3.2 ARMA(1, 1) Models

Example 3.2(continuous). Look for the minimizer of

$$S_n(\theta) = \sum_{t=1}^{n} [\varepsilon_t(\theta)]^2,$$

i.e., LSE of θ_0.

Recall this conditional LSE or MLE, given (Y_0, ε_0).

$$\varepsilon_1(\theta) = Y_1 - \phi Y_0 + \psi \varepsilon_0.$$

$$\begin{aligned} \varepsilon_t(\theta) &= Y_t - \phi Y_{t-1} + \psi \varepsilon_{t-1}(\theta) \\ &= \sum_{i=0}^{t-1} \psi^i (Y_{t-i} - \phi Y_{t-i-1}) + \psi^t \varepsilon_0. \end{aligned}$$

$\{\varepsilon_t(\theta)\}$ is not stationary !

Denote

$$\begin{aligned} \tilde{\varepsilon}_t(\theta) &= Y_t - \phi Y_{t-1} + \psi \varepsilon_{t-1}(\theta) \\ &= \sum_{i=0}^{\infty} \psi^i (Y_{t-i} - \phi Y_{t-i-1}). \end{aligned}$$

Then $\{\tilde{\varepsilon}_t(\theta)\}$ is stationary and ergodic, and

$$\tilde{\varepsilon}_t(\theta) = Y_t - \phi Y_{t-1} + \psi \tilde{\varepsilon}_{t-1}(\theta).$$

We study the minimizer of the following objective function

$$\tilde{S}_n(\theta) = \sum_{t=1}^{n} \tilde{\varepsilon}_t^2(\theta),$$

where $\theta = (\phi, \psi)'$ and

$$\tilde{\varepsilon}_t(\theta) = Y_t - \phi Y_{t-1} + \psi \tilde{\varepsilon}_{t-1}(\theta).$$

Assume that the parameter space is

$$\Theta = \{(\phi, \psi)' : |\phi| \le c, \ \ |\psi| \le c \quad \textbf{and} \quad \phi \ne \psi\}$$

for some $0 < c < 1$**.**

Denote

$$\tilde{\theta}_n = arg \min_{\theta \in \Theta} \tilde{S}_n(\theta).$$

Note that $\sup_{\theta \in \Theta} |\tilde{\varepsilon}_t(\theta)| \le c_0 \sum_{i=0}^{\infty} \rho^i |Y_{t-i}|$, where $\rho \in (0, 1)$.

$$E \sup_{\theta \in \Theta} |\tilde{\varepsilon}_t(\theta)|^2 \le c_0 \left[\sum_{i=0}^{\infty} \rho^i (E|Y_t|^2)^{1/2} \right]^2 < \infty.$$

Secondly, use the following expansion

$$\begin{aligned} \tilde{\varepsilon}_t(\theta) &= \varepsilon_t + \phi_0 Y_{t-1} - \psi_0 \varepsilon_{t-1} - \phi Y_{t-1} + \psi \tilde{\varepsilon}_{t-1}(\theta) \\ &= \varepsilon_t - (\phi - \phi_0) Y_{t-1} + [\psi \tilde{\varepsilon}_{t-1}(\theta) - \psi_0 \varepsilon_{t-1}], \end{aligned}$$

it follows that

$$
\begin{aligned}
E\tilde{\varepsilon}_t^2(\theta) &= \sigma^2 + E\{(\phi-\phi_0)Y_{t-1} - [\psi\tilde{\varepsilon}_{t-1}(\theta) - \psi_0\varepsilon_{t-1}]\}^2 \\
&\geq \sigma^2
\end{aligned}
$$

and "=" holds if and only if

$$(\phi-\phi_0)Y_{t-1} - \psi\tilde{\varepsilon}_{t-1}(\theta) + \psi_0\varepsilon_{t-1} = 0,$$

that is,

$$
\begin{aligned}
&(\phi-\phi_0-\psi+\psi_0)\varepsilon_{t-1} \\
&\quad +(\phi-\phi_0-\psi)(\phi_0 Y_{t-2} - \psi_0\varepsilon_{t-2}) \\
&\quad +\psi\phi Y_{t-2} - \psi^2\tilde{\varepsilon}_{t-2}(\theta) = 0.
\end{aligned}
$$

Since ε_{t-1} is independent of F_{t-2}, we have

$$
\begin{aligned}
&\phi-\phi_0-\psi+\psi_0 = 0 \quad \text{and} \\
&-\psi_0[\phi_0 Y_{t-2} - \psi_0\varepsilon_{t-2}] + \psi\phi Y_{t-2} - \psi^2\tilde{\varepsilon}_{t-2}(\theta) = 0,
\end{aligned}
$$

which implies that

$$(\phi\psi-\phi_0\psi_0)Y_{t-2} + \psi_0^2\varepsilon_{t-2} - \psi^2\tilde{\varepsilon}_{t-2}(\theta) = 0.$$

that is,

$$(\phi\psi-\phi_0\psi_0+\psi_0^2-\psi^2)\varepsilon_{t-2}^2 + g_{t-3}(\theta) = 0,$$

where $g_{t-3}(\theta)$ is F_{t-3}-measurable. Thus,

$$
\begin{aligned}
&\phi\psi-\phi_0\psi_0+\psi_0^2-\psi^2 = 0 \\
&\quad \Rightarrow \quad \psi(\phi-\psi) - \psi_0(\phi_0-\psi_0) = 0.
\end{aligned}
$$

Since $\phi - \psi = \phi_0 - \psi_0$, it follows that

$$(\psi - \psi_0)(\phi_0 - \psi_0) = 0.$$

Since $\phi_0 \neq \psi_0$, we have $\psi = \psi_0$ and then $\phi = \phi_0$. Thus, Assumption 3.1 holds.

By Theorem 3.1, we have

$$\tilde{\theta}_n \to \theta_0, \ a.s..$$

Initial value problem

Recall

$$\widehat{\theta}_n = arg \min_{\theta \in \Theta} S_n(\theta).$$

For any $\epsilon > 0$ and $\eta > 0$,

$$
\begin{aligned}
& P(|\widehat{\theta}_n - \theta_0| > \epsilon) \\
&= P(|\widehat{\theta}_n - \theta_0| > \epsilon, \frac{1}{n}[S_n(\widehat{\theta}_n) - S_n(\theta_0))] \le 0) \\
&\le P(\min_{|\theta-\theta_0|>\epsilon} \frac{1}{n}[S_n(\theta) - S_n(\theta_0))] \le 0) \\
&\le P(\min_{|\theta-\theta_0|>\epsilon} \frac{1}{n}[\tilde{S}_n(\theta) - \tilde{S}_n(\theta_0))] - \\
&\qquad -2\sup_{\theta\Theta} \frac{1}{n}|\tilde{S}_n(\theta) - S_n(\theta))] \le 0) \\
&\le P\Big(\min_{|\theta-\theta_0|>\epsilon} [E\tilde{\varepsilon}_t^2(\theta) - \tilde{\varepsilon}_t^2(\theta_0))] \\
&\qquad -2\sup_{\theta\Theta} |\frac{1}{n}\tilde{S}_n(\theta) - E\tilde{\varepsilon}_t^2(\theta))| - \\
&\qquad -2\sup_{\theta\Theta} \frac{1}{n}|\tilde{S}_n(\theta) - S_n(\theta))| \le 0\Big) \\
&\le \eta.
\end{aligned}
$$

Exercises:

$$(a). \qquad \sup_{\theta\Theta} \frac{1}{n}|\tilde{S}_n(\theta) - S_n(\theta))| = o_p(1).$$

$$(b). \qquad \sup_{\theta\Theta} |\frac{1}{n}\tilde{S}_n(\theta) - E\tilde{\varepsilon}_t^2(\theta))| = o_p(1).$$

We now consider Assumption 3.2.

$$\begin{aligned}
D_t(\theta) &= 2\frac{\partial \tilde{\varepsilon}_t(\theta)}{\partial \theta}\tilde{\varepsilon}_t(\theta); \\
\Omega &= E[D_t(\theta_0)D_t'(\theta_0)] = 4\sigma^2 E\left[\frac{\partial \tilde{\varepsilon}_t(\theta_0)}{\partial \theta}\frac{\partial \tilde{\varepsilon}_t(\theta_0)}{\partial \theta'}\right]; \\
P_t(\theta) &= 2\frac{\partial \tilde{\varepsilon}_t(\theta)}{\partial \theta}\frac{\partial \tilde{\varepsilon}_t(\theta)}{\partial \theta'} + 2\frac{\partial^2 \tilde{\varepsilon}_t(\theta)}{\partial \theta \partial \theta'}\tilde{\varepsilon}_t(\theta); \\
\Sigma &= EP_t(\theta_0) = 2E\left[\frac{\partial \tilde{\varepsilon}_t(\theta_0)}{\partial \theta}\frac{\partial \tilde{\varepsilon}_t(\theta_0)}{\partial \theta'}\right].
\end{aligned}$$

Then,

$$\Sigma^{-1}\Omega\Sigma^{-1} = \sigma^2\left\{E\left[\frac{\partial \tilde{\varepsilon}_t(\theta_0)}{\partial \theta}\frac{\partial \tilde{\varepsilon}_t(\theta_0)}{\partial \theta'}\right]\right\}^{-1}.$$

The first derivative of $\tilde{\varepsilon}_t(\theta)$ with respect to θ is

$$\begin{aligned}
\frac{\partial \tilde{\varepsilon}_t(\theta)}{\partial \phi} &= -Y_{t-1} + \psi\frac{\partial \tilde{\varepsilon}_{t-1}(\theta)}{\partial \phi} = -\sum_{i=0}^{\infty}\psi^i Y_{t-i-1}; \\
\frac{\partial \tilde{\varepsilon}_t(\theta)}{\partial \psi} &= \tilde{\varepsilon}_{t-1}(\theta) + \psi\frac{\partial \tilde{\varepsilon}_{t-1}(\theta)}{\partial \psi} = \sum_{i=0}^{\infty}\psi^i \tilde{\varepsilon}_{t-i-1}(\theta).
\end{aligned}$$

Thus, there exist constant c and $\rho \in (0,1)$ such that

$$\sup_{\Theta}\left|\frac{\partial \tilde{\varepsilon}_t(\theta)}{\partial \phi}\right| \le c\xi_{\rho t},$$

$$\sup_{\Theta}\left|\frac{\partial \tilde{\varepsilon}_t(\theta)}{\partial \psi}\right| \le c\xi_{\rho t} \quad \text{and} \quad \sup_{\Theta}\left|\frac{\partial^2 \tilde{\varepsilon}_t(\theta)}{\partial \theta_i \partial \theta_j}\right| \le c\xi_{\rho t},$$

where $\theta_i = \phi$ or ψ and $\rho \in (0,1)$. Thus,

$$\sup_{\Theta} \|P_t(\theta)\| \leq c\xi_{\rho t}^2.$$

Since $E\xi_{\rho t}^2 < \infty$, Assumption 2.2 holds.

If Ω is not positive definite, then, there exists a non-zero constant vector $(c_1, c_2)'$ such that

$$(c_1, c_2)\Omega(c_1, c_2)' = 0.$$

Without loss of generality, assume that $c_1 = 1$ and $c_2 = c$, we have

$$E\left\{\frac{\partial\tilde{\varepsilon}_t(\theta_0)}{\partial\phi} + c\frac{\partial\tilde{\varepsilon}_t(\theta_0)}{\partial\psi}\right\}^2 = 0.$$

Thus,

$$\frac{\partial\tilde{\varepsilon}_t(\theta_0)}{\partial\phi} + c\frac{\partial\tilde{\varepsilon}_t(\theta_0)}{\partial\psi} = 0,$$

that is,

$$\sum_{i=0}^{\infty} \psi^i(-Y_{t-i-1} + c\varepsilon_{t-i-1}) = 0,$$

which implies that

$$\begin{aligned}
&Y_{t-1} - c\varepsilon_{t-1} = \sum_{i=1}^{\infty} \psi^i(-Y_{t-i-1} + c\varepsilon_{t-i-1}) \in F_{t-2} \\
&\Rightarrow (1-c)\varepsilon_{t-1} + \phi_0 Y_{t-2} - \psi_0\varepsilon_{t-2} \\
&= \sum_{i=1}^{\infty} \psi_0^i(-Y_{t-i-1} + c\varepsilon_{t-i-1}).
\end{aligned}$$

If $c \neq 1$, other terms are F_{t-2}-measurable. Thus, $\varepsilon_{t-1} \in F_{t-2}$ is a contradiction. If $c = 1$, then we have

$$\begin{aligned}
&\phi_0 Y_{t-2} - \psi_0 \varepsilon_{t-2} = \sum_{i=1}^{\infty} \psi_0^i (-Y_{t-i-1} + \varepsilon_{t-i-1}), \\
&\Rightarrow (\phi_0 + \psi_0) Y_{t-2} - 2\psi_0 \varepsilon_{t-2} \\
&= \sum_{i=2}^{\infty} \psi_0^i (-Y_{t-i-1} + \varepsilon_{t-i-1}), \\
&\Rightarrow (\phi_0 - \psi_0)\varepsilon_{t-2} + (\phi_0 + \psi_0)(\phi_0 Y_{t-3} - \psi_0 \varepsilon_{t-3}) \\
&\quad \in F_{t-3}.
\end{aligned}$$

Since $\phi_0 \neq \psi_0$, $\varepsilon_{t-2} \in F_{t-3}$ is a contradiction. Thus, $\Omega > 0$.

Initial value problem (continuous)

$$\sum_{t=1}^{n} \frac{\partial \varepsilon_t^2(\widehat{\theta}_n)}{\partial \theta} = 0.$$

By Taylor's expansion,

$$\sum_{t=1}^{n} \frac{\partial \varepsilon_t^2(\widehat{\theta}_n)}{\partial \theta} - \sum_{t=1}^{n} \frac{\partial \varepsilon_t^2(\theta_0)}{\partial \theta} = \sum_{t=1}^{n} \frac{\partial^2 \varepsilon_t^2(\xi_n)}{\partial \theta \partial \theta'}(\widehat{\theta}_n - \theta_0).$$

$$\sqrt{n}(\widehat{\theta}_n - \theta_0) = -\left[\frac{1}{n}\sum_{t=1}^{n} \frac{\partial^2 \varepsilon_t^2(\xi_n)}{\partial \theta \partial \theta'}\right]^{-1} \frac{1}{\sqrt{n}}\sum_{t=1}^{n} \frac{\partial \varepsilon_t^2(\theta_0)}{\partial \theta}.$$

Exercises:

$$(a). \qquad \frac{1}{\sqrt{n}}\sum_{t=1}^{n} \frac{\partial \varepsilon_t^2(\theta_0)}{\partial \theta} - \frac{1}{\sqrt{n}}\sum_{t=1}^{n} D_t(\theta_0) = o_p(1).$$

$$(b). \qquad \sup_{\theta \Theta} \left|\frac{1}{n}\sum_{t=1}^{n} \frac{\partial^2 \varepsilon_t^2(\theta)}{\partial \theta \partial \theta'} - \frac{1}{n}\sum_{t=1}^{n} P_t(\theta)\right| = o_p(1).$$

Thus,

$$\sqrt{n}(\widehat{\theta}_n - \theta_0) \to_{\mathcal{L}} N(0, \Sigma^{-1}\Omega\Sigma^{-1}).$$