If we use A'_n to denote the last event then we have

(c) $$\{\omega : \pi\omega \in A_n \Delta A\} = \{\omega : \omega \in A'_n \Delta A\}$$

Combining (b) and (c) gives

(d) $$P(A_n \Delta A) = P(A'_n \Delta A)$$

Now $A - C \subset (A - B) \cup (B - C)$ and with a similar inequality for $C - A$ implies $A\Delta C \subset (A\Delta B) \cup (B \Delta C)$. The last inequality, (d), and (a) imply

$$P(A_n \Delta A'_n) \le P(A_n \Delta A) + P(A \Delta A'_n) \to 0$$

The last result implies

$$\begin{aligned} 0 &\le P(A_n) - P(A_n \cap A'_n) \\ &\le P(A_n \cup A'_n) - P(A_n \cap A'_n) = P(A_n \Delta A'_n) \to 0 \end{aligned}$$

so $P(A_n \cap A'_n) \to P(A)$. But A_n and A'_n are independent, so

$$P(A_n \cap A'_n) = P(A_n)P(A'_n) \to P(A)^2$$

(Recall $P(A'_n) = P(A_n)$.) This shows $P(A) = P(A)^2$, and proves (1.1). □

A typical application of (1.1) is

(1.2) Theorem. For a random walk on $\mathbf{R}$, there are only four possibilities, one of which has probability one.

(i) $S_n = 0$ for all n.
(ii) $S_n \to \infty$.
(iii) $S_n \to -\infty$.
(iv) $-\infty = \liminf S_n < \limsup S_n = \infty$.

$\{\omega : S_n(\omega) \in B \text{ i.o.}\} \in \mathcal{E}$
$\{\omega : \limsup S_n(\omega) = c\} \in \mathcal{E}$

Proof (1.1) implies $\limsup S_n$ is a constant $c \in [-\infty, \infty]$. Let $S'_n = S_{n+1} - X_1$. Since S'_n has the same distribution as S_n it follows that $c = c - X_1$. If c is finite, subtracting c from both sides we conclude $X_1 \equiv 0$ and (i) occurs. Turning the last statement around we see that if $X_1 \not\equiv 0$ then $c = -\infty$ or ∞. The same analysis applies to the liminf. Discarding the impossible combination $\limsup S_n = -\infty$ and $\liminf S_n = +\infty$ we have proved the result. □

EXERCISE 1.1. **Symmetric random walk.** Let $X_1, X_2, \ldots \in \mathbf{R}$ be i.i.d. with a distribution that is symmetric about 0 and nondegenerate (i.e., $P(X_i = 0) < 1$). Show that we are in case (iv) of (1.2).

Lévy's ~~0-1~~ law:

if $\mathcal{F}_n \uparrow \mathcal{F}_\infty$ and $A \in \mathcal{F}_\infty$, then

$$E(I_A \mid \mathcal{F}_n) \longrightarrow I_A \text{ a.s.}$$

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if $A \in \mathcal{F}$ and $T^{-1}A = A \Rightarrow T^n A = A$.

$A = \{(w_0, w_1, \ldots) \in A\} = \{(Y_0, Y_1, \ldots)(\omega) : \omega \in A\}$

$= \{(w_n, w_{n+1}, \ldots) \in A\} = \{T^n w : w \in A\}$

Let $\mathcal{F}_n = \sigma(X_0, X_1, \ldots X_n)$.

$E_\lambda(I_A \mid \mathcal{F}_n) = E[I_{(T^n A)} \mid \mathcal{F}_n]$

$= E_\lambda[I_{\{(X_n, X_{n+1}, \ldots)(\omega) : \omega \in A\}} \mid \mathcal{F}_n]$

$= E_\lambda[I_A \mid X_n] = h(X_n) \longrightarrow I_A$ a.s.

$h(x) = E_\lambda[I_A \mid X_n = x]$

Assume $X_n \in S = \{0, 1, 2, \ldots\}$.

For a sequence of $\{X_n\}$ s.t. $X_n = x_n$

$h(x_n) \longrightarrow I_A = \begin{cases} 0 \\ 1 \end{cases}$ as $n \to \infty$

$h(x_n) = h(0)$ i.o. $\Rightarrow h(0) = 0$ or 1.

[recurrent]

If $h(x_n) \to 0$, $\exists$ $h(x_n) = h(1)$ i.o. $\Rightarrow h(1) = 0$

$h(x_n) = h(i)$ i.o. $\quad h(i) = 0$

if $h(x) \to 1$, $\exists$ $h(x_n) = h(i)$ i.o. $\Rightarrow h(i) = 1$

so $h(X_n) \equiv 0$ or 1. $\Rightarrow E_\lambda I_A = 0$ or 1.