## 4.3. Mixing portmanteau test

- Null model:

$$Y_t = m(\theta, X_{t-1}) + \eta_t \sqrt{h(\theta, X_{t-1})}. \qquad (1)$$

Given observations $\{Y_1, \ldots, Y_n\}$ and the initial value $Y_0$, the conditional log-LF (ignoring a constant) can be written as

$$L_n(\theta) = \sum_{t=1}^{n} \ell_t(\theta)$$

$$\ell_t(\theta) = -\left[\frac{1}{2} \log h_t(\theta) + \frac{\varepsilon_t^2(\theta)}{2h_t(\theta)}\right],$$

where $m_t(\theta) = m(\theta, Y_{t-1}), h_t(\theta) = h(\theta, Y_{t-1})$ and $\varepsilon_t(\theta) = Y_t - m_t(\theta)$.

The maximizer of $L_n(\theta)$ on the parameter spaces $\Theta$, denoted by $\widehat{\theta}_n$, is the QMLE of $\theta_0$. Denote

$$\begin{aligned}
\eta_t(\theta) &= \frac{\varepsilon_t(\theta)}{\sqrt{h_t(\theta)}}, \\
D_t(\theta) &= \frac{\partial \ell_t(\theta)}{\partial \theta} = -U_t(\theta)\xi_t(\theta) \quad \text{and} \\
P_t(\theta) &= \frac{\partial^2 \ell_t(x)}{\partial \theta \partial \theta'} = -U_t(\theta)U_t^{'}(\theta) - R_t(\theta),
\end{aligned}$$

where

$$\begin{aligned}
\xi_t(\theta) &= \left(\eta_t(\theta), \frac{1}{\sqrt{2}}\left[1-\eta_t^2(\theta)\right]\right)', \\
U_t(\theta) &= \left[\frac{1}{\sqrt{h_t(\theta)}}\frac{\partial \varepsilon_t(\theta)}{\partial(\theta)}, \frac{1}{\sqrt{2}h_t(\theta)}\frac{\partial h_t(\theta)}{\partial \theta}\right], \\
R_t(\theta) &= \frac{1}{\partial h_t(\theta)}\left[\frac{\partial^2 h_t(\theta)}{\partial\theta\partial\theta'} - \frac{1}{h_t(\theta)}\frac{\partial h_t(\theta)}{\partial \theta}\frac{\partial h_t(\theta)}{\partial \theta'}\right] \\
&\quad \left[1 - \frac{\varepsilon_t^2(\theta)}{h_t(\theta)}\right].
\end{aligned}$$

Furthermore, let $J = E(\xi_t(\theta_0)\xi_t'(\theta_0))$,

$\Omega = E\left[U_t(\theta_0)JU_t^{'}(\theta_0)\right]$ and $\Sigma = E\left[U_t(\theta_0)U_t'(\theta_0)\right]$.

We assume:

$\Omega$ and $\Sigma$ are positive definite and the estimator

$\widehat{\theta}_n$ has the following expansion:

$$\sqrt{n}(\widehat{\theta}_n - \theta_0) = \frac{\Sigma^{-1}}{\sqrt{n}} \sum_{t=1}^{n} D_t(\theta_0) + o_p(1).$$

The assumptions of Theorem 3.1 hold.

The lag-$\ell$ (standardized) residual ACF can be defined as

$$\tilde{\rho}_\ell = \sum_{t=\ell+1}^{n} \widehat{\eta}_t \widehat{\eta}_{t-\ell} / \sqrt{\sum_{t=\ell+1}^{n} \widehat{\eta}_{t-\ell}^2 \sum_{t=1}^{n} \widehat{\eta}_t^2}.$$

The lag-$\ell$ squared-residual (standardized) ACF as

$$\tilde{r}_\ell = \sum_{t=\ell+1}^{n} \left(\widehat{\eta}_t^2 - \tilde{\varepsilon}\right)\left(\widehat{\eta}_{t-\ell}^2 - \tilde{\varepsilon}\right) / \sum_{t=1}^{n} \left(\widehat{\eta}_t^2 - \tilde{\varepsilon}\right)^2,$$

where

$$\tilde{\varepsilon} = n^{-1} \sum_{t=1}^{n} \widehat{\eta}_t^2 \text{ and } \ell = 1, 2, \cdots, M.$$

For the null model (1), the Ljoung-Box and McLeod-Li tests are still used in SAS.

Ling and Li (1997) showed that

$$\begin{aligned}\sqrt{n}\,(\widehat{\rho}_1, \cdots, \widehat{\rho}_M)' \; &= \; \sqrt{n}\,(\rho_1, \cdots, \rho_M)' \\ &\quad -X_\rho \sqrt{n}(\widehat{\theta}_n - \theta_0) + o_p(1),\end{aligned}$$

for any given integer $M$, where

$$\begin{aligned} \rho_\ell &= \frac{1}{n}\sum_{t=\ell+1}^{n} \eta_t\eta_{t-\ell}, \\ X_\rho &= (X_{\rho 1}, X_{\rho 2}, \cdots, X_{\rho M})', \\ X_{\rho\ell} &= E\left[\frac{\partial\varepsilon_t(\theta_0)}{\partial\theta}\frac{\eta_{t-\ell}}{\sqrt{h_t}}\right], \;\; \ell = 1, 2, \cdots, M. \end{aligned}$$

Li and Mak (1994) and Ling and Li (1997) showed that

$$\begin{aligned} \sqrt{n}(\widehat{r}_1, \cdots, \widehat{r}_M)' &= \sqrt{n}(r_1, \cdots, r_M)' \\ &\quad - \frac{1}{\sigma_\varphi}X_r\sqrt{n}(\widehat{\theta}_n - \theta_0) + o_p(1), \end{aligned}$$

where $X_r = (X_{r1}, X_{r2}, \cdots, X_{rM})'$ and $X_{r\ell} = E\left\{h_t^{-1}[\partial h_t(\theta_0)/\partial\theta](\eta_{t-\ell}^2 - 1)\right\}$.

Denote $\widehat{\rho} = (\widehat{\rho}_1, \ldots, \widehat{\rho}_M)'$ and $\widehat{r} = (\widehat{r}_1, \cdots, \widehat{r}_M)'$.

We get the following expansion:

$$\sqrt{n}\begin{pmatrix} \widehat{\rho} \\ \widehat{r} \end{pmatrix} = \sqrt{n}V\begin{bmatrix} \rho \\ r \\ \frac{1}{n}\sum_{t=1}^{n} D_t(\theta_0) \end{bmatrix} + o_p(1),$$

where $V = \begin{bmatrix} I_M & 0 & -X_\rho\Sigma^{-1} \\ 0 & I_M & -X_r\Sigma^{-1}/\sigma_\varphi \end{bmatrix}$.

Denote

$$Z_n = \sqrt{n}\left[\rho', r', \frac{1}{n}\sum_{t=1}^{n} D_t'(\theta_0)\right]'.$$

It is easy to show that

$$\begin{aligned}
Z_n &= \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left[\eta_t\eta_{t-1},\cdots,\eta_t\eta_{t-M},\right.\\
&\quad \frac{\left(\eta_t^2-1\right)\left(\eta_{t-1}^2-1\right)}{\sigma_\varphi},\\
&\quad \left.\cdots,\frac{\left(\eta_t^2-1\right)\left(\eta_{t-M}^2-1\right)}{\sigma_\varphi}, D_t'(\theta_0)\right]' + o_p(1)\\
&\equiv \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\nu_t + o_p,
\end{aligned}$$

where $\nu_t$ is a martingale difference in terms of $\mathcal{F}_t$ with

$$E(\nu_t\nu_t') \equiv \tilde{\Omega}.$$

Thus,

$$Z_n \to_{\mathcal{L}} N(0,\tilde{\Omega}) \text{ and } \sqrt{n}\begin{pmatrix}\widehat{\rho}\\ \widehat{r}\end{pmatrix} \to_{\mathcal{L}} N(0, V\tilde{\Omega}V').$$

Wong and Ling (2005) define the following statistic:

$$Q_M = n\begin{pmatrix}\rho\\ r\end{pmatrix}'(V\Omega V^1)^{-1}\begin{pmatrix}\widehat{\rho}\\ \widehat{r}\end{pmatrix}.$$

$$Q_M \longrightarrow_{\mathcal{L}} \chi^2(2M) \;\; as \;\; n \to \infty.$$

We can use $Q_M$ to test the hypothesis $H_0$ : Model (1) is correct.

When $\eta_t$ is $N(0,1)$, $\Omega = I$, and

$$\hat{\Omega} = \begin{bmatrix} I_M & 0 & X_\rho \\ 0 & I_M & X_r \\ X_\rho' & X_r' & \Sigma^{-1} \end{bmatrix}$$

$$V\hat{\Omega}V' = \begin{bmatrix} I_M - X_\rho \Sigma^{-1} X_\rho' & 2^{-1} X_\rho \Sigma^{-1} X_r' \\ 2^{-1} X_r \Sigma^{-1} X_\rho' & I_M - \frac{1}{4} X_r \Sigma^{-1} X_r' \end{bmatrix}.$$

When $X_\rho \approx 0$, $Q_M$ can be further simplified as

$$Q_{0M} \approx n \sum_{l=1}^{M} \hat{\rho}_l^2 + n\hat{r}' \left[ I_M - \frac{1}{4} X_r \Sigma^{-1} X_r' \right]^{-1} \hat{r}.$$

This is the mixed statistic of Box-Pierce and Li-Mak test statistics.

In many time series model $X_{\rho l} \approx 0$ when $l$ is large enough. In this case, we can use the following modified statistics.

$$
\begin{aligned}
Q_{1M} &\approx n \sum_{l=L_0}^{M} \widehat{\rho}_l^2 + n\widehat{r}' \left[ I_M - \frac{1}{4} X_r \Sigma^{-1} X_r' \right]^{-1} \widehat{r} \\
&\qquad \rightarrow_{\mathcal{L}} \chi^2(2M - L_0 + 1).
\end{aligned}
$$

The $Q_{1M}$ statistic can be used for diagnostic checking the mean and variance part of model (2.1) jointly.

This statistic is very simple.

Simulation results show that $Q_{1M}$ seems to work reasonably well when $L_0 = 1$.

**Simulation Study**:

We further modify $Q_{1M}$ to rectify the conservative nature of BP statistic as follows.

$$Q_s \approx n(n+2)\sum_{l=1}^{M}\frac{\hat{\rho}_l^2}{n-l} + \ n(n+2)\sum_{l=1}^{M}\frac{\hat{r}_l^2}{n-l}.$$

Replications are 1000 and the sample size $n =$ 100, 200 and 400

The first null model is the AR(1) model,

$$(a). \qquad Y_t = 0.5Y_{t-1} + \varepsilon_t, \ \ \varepsilon_t \sim iidN(0,1).$$

The following five alternative models are used to study the powers of $Q_M$ and $Q_s$:

$$
\begin{aligned}
(b). \quad & Y_t = 0.2Y_{t-1} + \varepsilon_t, \\
& \quad E\left(\varepsilon_t^2 \middle| F_{t-1}\right) = h_t \\
& \quad h_t = 0.2 + 0.2\varepsilon_{t-1}^2 + 0.2\varepsilon_{t-2}^2. \\
(c). \quad & Y_t = 0.3Y_{t-1} + \varepsilon_t, \\
& \quad h_t = 0.3 + 0.3\varepsilon_{t-1}^2 + 0.3h_{t-1}. \\
(d). \quad & Y_t = 0.6Y_{t-1} + 0.3Y_{t-1}\varepsilon_{t-1} + \varepsilon_t, \\
& \quad \varepsilon_t \sim iidN(0,1). \\
(e). \quad & Y_t = 0.7Y_{t-1} + 0.2Y_{t-2} + \varepsilon_t, \\
& \quad h_t = 0.01 + 0.2\varepsilon_{t-1}^2.
\end{aligned}
$$

Following Li and Mak (1994), we choose $M$=6 in the simulations.

Table I. Size and power of $Q_M$, $Q_S$, $Q_{LB}$ and $Q_{ML}$

| | Models | (a) | (b) | (c) | (d) | (e) |
|---|---|---|---|---|---|---|
| 100 | $Q_M$ | 0.060 | 0.346 | 0.363 | 0.435 | 0.362 |
| | $Q_S$ | 0.070 | 0.359 | 0.374 | 0.445 | 0.379 |
| | $Q_{LB}$ | 0.062 | 0.108 | 0.105 | 0.244 | 0.324 |
| | $Q_{ML}$ | 0.060 | 0.381 | 0.401 | 0.377 | 0.226 |
| 200 | $Q_M$ | 0.048 | 0.635 | 0.683 | 0.762 | 0.659 |
| | $Q_S$ | 0.054 | 0.639 | 0.686 | 0.769 | 0.665 |
| | $Q_{LB}$ | 0.050 | 0.104 | 0.114 | 0.344 | 0.560 |
| | $Q_{ML}$ | 0.056 | 0.673 | 0.717 | 0.720 | 0.428 |
| 400 | $Q_M$ | 0.048 | 0.908 | 0.923 | 0.976 | 0.937 |
| | $Q_S$ | 0.050 | 0.912 | 0.924 | 0.976 | 0.939 |
| | $Q_{LB}$ | 0.054 | 0.096 | 0.106 | 0.560 | 0.863 |
| | $Q_{ML}$ | 0.044 | 0.939 | 0.945 | 0.962 | 0.710 |

The second null model is AR(1)-ARCH(1) model:

$(f)$. $\quad Y_t = 0.5Y_{t-1} + \ \varepsilon_t \ , \ h_t = 0.01 + 0.4\varepsilon_{t-1}^2.$

The following five models are the alternative models:

$$
\begin{aligned}
(g). \quad & Y_t = 0.5Y_{t-1} + \ \varepsilon_t, \\
& h_t = 0.01 + 0.4\varepsilon_{t-1}^2 + 0.3\varepsilon_{t-2}^2, \\
(h). \quad & Y_t = 0.5Y_{t-1} + \ \varepsilon_t, \\
& h_t \ = 0.01 + 0.4\varepsilon_{t-1}^2 + 0.5h_{t-1}, \\
(i). \quad & Y_t = 0.5Y_{t-1} + 0.2Y_{t-2} + \ \varepsilon_t, \\
& h_t = 0.01 + 0.4\varepsilon_{t-1}^2, \\
(j). \quad & Y_t = 0.5Y_{t-1} + 0.1Y_{t-2} + \ \varepsilon_t, \\
& h_t = 0.01 + 0.4\varepsilon_{t-1}^2 + 0.1\varepsilon_{t-2}^2, \\
(k). \quad & Y_t = 0.5Y_{t-1} + 0.2Y_{t-2} + \ \varepsilon_t, \\
& h_t = 0.01 + 0.4\varepsilon_{t-1}^2 + 0.2\varepsilon_{t-2}^2.
\end{aligned}
$$

Table II. Size and power of $Q_M$, $Q_{1M}$, $Q_{BP}$, and $Q_{LM}$

| M. | | (f) | (g) | (h) | (i) | (j) | (k) |
|---|---|---|---|---|---|---|---|
| 100 | $Q_M$ | 0.057 | 0.248 | 0.289 | 0.220 | 0.142 | 0.369 |
| | $Q_{1M}$ | 0.040 | 0.241 | 0.295 | 0.184 | 0.112 | 0.348 |
| | $Q_{BP}$ | 0.030 | 0.076 | 0.064 | 0.222 | 0.102 | 0.324 |
| | $Q_{LM}$ | 0.036 | 0.277 | 0.333 | 0.073 | 0.079 | 0.195 |
| 200 | $Q_M$ | 0.061 | 0.492 | 0.472 | 0.406 | 0.235 | 0.706 |
| | $Q_{1M}$ | 0.047 | 0.481 | 0.477 | 0.362 | 0.195 | 0.685 |
| | $Q_{BP}$ | 0.033 | 0.091 | 0.075 | 0.499 | 0.139 | 0.602 |
| | $Q_{LM}$ | 0.055 | 0.547 | 0.509 | 0.094 | 0.164 | 0.427 |
| 400 | $Q_M$ | 0.053 | 0.782 | 0.548 | 0.723 | 0.441 | 0.953 |
| | $Q_{1M}$ | 0.047 | 0.782 | 0.558 | 0.735 | 0.398 | 0.950 |
| | $Q_{BP}$ | 0.038 | 0.079 | 0.092 | 0.871 | 0.261 | 0.923 |
| | $Q_{LM}$ | 0.055 | 0.844 | 0.561 | 0.154 | 0.300 | 0.727 |