3.3 GARCH(1, 1) Models

Example 3.3(continuous). Look for the maximizer of

$$L_n(\theta) \;=\; -\frac{1}{2}\sum_{t=1}^{n}\left[\log h_t(\theta) + \frac{\varepsilon_{t-1}^2}{h_t(\theta)}\right].$$

i.e., MLE of θ_0.

Recall this conditional LSE or MLE, given (h_0, ε_0).

$$h_1(\theta) = \omega + \alpha_1\varepsilon_0^2 + \beta_1 h_0.$$

$$\begin{aligned} h_t(\theta) &= \omega + \alpha_1\varepsilon_{t-1}^2 + \beta_1 h_{t-1}(\theta). \\ &= \sum_{i=0}^{t-1}\beta_1^i(\omega + \alpha_1\varepsilon_{t-i}^2) + \beta^t h_0. \end{aligned}$$

$\{h_t(\theta)\}$ is not stationary.

Denote

$$\tilde{h}_t(\theta) = \sum_{i=0}^{\infty}\beta_1^i(\omega + \alpha_1\varepsilon_{t-i}^2).$$

Then $\tilde{h}_t(\theta_0) = h_t$, $\{\tilde{h}_t(\theta)\}$ is stationary and ergodic, and

$$\tilde{h}_t(\theta) \;=\; \omega + \alpha_1\varepsilon_{t-1}^2 + \beta_1\tilde{h}_{t-1}(\theta).$$

We study the maximizer of the following objective function

$$\tilde{L}_n(\theta) \;=\; -\frac{1}{2}\sum_{t=1}^{n}\left[\log \tilde{h}_t(\theta) + \frac{\varepsilon_{t-1}^2}{\tilde{h}_t(\theta)}\right].$$

Assumption:

(a) $\Theta = \{\theta : E \ln(\beta+\alpha\eta_t^2) < 0, \alpha_0 \geq c, \alpha \geq c$ **and** $\beta \geq c\}$.

(b). η_t **has a bounded density in some neighborhood of 0.**

When using MLE to estimate θ_0, we take

$$l_t(\theta) = -\log \tilde{h}_t(\theta) - \frac{\varepsilon_t^2}{\tilde{h}_t(\theta)}.$$

$$\tilde{\theta}_n = arg \max_{\Theta} \sum_{t=1}^{n} l_t(\theta).$$

We now consider Assumption 2.1.

$$\begin{aligned}
El_t(\theta) &= -E\log \tilde{h}_t(\theta) - E\frac{\tilde{h}_t(\theta_0)}{\tilde{h}_t(\theta)} \\
&= -\left[-E\log\frac{\tilde{h}_t(\theta_0)}{\tilde{h}_t(\theta)} + E\frac{\tilde{h}_t(\theta_0)}{\tilde{h}_t(\theta)}\right] - \log \tilde{h}_t(\theta_0) \\
&= -\left[-E\log M_t + EM_t\right] - \log \tilde{h}_t(\theta_0),
\end{aligned}$$

where $M_t = \tilde{h}_t(\theta_0)/\tilde{h}_t(\theta)$.

Note that, for any $x > 0$, $f(x) \equiv -\log x + x \geq 1$ and hence

$$-E \log M_t + EM_t \geq 1.$$

When $M_t = 1$, we have $f(M_t) = f(1) = 1$. If $M_t \neq 1$, then

$$f(M_t) > f(1).$$

Thus, $Ef(M_t) \geq Ef(1)$ with equality only if $M_t = 1$ with probability 1.

Thus, $El_t(\theta)$ reaches its maximum $-1 - \log \tilde{h}_t(\theta_0)$ and this occurs if and only if

$$\tilde{h}_t(\theta) = \tilde{h}_t(\theta_0) = h_t.$$

Thus,

$$\begin{aligned} \omega + \alpha\varepsilon_{t-1}^2 + \beta\tilde{h}_{t-1}(\theta) &= \omega_0 + \alpha_0\varepsilon_{t-1}^2 + \beta_0 h_{t-1} \\ \Rightarrow (\alpha - \alpha_0)\varepsilon_{t-1}^2 &= (\omega_0 - \omega) - \beta\tilde{h}_{t-1}(\theta) + \beta_0 h_{t-1} \\ \Rightarrow (\alpha - \alpha_0)\eta_{t-1}^2 &= \left[(\omega_0 - \omega) - \beta\frac{\tilde{h}_{t-1}(\theta)}{h_{t-1}} + \beta_0\right]. \end{aligned}$$

Since the left-hand-side is F_{t-2}-measurable, we have $\alpha = \alpha_0$. Thus,

$$
\begin{aligned}
&\omega + \beta\tilde{h}_{t-1}(\theta) = \omega_0 + \beta_0 h_{t-1} \\
&\Rightarrow \omega + \beta\omega + \alpha_0\beta\varepsilon_{t-2}^2 + \beta^2\tilde{h}_{t-2}(\theta) \\
&\quad = \omega_0 + \beta_0\omega_0 + \alpha_0\beta_0\varepsilon_{t-2}^2 + \beta_0^2 h_{t-2} \\
&\Rightarrow \alpha_0(\beta - \beta_0)\varepsilon_{t-2}^2 \\
&\quad = \omega_0 - \omega + \beta_0\omega_0 - \beta\omega + \beta_0^2 h_{t-2} - \beta^2\tilde{h}_{t-2}(\theta).
\end{aligned}
$$

Similarly, we have $\beta = \beta_0$. Furthermore

$$
\omega = \omega_0 + \beta_0 h_{t-1} - \beta_0\tilde{h}_{t-1}(\theta) = \omega_0.
$$

We next show that, for any integer $m \geq 1$,

$$E \sup_{\Theta} \left| \frac{h_t}{\tilde{h}_t(\theta)} \right|^m < \infty.$$

We only give the proof when $m = 1$. Other case is similar.

$$\begin{aligned} h_t &= \omega_0 + \alpha_0 \varepsilon_{t-1}^2 + \beta_0 h_{t-1} \\ &\leq c(1 + \varepsilon_{t-1}^2 + \varepsilon_{t-2}^2 + \varepsilon_{t-3}^2 + h_{t-3}) \\ \tilde{h}_t(\theta) &\geq c(1 + \varepsilon_{t-1}^2 + \varepsilon_{t-2}^2 + \varepsilon_{t-3}^2). \end{aligned}$$

Furthermore, we have

$$\varepsilon_{t-1}^2 + \varepsilon_{t-2}^2 + \varepsilon_{t-3}^2 \geq c(\eta_{t-1}^2 + \eta_{t-2}^2 + \eta_{t-3}^2) h_{t-3}.$$

Thus,

$$\begin{aligned} \frac{h_t}{\tilde{h}_t(\theta)} &\leq c + \frac{c h_{t-3}}{\varepsilon_{t-1}^2 + \varepsilon_{t-2}^2 + \varepsilon_{t-3}^2} \\ &\leq c + \frac{c}{\eta_{t-1}^2 + \eta_{t-2}^2 + \eta_{t-3}^2}. \end{aligned}$$

$$
\begin{aligned}
& E\frac{1}{\eta_{t-1}^2+\eta_{t-2}^2+\eta_{t-3}^2} \\
&= \int_0^\infty P(\eta_{t-1}^2+\eta_{t-2}^2+\eta_{t-3}^2<\frac{1}{y})dy \\
&\le N+\int_N^\infty P(\eta_{t-1}^2+\eta_{t-2}^2+\eta_{t-3}^2<\frac{1}{y})dy \\
&\le N+\int_N^\infty [P(\eta_{t-1}^2<\frac{1}{y})]^3 dy \\
&= N+\int_N^\infty \left[\int_{-\sqrt{y^{-1}}}^{\sqrt{y^{-1}}} f(x)dx\right]^3 dy \\
&\le N+c\int_N^\infty y^{-3/2}dy<\infty,
\end{aligned}
$$

as N is large enough, where c is a constant. Thus,

$$
E\sup_{\Theta}\left|\frac{h_t}{\tilde{h}_t(\theta)}\right|<\infty.
$$

Thus, Assumption 2.1 holds. By Theorem 2.1,

$$
\tilde{\theta}_n \to \theta_0 \ a.s..
$$

As for ARMA(1,1) model, we can show that the initial values do not affect $\hat{\theta}_n$ asymptotically, i.e.

$$
\hat{\theta}_n \to \theta_0 \ a.s..
$$

We now consider Assumption 2.2.

$$
\begin{aligned}
D_t(\theta) &= -\frac{1}{2\tilde{h}_t(\theta)}\frac{\partial \tilde{h}_t(\theta)}{\partial \theta}\left[1-\frac{\varepsilon_t^2}{\tilde{h}_t(\theta)}\right], \\
P_t(\theta) &= -\frac{1}{2\tilde{h}_t(\theta)}\frac{\partial \tilde{h}_t(\theta)}{\partial \theta}\frac{\partial \tilde{h}_t(\theta)}{\partial \theta'} - R_t(\theta), \\
R_t(\theta) &= \frac{1}{\tilde{h}_t(\theta)}\left[\frac{\partial^2 \tilde{h}_t(\theta)}{\partial \theta \partial \theta'}\right. \\
&\quad \left. -\frac{1}{\tilde{h}_t(\theta)}\frac{\partial \tilde{h}_t(\theta)}{\partial \theta}\frac{\partial \tilde{h}_t(\theta)}{\partial \theta'}\right]\left[1-\frac{\varepsilon_t^2}{\tilde{h}_t(\theta)}\right].
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \tilde{h}_t(\theta)}{\partial \omega} &= 1+\beta\frac{\partial \tilde{h}_{t-1}(\theta)}{\partial \omega} = \frac{1}{1-\beta}, \\
\frac{\partial \tilde{h}_t(\theta)}{\partial \alpha} &= \varepsilon_{t-1}^2+\beta\frac{\partial \tilde{h}_{t-1}(\theta)}{\partial \alpha} = \sum_{i=0}^{\infty}\beta^i\varepsilon_{t-i-1}^2, \\
\frac{\partial \tilde{h}_t(\theta)}{\partial \beta} &= \tilde{h}_{t-1}(\theta)+\beta\frac{\partial \tilde{h}_{t-1}(\theta)}{\partial \beta} \\
&= \sum_{i=0}^{\infty}\beta^i\tilde{h}_{t-i-1}(\theta), \\
&= \frac{\omega}{1-\beta}+\alpha\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\beta^{i+j}\varepsilon_{t-i-j-2}^2.
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{\tilde{h}_t(\theta)}\frac{\partial \tilde{h}_t(\theta)}{\partial \alpha} &= \sum_{i=0}^{\infty} \frac{\beta^i \varepsilon_{t-i-1}^2}{\omega + \alpha\beta^i \varepsilon_{t-i-1}^2} \\
&\leq c\sum_{i=0}^{\infty} \frac{\beta^i \varepsilon_{t-i-1}^2}{1 + \beta^i \varepsilon_{t-i-1}^2} \leq c\sum_{i=0}^{\infty} \left[\frac{\beta^i \varepsilon_{t-i-1}^2}{1 + \beta^i \varepsilon_{t-i-1}^2}\right] \\
&\leq c\sum_{i=0}^{\infty} \beta^{i\tau} |\varepsilon_{t-i-1}|^{2\tau} \leq c\sum_{i=0}^{\infty} \rho^{i} |\varepsilon_{t-i-1}|^{2\tau}
\end{aligned}
$$

for any $\tau \in (0,1)$, where $\rho \in (0,1)$ and c are constants. Thus

$$
E\sup_{\Theta}\left|\frac{1}{\tilde{h}_t(\theta)}\frac{\partial \tilde{h}_t(\theta)}{\partial \alpha}\right|^m \leq c\left\{\sum_{i=0}^{\infty} \rho^{i\tau}\left[E|\varepsilon_{t-i}|^{m\tau}\right]^{\frac{1}{m}}\right\}^m < \infty
$$

as τ is small enough.

$$
\begin{aligned}
\frac{1}{\tilde{h}_t(\theta)}\frac{\partial \tilde{h}_t(\theta)}{\partial \beta} &\leq c + c\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{\beta^{i+j} \varepsilon_{t-i-j-2}^2}{\omega + \alpha\beta^{i+j+1} \varepsilon_{t-i-j-2}^2} \\
&\leq c + c\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{\beta^{i+j} \varepsilon_{t-i-j-2}^2}{1 + \beta^{i+j} \varepsilon_{t-i-j-2}^2} \\
&\leq c + c\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \left[\frac{\beta^{i+j} \varepsilon_{t-i-j-2}^2}{1 + \beta^{i+j} \varepsilon_{t-i-j-2}^2}\right]^{\tau} \\
&\leq c + c\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \rho^{(i+j)\tau} |\varepsilon_{t-i-j-2}|^{\tau}
\end{aligned}
$$

for any $\tau \in (0,1)$, where $\rho \in (0,1)$ and c are constants. Thus, for any $m \geq 1$,

$$
\begin{aligned}
& E \sup_{\Theta} \left| \frac{1}{\tilde{h}_t(\theta)} \frac{\partial \tilde{h}_t(\theta)}{\partial \beta} \right|^m \\
& \leq \left\{ c + c \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \rho^{(i+j)\tau} \left[E|\varepsilon_{t-i-j}|^{m\tau} \right]^{\frac{1}{m}} \right\}^m \\
& < \infty
\end{aligned}
$$

as τ is small enough. Furthermore, we can show that, for any $m \geq 1$,

$$E \sup_{\Theta} \left| \frac{1}{\tilde{h}_t^2(\theta)} \frac{\partial \tilde{h}_t(\theta)}{\partial \alpha} \frac{\partial \tilde{h}_t(\theta)}{\partial \beta} \right|^m < \infty$$

and

$$E \sup_{\Theta} \left\| \frac{\partial l_t(\theta)}{\partial \theta} \frac{\partial l_t(\theta)}{\partial \theta'} \right\|^m < \infty.$$

Using the similar method, we can show that

$$E \sup_{\Theta} \left| \frac{1}{\tilde{h}_t(\theta)} \frac{\partial^3 \tilde{h}_t(\theta)}{\partial \theta_i \partial \theta_j \partial \theta_k} \right|^m < \infty$$

for any m and θ_i is ω, α or β. Thus, we can claim that

$$E \sup_{\Theta} |P_t(\theta)| < \infty.$$

For the convariance matrix, we have

$$\begin{aligned}
\Omega &= \frac{E(1-\eta_t^2)^2}{4} E\left[\frac{1}{\tilde{h}_t^2} \frac{\partial \tilde{h}_t(\theta_0)}{\partial \theta} \frac{\partial \tilde{h}_t(\theta_0)}{\partial \theta'}\right], \\
\Sigma &= \frac{1}{2} E\left[\frac{1}{\tilde{h}_t^2} \frac{\partial \tilde{h}_t(\theta_0)}{\partial \theta} \frac{\partial \tilde{h}_t(\theta_0)}{\partial \theta'}\right], \\
\Sigma^{-1}\Omega\Sigma^{-1} &= \frac{E\eta_t^4 - 1}{2} (2\Sigma)^{-1} \equiv \kappa (2\Sigma)^{-1}.
\end{aligned}$$

Thus,

$$\sqrt{n}(\tilde{\theta}_n - \theta_0) \to_{\mathcal{L}} N(0, \kappa(2\Sigma)^{-1}).$$

As for ARMA(1,1) model, we can show that the initial values do not affect $\widehat{\theta}_n$ asymptotically, i.e.

$$\sqrt{n}(\widehat{\theta}_n - \theta_0) \to_{\mathcal{L}} N(0, \kappa(2\Sigma)^{-1}).$$

When $\eta_t \sim N(0,1)$, $E\eta_t^4 = 3$ and $\kappa = 1$. When $\eta_t \not\sim N(0,1)$, $\widehat{\theta}_n$ is called the QMLE.