1}{n} \min_{\substack{k-k_0 \geq M \\ \lceil n\hat{\tau}\rceil \leq k \leq \lceil n\hat{\tau}_1\rceil}} \min_{\phi_2} \left[\sum_{t=k+1}^{n} [X_t(\phi_2) - EX_t(\phi_2)] + (n-k) EX_t(\phi_2) \right]$$

$$\geq \frac{1}{n} \min_{k-k_0 \geq M} (n-k) \min_{\phi_2} EX_t(\phi_2)$$

$$- \frac{1}{n} \max_{\substack{k_0 \leq k \leq \lceil n\hat{\tau}_1\rceil \\ \lceil n\hat{\tau}\rceil}} \max_{\phi_2} \left| \sum_{t=k+1}^{n} [X_t(\phi_2) - EX_t(\phi_2)] \right|$$

$$\geq -\frac{1}{n} \max_{\lceil n\hat{\tau}_0\rceil \leq k \leq \lceil n\hat{\tau}\rceil} \max_{\phi_1} \left| \frac{n - \lceil n\hat{\tau}_1\rceil}{n-k} \sum_{t=k+1}^{\hat{n}} [X_t(\phi_1) - EX_t(\phi_1)] \right|$$

$$\geq -(1-\hat{\tau}_1) \max_{n-k \geq n - \lceil n\hat{\tau}_1\rceil} \frac{1}{n-k} \max_{\phi_1} \left| \sum_{t=k+1}^{\hat{n}} [X_t(\phi_1) - EX_t(\phi_1)] \right|$$

$$= o_p(1) \qquad (6)$$

(3) + (6) + (5), we have

$$P(\hat{k}_n - k_0 > M, |\hat{\phi}_1 - \phi_{10}| \geq \delta, \hat{\tau}_n \in [\hat{\tau}_0, \hat{\tau}_1]) < \varepsilon$$

as $n \to \infty$

$(*)$

Thus, (1) holds.

Proof of Lemma.

Denote $\tilde{k} = \lceil n\tilde{\tau} \rceil$ and $\tilde{k}_1 = \lceil n\tilde{\tau}_1 \rceil$. We only need to prove that

$$P(\hat{k}_n \le \tilde{k}) \to 0 \qquad (7)$$

$$P(\hat{k}_n \ge \tilde{k}_1) \to 0 \qquad (8).$$

To (7),

$$P(\hat{k}_n \le \tilde{k})$$

$$= P\left(\hat{k}_n \le \tilde{k},\ \min_{k} \min_{\phi_1, \phi_2} \left[L_n(\phi_1, \phi_2, k) - L_n(\phi_{10}, \phi_{20}, k_0)\right] \le 0\right)$$

$$\le P\left(\min_{k \le \tilde{k}} \min_{\phi_1, \phi_2} \left[\sum_{t=1}^{k} X_t(\phi_1) + \sum_{t=k+1}^{k_0} X_t(\phi_2) + \sum_{t=k_0+1}^{\hat{n}} X_t(\phi_2)\right] \le 0\right)$$

$$\frac{1}{n} \max_{1 \le k \le \lceil n\tilde{\tau} \rceil} \max_{\phi_1} \left[\sum_{t=1}^{k} X_t(\phi_1)\right]$$

$$\ge \frac{1}{n} \min_{1 \le k \le \lceil n\tilde{\tau} \rceil} \min_{\phi_1} E X_t(\phi_1) - \frac{1}{n} \max_{1 \le k \le \lceil n\tilde{\tau} \rceil} \max_{\theta} \left|\sum_{t=1}^{k} \left[X_t(\phi_1) - E X_t(\phi_1)\right]\right|$$

$$= o_p(1) \qquad (?)$$

(9)

$$\frac{1}{n} \max_{1 \le k \le [n\tilde{\tau}]} \max_{\Theta} \left| \sum_{t=1}^{k} [X_t(\Phi_1) - EX_t(\Phi_1)] \right|$$

$$= \max \left\{ \frac{1}{n} \max_{1 \le k \le \log n} \max_{\Theta} \left| \sum_{t=1}^{k} [X_t(\Phi_1) - EX_t(\Phi_1)] \right|, \right.$$

$$\left. \frac{1}{n} \max_{\log n \le k \le [n\tilde{\tau}]} \max_{\Theta} \left| \sum_{t=1}^{k} [X_t(\Phi_1) - EX_t(\Phi_1)] \right| \right\}$$

$$= o_p(1)$$

<1> $U_\delta = \{ \Phi_2 : |\Phi_2 - \Phi_{20}| < \delta \}$

$$\frac{1}{n} \min_{k \le \hat{k}} \min_{\Phi_2 \in U_2} \sum_{t=k+1}^{k_0} X_t(\Phi_2)$$

$$\ge \frac{1}{n} \min_{k \le \hat{n}} \min_{\Phi_2 \in U_2} \left[\left\{ \sum_{t=k+1}^{k_0} [X_t(\Phi_1) - EX_t(\Phi_1)] \right\} + (k_0 - n) EX_t(\Phi_1) \right]$$

$$\ge \frac{k_0 - \hat{k}}{n} \min_{\Phi_2 \in U_2} EX_t(\Phi_2)$$

$$- \frac{1}{n} \max_{k \le \hat{n}} \max_{\Phi_2} \left| \sum_{t=k+1}^{k_0} [X_t(\Phi_2) - EX_t(\Phi_2)] \right|$$

$$\ge (\tau_0 - \tilde{\tau}) \min_{\Theta_2 \in U_2} EX_t(\Phi_2) + o_p(1)$$

$$\min_{\phi_2 \in U_2} E X_t(\phi_2) = \min_{\phi_2 \in U_2} \left[E(y_t - \phi_2 y_{t-1})^2 - E \varepsilon_t^2 \right]$$

$$= \min_{\phi_2 \in U_2} (\phi_2 - \phi_0)^2 E y_t^2 \geq c > 0 \text{ as } \delta \text{ is small.}$$

Thus, $$\frac{1}{n} \min_{k \leq \tilde{n}} \min_{\phi_2 \in U_2} \sum_{t=k+1}^{k_0} X_t(\phi_2) \geq c + o_p(1).$$

$$\frac{1}{n} \min_{k \leq \tilde{n}} \min_{\phi_2 \in U_2} \sum_{t=k+1}^{n} X_t(\phi_2)$$

$$= \frac{1}{n} \min_{k \leq \tilde{n}} \min_{\phi_2 \in U_2} \left\{ \sum_{t=k_0+1}^{n} \left[X_t(\phi_2) - E X_t(\phi_2) \right] + (n - k_0) E X_t(\phi_2) \right\}$$

$$\geq \min_{\phi_2 \in U_2} E X_t(\phi_2) + \frac{1}{n} \max_{\phi_2} \left| \sum_{t=k_0+1}^{n} \left[X_t(\phi_2) - E X_t(\phi_2) \right] \right|$$

$$= 0 + o_p(1).$$

<2> $\bar{U}_\delta = \{ \phi_2 : |\phi_{20} - \phi_{21}| > \delta \}$

$$\frac{1}{n} \min_{k \le \tilde{n}} \min_{\phi_2 \in \bar{U}_2} \sum_{t=k+1}^{k_0} X_t(\phi_2)$$

$$\ge \frac{1}{n} \min_{k \le \tilde{n}} \min_{\phi_2} \left[E X_t(\phi_2)\right](k_0 - k)$$

$$- \frac{1}{n} \max_{1 \le k \le \tilde{n}} \max_{\phi_2} \left| \sum_{t=k+1}^{k_0} \left[X_t(\phi_2) - E X_t(\phi_2)\right] \right|$$

$$= 0 + o_p(1)$$

$$\frac{1}{n} \min_{k \le \tilde{n}} \min_{\phi_2 \in \bar{U}_0} \left[\sum_{t=k_0+1}^{n} \left[X_t(\phi_2) - E X_t(\phi_2)\right] + (n-k_0) E X_t(\phi_2) \right]$$

$$\ge (1-\tau_0) \min_{\phi_2 \in \bar{U}_0} E X_t(\phi_2) - \frac{1}{n} \max_{k \le \tilde{n}} \max_{\phi_2} \left| \sum_{t=k_0+1}^{n} \left[X_t(\phi_2) - E X_t(\phi_2)\right] \right|$$

$$E X_t(\phi_2) = E(y_t - \phi_2 y_{t-1})^2 - E\varepsilon_t^2 = (\phi_2 - \phi_{20})^2 E y_{t-1}^2$$

$$\min_{\phi_2 \in \bar{U}_0} E X_t(\phi_2) \ge \delta E y_{t-1}^2 \equiv C.$$

$$\ge C + o_p(1).$$

Thus, $P(\hat{k}_n \le \tilde{n}) \le P(o_p(1) + C + o_p(1) \le 0)$

$\longrightarrow 0$ as $n \to \infty$ ⊨

$$\sqrt{k_0}(\hat{\phi}_1 - \phi_{10}) = \sum_{t=1}^{k} y_{t-1}\varepsilon_t \Big/ \sum_{t=1}^{k} y_{t-1}^2$$

$$= \frac{\left(\sum_{t=1}^{k_0} y_{t-1}\varepsilon_t + \sum_{t=k_0+1}^{k_0} y_{t-1}\varepsilon_t\right)\frac{1}{\sqrt{k_0}}}{\frac{1}{k_0}\left(\sum_{t=1}^{k_0} y_{t-1}^2 + \sum_{t=k_0+1}^{k} y_{t-1}^2\right)}$$

$$= \frac{\left(\sum_{t=1}^{k_0} y_{t-1}\varepsilon_t\right)/\sqrt{n}}{E y_{t-1}^2 + o_p(1)} + o_p(1)$$

$$\xrightarrow{d} N\left(0, \sigma^2 (E y_{t-1}^2)^{-1}\right).$$

$$\sqrt{n-k_0}\,(\hat{\phi}_2 - \phi_{20}) = \sqrt{n-k_0}\left(\frac{\sum_{t=k+1}^{n} (y_t - \phi_{20} y_{t-1}) y_{t-1}}{\sum_{t=k_0+1}^{r} y_{t-1}^2}\right.$$

$$= \sqrt{n-k_0}\left[\frac{\sum_{t=k_0+1}^{n} \varepsilon_t y_{t-1} + (\phi_{10} - \phi_{20}) \sum_{t=k+1}^{k_0} y_{t-1}^2}{\sum_{t=k_0+1}^{r} y_{t-1}^2 + \sum_{t=k+1}^{k_0} y_{t-1}^2}\right]$$

$$\longrightarrow N\left(0, \sigma (E y_{t-1}^2)^{-1}\right).$$

(14)

$$L_n(\hat{\phi}_1, \hat{\phi}_2, k)$$

$$= S_{1k_0}(\hat{\phi}_1) + S_{2k_0}(\hat{\phi}_2) + D_1(\hat{\phi}_2, k)$$

$$= A_n + B_n + o_p(1) + D_1(\hat{\phi}_2, k)$$

$$D_1(\hat{\phi}_2, k) = \sum_{t=k+1}^{k_0} \left[(y_t - \hat{\phi}_2 y_{t-1})^2 - \varepsilon_t^2 \right]$$

$$= \sum_{t=k+1}^{k_0} \left\{ \left[\varepsilon_t - (\hat{\phi}_2 - \phi_{10}) y_{t-1} \right]^2 - \varepsilon_t^2 \right\}$$

$$= -\cancel{2(\hat{\phi}_2 - \phi_{10}) \sum_{t=k+1}^{k_0} y_{t-1} \varepsilon_t} + \cancel{(\hat{\phi}_2 - \phi_{10})^2 \sum_{t=k+1}^{k_0}}$$

$$= \sum_{t=k+1}^{k_0} \left[\left[\underbrace{\varepsilon_t + (\phi_{10} - \phi_{20}) y_{t-1}} + (\hat{\phi}_2 - \phi_{20}) y_{t-1} \right]^2 - \varepsilon_t^2 \right.$$

$$= \sum_{t=k+1}^{k_0} \left\{ \left[\varepsilon_t + (\phi_{10} - \phi_{20}) y_{t-1} \right]^2 - \varepsilon_t^2 \right.$$

$$- 2(\hat{\phi}_2 - \phi_{20}) \sum_{t=k+1}^{k_0} y_{t-1} \left[\varepsilon_t + (\phi_{10} - \phi_{20}) y_{t-1} \right]$$

$$+ (\hat{\phi}_2 - \phi_{20}) \sum_{t=k+1}^{k_0} y_{t-1}^2$$

$$= \sum_{t=k+1}^{k_0} \left\{ \left[\varepsilon_t + (\phi_{10} - \phi_{20}) y_{t-1} \right]^2 - \varepsilon_t^2 \right\} + o_p(1)$$

So

$$\hat{k} = \arg\min_{k \in \{1,2,\dots,n\}} \sum_{t=k+1}^{k_0} \left\{ \left[\varepsilon_t + (\phi_{10} - \phi_{20}) y_{t-1} \right]^2 - \varepsilon_t^2 \right\} + o_p(1)$$

Let $k_* - k_0 = J$

$$= \arg\min_{\substack{k \in \{1,2,\dots,n\} \\ J = \{-1,-2,\dots,-k_0\}}} \sum_{t=-J}^{-1} \left\{ \left[\varepsilon_{t+k_0+1} + (\phi_{10} - \phi_{20}) y_{t+k_0} \right]^2 - \varepsilon_{t+k_0+1}^2 \right\}$$

$$\hat{k} - k_0 = \arg\min_{J \in \{1,2,\dots,n-k_0\}} \sum_{t=-J}^{-1} \left\{ \left[\varepsilon_t + (\phi_{10} - \phi_{20}) y_{t-1} \right]^2 - \varepsilon_t^2 \right\} + o_p(1)$$

$$\longrightarrow \arg_{J \in \{1,2,\dots\}} \sum_{t=-J}^{-1} \left\{ \left[\varepsilon_t + (\phi_{10} - \phi_{20}) y_{t-1} \right]^2 - \varepsilon_t^2 \right\}.$$

When $k < k_0$

$$\hat{k} = \arg\min_{k \in \{1,2,\dots\}} \sum_{t=1}^{J} \left\{ \left[\varepsilon_t + (\phi_{20} - \phi_{10}) y_{t-1} \right]^2 - \varepsilon_t^2 \right]$$