Testing for the threshold AR model

$$y_t = \phi_{10} y_{t-1} I(y_{t-1} > r) + \phi_{20} y_{t-1} I(y_{t-1} \le r) + \varepsilon_t$$

$H_0: \phi_{10} = \phi_{20}$ v.s $H_1: \phi_{10} \neq \phi_{20}$.

$$L(\phi_1, \phi_2 r) = \sum_{t=1}^{n} (y_t - \phi_1 y_{t-1})^2 I(y_{t-1} > r)$$
$$+ \sum_{t=1}^{n} (y_t - \phi_2 y_{t-1})^2 I(y_{t-1} \le r)$$

$$\min_{\phi_1, \phi_2} L(\phi_1, \phi_2, r) = L[\hat{\phi}_1(r), \hat{\phi}_2(r), r]$$

Under H_0,

- $$y_t = \phi y_{t-1} + \varepsilon_t$$

$$L(\phi) = \sum_{t=1}^{n} (y_t - \phi y_{t-1})^2. \quad \hat{\phi} = \sum_{t=1}^{n} y_t y_{t-1} \Big/ \sum_{t=1}^{n} y_{t-1}^2$$

$$\min_{\phi} L(\phi) = \sum_{t=1}^{n} (y_t - \hat{\phi} y_{t-1})^2$$
$$= \sum_{t=1}^{n} [\varepsilon_t - (\hat{\phi} - \phi_0) y_{t-1}]^2$$
$$= \sum_{t=1}^{n} \varepsilon_t^2 - (\hat{\phi} - \phi_0)^2 \sum_{t=1}^{n} y_{t-1}^2$$
$$= \sum_{t=1}^{n} \varepsilon_t^2 - \left(\sum_{t=1}^{n} y_{t-1}\varepsilon_t\right)^2 \Big/ \sum_{t=1}^{n} y_{t-1}^2 .$$

$$LT(r) = \min_{\phi} L(\phi) - \min_{\phi_1, \phi_2} L(\phi_1, \phi_2, r)$$

$$L(\hat{\phi}_1(r), \hat{\phi}_2(r), r) = \sum_{t=1}^{n} (y_t - \hat{\phi}_1(r) y_{t-1})^2 I(y_{t-1} > r) + \sum_{t=1}^{n} (y_t - \hat{\phi}_2(r) y_{t-1})^2 I(y_{t-1} \le r)$$

$$\sum_{t=1}^{n} [y_t - \hat{\phi}_1(r) y_{t-1}]^2 I(y_{t-1} > r)$$

$$= \sum_{t=1}^{r} [\varepsilon_t - (\hat{\phi}_1(r) - \phi_{1,0}) y_{t-1}]^2 I(y_{t-1} > r)$$

$$= \sum_{t=1}^{r} \varepsilon_t^2 \, I(y_{t-1} > r) - 2 (\hat{\phi}_1(r) - \phi_{1,0}) \sum_{t=1}^{n} y_{t-1} \varepsilon_t I(y_{t-1} > r) + (\hat{\phi}_1(r) - \phi_{1,0})^2 \sum_{t=1}^{r} y_{t-1}^2 I(y_{t-1} > r)$$

$$\hat{\phi}_1(r) - \phi_{1,0} = \frac{\sum_{t=1}^{n} [y_t - \phi_{1,0} y_{t-1}] y_{t-1} I(y_{t-1} > r)}{\sum_{t=1}^{n} y_{t-1}^2 I(y_{t-1} > r)} = \frac{\sum_{t=1}^{n} y_{t-1} \varepsilon_t I(y_{t-1} > r)}{\sum_{t=1}^{n} y_{t-1}^2 I(y_{t-1} > r)}$$

$$= \sum_{t=1}^{n} \varepsilon_t^2 I(y_{t-1} > r) - \frac{\left[\sum_{t=1}^{n} y_{t-1} \varepsilon_t I(y_{t-1} > r)\right]^2}{\sum_{t=1}^{n} y_{t-1}^2 I(y_{t-1} > r)}$$

Similarly,

$$\sum_{t=1}^{n} [y_t - \hat{\phi}_2(r) y_{t-1}]^2 I(y_{t-1} \le r)$$

$$= \sum_{t=1}^{n} \varepsilon_t^2 I(y_t \le r) - \frac{\left[\sum_{t=1}^{n} y_{t-1}\varepsilon_t I(y_{t-1} \le r)\right]^2}{\sum_{t=1}^{n} y_{t-1}^2 I(y_{t-1} \le r)}.$$

Thus,

$$L(\hat{\phi}_1(r), \hat{\phi}_2(r), \cancel{r}) = \frac{\left[\sum_{t=1}^{n} y_{t-1}\varepsilon_t I(y_t > r)\right]^2}{\sum_{t=1}^{n} y_{t-1}^2 I(y_t > r)}$$

$$+ \frac{\left[\sum_{t=1}^{n} y_{t-1}\varepsilon_t I(y_{t-1} \le r)\right]^2}{\sum_{t=1}^{n} y_{t-1}^2 I(y_{t-1} \le r)}$$

$$- \left(\sum_{t=1}^{n} y_{t-1}\varepsilon_t\right)^2 \Big/ \sum_{t=1}^{n} y_{t-1}^2$$

$$\frac{1}{n}\sum_{t=1}^{n} y_{t-1}^2 = \sigma_y^2 + o(1), \quad \frac{1}{n}\sum_{t=1}^{n} y_{t-1}^2 I(y_{t-1} \le r) = \sigma_y^2(r) + o(1)$$

$$\sigma_y^2 = E y_{t-1}^2 \qquad \sigma_y^2(r) = E\, y_{t-1}^2 I(y_{t-1} \le r)$$

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Thus,

$$L(\hat{f}_1(r), \hat{f}_2(r), r) = \frac{1}{\sigma_y^2 - \sigma_y^2(r)} \left[\frac{1}{\sqrt{n}} \sum_{t=1}^{n} y_{t-1} \varepsilon_t I(y_{t-1} > r) \right]^2$$

$$+ \frac{1}{\sigma_y^2(r)} \left[\frac{1}{\sqrt{n}} \sum_{t=1}^{n} y_{t-1} \varepsilon_t I(y_{t-1} \le r) \right]^2$$

$$- \frac{1}{\sigma_y^2} \left[\frac{1}{\sqrt{n}} \sum_{t=1}^{n} y_{t-1} \varepsilon_t \right]^2 .$$

$$S_n(r) = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} y_{t-1} \varepsilon_t I(y_{t-1} \le r)$$

$$\Rightarrow \sigma B(\sigma^2(r)) \quad \text{in } D[-M, M] \text{ for } \forall M >$$

(?)

① $\{ S_n(r) : r \in [-M, M] \}$ is tight.

② For $\forall r, u$, the finite dimensional distribution of $S_n(r)$ is normal. In fact for $\forall\ c = (c_1, c_2)' \neq 0$, $(u > r)$

$$c_1 S_n(r) + c_2 S_n(u) = (c_1 + c_2) S_n(r) + c_2 \frac{1}{\sqrt{n}} \sum_{t=1}^{n} y_{t-1} \varepsilon_t I(r < y_{t-1} \le u)$$

$$= \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \left[(c_1 + c_2) y_{t-1} \varepsilon_t I(y_{t-1} \le r) + c_2 y_{t-1} \varepsilon_t I(y_{t-1} \in (r, u]) \right]$$

$$\longrightarrow N\left(0,\ (c_1+c_2)^2 \sigma^2 \sigma_y^2(r) + c_2^2 \sigma^2 [\sigma_y^2(a) - \sigma_y^2(r)]\right)$$

$$= N\left(0,\ c_1^2 \sigma^2 \sigma_y^2(r) + 2 c_1 c_2 \sigma^2 \sigma_y^2(r) + c_2^2 \sigma^2 \sigma_y^2(a)\right)$$

$$= N\left(0\ \ (c_1, c_2) \begin{bmatrix} \sigma_y^2(r) & \sigma_y^2(r) \\ \sigma_y^2(r), & \sigma_y^2(a) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \sigma^2 \right)$$

Thus,

$$\begin{pmatrix} S_n(r) \\ S_n(a) \end{pmatrix} \xrightarrow{d} N\left(0,\ \begin{bmatrix} \sigma_y^2(r), & \sigma_y^2(r) \\ \sigma_y^2(r) & \sigma_y^2(a) \end{bmatrix} \sigma^2 \right)$$

Let $\tau = \frac{\sigma_y^2(r)}{\sigma_y^2}$. Then $\tau \in [\overset{a}{0}, \overset{b}{1}]$.

$$E[B(\tau) B(\tau')] = (\tau \wedge \tau') \sigma^2 \sigma_y^2$$

$B(\tau)$ is an independent ~~interval~~ increment Gaussian process and hence it is a Brownian motion.

$$\sigma^{-2} L(\hat{f}_1(r), \hat{f}_2(r), r) \xrightarrow{d} \frac{1}{1-\tau}[B(1) - B(\tau)]^2 + \frac{1}{\tau} B^2(\tau) - B^2(1)$$

$$= \frac{[\tau B(1) - B(\tau)]^2}{\tau(1-\tau)} \quad \text{in } D[\overset{a}{0}, \overset{b}{1}]$$

$a = \sigma^2(-M)/\sigma_y^2$

$b = \sigma^2(M)/\sigma_y^2$

$$\hat{\sigma}_n^2 = \frac{1}{n} \sum_{t=1}^{n} (y_t - \hat{\phi} y_{t-1})^2$$

$$\sup_{a \le \tau \le b} \hat{\sigma}_r^{-2} L_r(\hat{\phi}_1(r), \hat{\phi}_2(r), \hat{\beta})$$

$$\longrightarrow \sup_{a \le \tau \le b} \frac{[\tau B(1) - B(\tau)]^2}{\tau(1-\tau)}$$

Let $B_s = [t B(1) - B(s)]$.

$$P\left(\sup_{a \le \tau \le b} B_\tau^2 > b^2 \right)$$

$$= P\left(\sup_{0 \le t \le \alpha} |U(t)| > b \right), \quad \alpha = \frac{1}{2} \ln \left[\frac{b(1-a)}{a(1-b)} \right].$$

$$\simeq \left[\frac{2}{\pi}\right]^{\frac{1}{2}} \exp\left\{-\frac{b^2}{2}\right\} \left(\alpha b - \frac{\alpha}{b} + \frac{1}{b}\right).$$

where $U(t)$ is a stationary Ornstein-Uhlenbeck process with $EU(t) = 0$

and $EU(s)U(t) = e^{-|t-s|}$.