Assume that $\mathcal{B}$ is transient. Then

$$\sum_{m=1}^{\infty} \tilde{P}_{ij}^{m} \le \sum_{m=1}^{\infty} P_{ij}^{m} < \infty , \quad j > 0 .$$

Thus, $\lim_{m\to\infty} \hat{P}_{ij}^{m} = 0 \quad , \ j > 0 .$ $\quad \tilde{D}_{i0}^{m} \to \tilde{\lambda}_i(C_0)\tilde{\lambda}_0$

By (*), we have $\quad$ By Theorem 3.1 on P48

$$\tilde{\lambda}_i(C_0) y_0 + \min_{r \ge M}\{y_i\}\,(1 - \hat{\lambda}_i(C_0)) \le y_i ,$$

$$\Rightarrow 1 - \hat{\lambda}_i(C_0) \le \frac{1}{\min_{r\ge M}\{y_r\}} \{y_i - \hat{\lambda}_i(C_0) y_0\} \le \varepsilon K ,$$

where $K = y_i - \hat{\lambda}_i(C_0) y_0$. Let $\varepsilon \to 0$. We have

$$\tilde{\lambda}_i(C_0) = 1 \text{ for all } i .$$

But $\hat{\lambda}_i(C_0) = \sum_{n=1}^{\infty} \hat{\lambda}_i^{n}(C_0) = \sum_{n=1}^{\infty} \hat{f}_{i0}^{n} = \sum_{n=1}^{\infty} f_{i0}^{n} < 1 .$ □

$\underline{\text{as } i > 0}$

on $\mathcal{B}_0$

$\{X_n\}$ — MC $(R, \mathcal{B}, P)$

$n = 0, 1, 2, \ldots$

$X_n \in R$

$$P(X_{n+m} \in A \mid X_m = x) = P(X_n \in A \mid X_0 = x)$$

$$= P^n(x, A)$$ time-homogeneity

$$P(X_{n+m} \in A \mid X_j, j \le m, X_m = x)$$
$$= P(X_{n+m} \in A \mid X_m = x)$$
Markov property

① $\exists$ a probability measure π on $(R, \mathcal{B})$ s.t.

for $\forall A \in \mathcal{B}$,

$$\pi(A) = \int_R P(x, A)\, \pi(dx)$$

② $$P^n(x, A) \longrightarrow \pi(A) \quad \text{as } n \to \infty$$

for $\forall x \in R$, and $A \in \mathcal{B}$.

The hitting time from a point to a set.

$$\tau_A = \inf\{n : X_n \in A\}. \quad \left[\ T_j = \inf\{n \geq 1, X_n = j\} \ \text{P35} \right]$$

φ-irreducibility:

$\exists$ a measure φ on $(R, \mathcal{B})$, s.t.

$\varphi(A) > 0$ and for $\forall x \in R$, . . . $(*)$

$$L(x, A) \equiv P_x(\tau_A < \infty) = \sum_{n=1}^{\infty} P_x(\tau_A = n) > 0,$$

where $P_x(A) = P(x, A)$. We say that A is accessible. If, for $\forall A \in \mathcal{B}$, $(*)$ holds, then

$\{X_n\}$ is φ-irreducible. $\square$

Recurrence:

A set A is called recurrent if

$$\sum_{n=1}^{\infty} P_x^n(x, A) = E_x\left[\sum_{n=1}^{\infty} I(X_n \in A)\right] = \infty \quad \text{for all } x \in A.$$

$$\left[\ \eta_A = \sum_{n=1}^{\infty} I(X_n \in A)\ \right]$$

$$Q(x, A) \equiv P_x(\eta_A = \infty) = 1 \quad \text{for all } x \in A$$

[Harris recurrent]

$\{X_n\}$ is called Harris recurrent if it is φ-irreducible and every set $A \in \mathcal{B}$ is a Harris recurrent when $\varphi(A) > 0$.

Theorem. If $\{X_n\}$ is recurrent and φ-irreducible, then $\exists$ a invariant measure λ s.t.

$$\lambda(A) = \int_{R} P(x, A)\, \lambda(dx) \qquad (**)$$

If $\lambda(R) < \infty$, then $\exists$ a invariant probability measure λ s.t. $(**)$ holds.

Furthermore, if $\{X_n\}$ is aperiodic, and there exists some small set C, some $b < \infty$, and a non-negative function V finite at some one $x_0 \in R$, satisfying

$$\Delta V(x) \equiv E\left(V(X_n) \,\middle|\, X_{n-1} = x\right) - V(x)$$
$$\leq -1 + b\, I_C(x), \quad x \in R,$$

then λ is unique probability measure s.t

$$\sup_{A \in \mathcal{B}} \left| P^n(x, A) - \lambda(A) \right| \to 0 \quad \text{as } n \to \infty$$

for $\forall x \in R$.

Small sets:

A set $C \in \mathcal{B}$ is called a small set if $\exists$ an $m > 0$, and a non-trivial measure ν_m ~~$\mathcal{B}$~~ on $(\mathbb{R}, \mathcal{B})$ s.t. for $\forall x \in C$, $B \in \mathcal{B}$,

$$P^m(x, B) \geq \nu_m(B).$$

If $\{X_n\}$ is φ-irreducible, a small set C exists.

There are some other condition to check this.

Example $\quad X_n = X_{n-1} a + \varepsilon_n$, $\quad \varepsilon_n \sim$ iid $\sim N(0,1)$ or $f(x)$

$|a| < 1$.

Let $C = [0, 1]$.

$$\nu_m(B) = \int_B f(x)\,dx \qquad \text{(}f(x)\text{ above)}$$

ν_m is a measure on $(\mathbb{R}, \mathcal{B})$.

$$P(x, B) = P(X_n \in B \mid X_{n-1} = x)$$
$$= P(ax + \varepsilon_n \in B) = \int_{ax+y \in B} f(y)\,dy$$

When $B = (-\infty, y)$, $P(x, B) \geq \int_{-b}^{y-|a|} f(y)\,dy = \int_{-\infty}^{y} f(z+|a|)\,dz = \nu_m(B)$.

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