①

$$P(Y_{n+1} \le y \mid Y_n = x, Y_{n-1} = x_{n-1}, \cdots, Y_0 = x_0)$$

$$= P(Y_{n+1} \le y \mid Y_n = x)$$

$$\equiv P(x, B), \quad B = (-\infty, y]$$

$$\equiv p(x, y)$$

— stationary transition probability, which does not depend on n.

$$G(y) = \pi(B)$$

$$= \int_{-\infty}^{+\infty} P(x, B)\, d\pi(x) \quad [dG(x)]$$

$$= \int_{-\infty}^{+\infty} p(x, y)\, d\pi(x) \quad [dG(x)]$$

If $p(x, y)$ is derivative in terms of y. (i.e. conditional density $f(y|x)$), then

$$P(x, B) \equiv p(x, y) = \int_{-\infty}^{y} f(y|x)\, dy.$$

In this case,

$$G(y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{y} f(y|x)\, dy\, dG(x).$$

If $G(y)$ has a density $g(y)$, then

$$g(y) = \int_{-\infty}^{+\infty} f(y|x)\, g(x)\, dx. \qquad \square$$

When $Y_n \in \Omega = \{0, 1, 2, \dots N, \}$.

$$\begin{cases} \lambda(i) = \sum_{k=1}^{N} P(i,j)\, \lambda(j). \\ \sum_{k=1}^{N} \lambda(i) = 1 \end{cases}$$

have a unique solution with $\lambda(i) > 0$.

$$P(x, B) = P(x, y) = \int_{-\infty}^{y} f(y|x)\, dy.$$

$$P^2(x, B) = P(Y_{n+2} \in B \mid Y_n = x)$$

$$= E\left[I(Y_{n+2} \in B) \mid Y_n = x \right]$$

$$= E\left\{ E\left[I(Y_{n+2} \in B) \mid Y_{n+1}, Y_n = x \right] \mid Y_n = x \right\}$$

$$= E\left\{ E\left[I(Y_{n+2} \in B) \mid Y_{n+1} \right] \mid Y_n = x \right\}$$

$$= E\left[\underbrace{P(Y_{n+2} \in B \mid Y_{n+1})}_{u(Y_{n+1})} \mid Y_n = x \right]$$

$$= \int_{-\infty}^{+\infty} \underbrace{\left[\int_{-\infty}^{y} f(y|z)\, dy \right]}_{u(z)} f(z|x)\, dz$$

$$= \int_{-\infty}^{y} \underbrace{\int_{-\infty}^{+\infty} f(y|z)\, f(z|x)\, dz}\, dy$$

Assume

$Y_t = \phi Y_{t-1} + \varepsilon_t$, $\varepsilon_t \sim$ iid with a density $f(x)$.

$B = (-\infty, y]$.

$$P(x, B) = P(x, y) = P(Y_{n+1} \le y \mid Y_n = x, Y_{n-1} = x_{n-1}, \dots, Y_0 = x)$$

$$= P(Y_{n+1} \le y \mid Y_n = x)$$

$$= P(\varepsilon_{n+1} \le y - \phi_1 x)$$

$$= \int_{-\infty}^{y-\phi_1 x} f(z)\,dz = \int_{-\infty}^{y} f(y - \phi_1 x)\,dy$$

$P_y(x, y) = f(y - \phi_1 x)$ — conditional density

$$P^2(x, B) = P(Y_{n+2} \le y \mid Y_n = x)$$

$$= P(\phi^2 x + \phi \varepsilon_{n+1} + \varepsilon_{n+2} \le y)$$

$$= \int_{-\infty}^{+\infty} P(\phi^2 x + \phi z + \varepsilon_{n+2} \le y) f(z)\,dz$$

$$= \int_{-\infty}^{+\infty} \left\{ \int_{-\infty}^{y - \phi^2 x - \phi z} f(z_1)\,dz_1 \right] f(z)\,dz$$

$$= \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{y} f(z_1 - \phi^2 x - \phi z)\,dz_1 \right] f(z)\,dz$$

$$= \int_{-\infty}^{y} \left\{ \int_{-\infty}^{+\infty} f(z_1 - \phi^2 x - \phi z) f(z)\,dz \right] dz$$

GARCH: $\begin{cases} \varepsilon_t = \eta_t \sqrt{h_{t-1}}, \\ h_t = \alpha_0 + \alpha \varepsilon_{t-1}^2 + \beta h_{t-1}. \end{cases}$

ε_t is not M.C.

$$h_t = \alpha_0 + \alpha \eta_{t-1}^2 h_{t-1} + \beta h_{t-1} = \alpha_0 + (\alpha \eta_{t-1}^2 + \beta) h_{t-1}$$

is a M.C.

Assume $\eta_t^2 \sim$ i.i.d. with density $f(x)$.

$B = (-\infty, y]$, $\mathcal{F}_n = \sigma(h_n, h_{n-1}, \cdots)$.

$$P(h_{n+1} \in B \mid \mathcal{F}_n) = P(h_{n+1} \in B \mid h_n, h_{n-1}, \cdots h_0)$$

$$= P(h_{n+1} \in B \mid h_n).$$

$$P(h_{n+1} \le y \mid h_n = x, h_{n-1} = x_{n-1}, \cdots h_0 = x_0)$$

$$= P(h_{n+1} \le y \mid h_n = x)$$

$$= P(\alpha_0 + (\alpha \eta_n^2 + \beta) x \le y)$$

$$= \int_{-\infty}^{[y - \alpha_0 - \beta x]/\alpha} f(z)\, dz$$

$$= \int_{-\infty}^{y} \frac{1}{\alpha} f\left(\frac{u - \alpha_0 - \beta x}{\alpha}\right) du$$

$$= P(x, y)$$

$$P'(x, y) = \frac{1}{\alpha} f\left(\frac{y - \alpha_0 - \beta x}{\alpha}\right).$$

$$P^2(x, B) = P(h_{n+2} \le y \mid h_n = x)$$

$$= P\{\alpha_0 + (\alpha\eta_{n+1}^2 + \beta)\alpha_0 + (\alpha\eta_{n+1}^2 + \beta)(\alpha\eta_n^2 + \beta)x \le y\}$$

$$= \int_0^\infty P\{\alpha_0 + (\alpha z^2 + \beta)\alpha_0 + (\alpha z^2 + \beta)(\alpha\eta_n^2 + \beta)x \le y\} f(z)\,dz$$

$$= \int_0^\infty \int_{\alpha_0 + (\alpha z^2 + \beta)\alpha_0 + (\alpha z^2 + \beta)(\alpha z_1^2 + \beta)x \le y} f(z) f(z_1)\,dz\,dz_1 .$$

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or.

$$P^2(x, B) = \int_{-\infty}^{y} \int_{-\infty}^{\infty} f(y \mid z) f(z \mid x)\,dz\,dy$$

$$= \int_0^y \int_0^\infty \frac{1}{\alpha} f\left(\frac{y - \alpha_0 - \beta z}{\alpha}\right) \frac{1}{\alpha} f\left(\frac{z - \alpha_0 - \beta x}{\alpha}\right) dz\,dy$$